Fourierova transformace distribucí

Def. Je-li $T \in \mathcal{S}'$, pak def. $\hat{T} = \mathcal{F}[T]$ předpisem
$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle \quad \forall \hat{\varphi} \in \mathcal{S}$$

Příklady typu: Mějme distribuci T $\rightarrow$ spočtěme $\hat{T}$

Př $T = T_f \leftarrow f \in L^1(\mathbb{R}^d)$ $\leadsto$ $T = f$

$$\langle \hat{T}_f, \varphi \rangle \underset{\text{def.}}{=} \langle T_f, \hat{\varphi} \rangle = \int_{\mathbb{R}^d} f(s)\hat{\varphi}(s)\,ds = \int_{\mathbb{R}^d} f(s)\left(\int_{\mathbb{R}^d} \varphi(x) e^{-2\pi i x s}\,dx\right) ds$$

$$\underset{\substack{\text{Fubini}\\ \mathbb{R}^d \times \mathbb{R}^d}}{=} \int_{\mathbb{R}^d} \varphi(x) \underbrace{\left(\int f(s) e^{-2\pi i x s}\,ds\right)}_{\hat{f}(x)} dx = \langle \hat{f}, \varphi \rangle = \langle T_{\hat{f}}, \varphi \rangle$$

$\widehat{T_f} = T_{\hat{f}}$ ← $\hat{f}$ ve smyslu klasickém pro $f \in L^1$

↑ ve smyslu distribucí

Př $T = \delta \rightarrow \hat{T} \equiv \hat{\delta} = 1$

$\check{T} \equiv \check{\delta} = 1$

$$\langle \hat{\delta}, \varphi \rangle = \langle \delta, \hat{\varphi} \rangle = \hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(x) e^{-2\pi i x s}\,dx \Big|_{s=0} = \int_{\mathbb{R}} \varphi(x)\,dx$$

$$= \int_{\mathbb{R}} 1 \cdot \varphi(x)\,dx = \langle 1, \varphi \rangle$$

Př $T = 1 \equiv T_1 \rightarrow \hat{T}$

$\quad f = 1 \notin L^1$

Tvrz. Na $\mathcal{S}'$ platí inv. Fourierův vzorec
$\forall T \in \mathcal{S}'$: $\quad \mathcal{F}^{-1}[\mathcal{F}[T]] = T$

$\mathcal{F}[\delta] = 1$ & $\underline{\mathcal{F}^{-1}[\delta] = 1}$ $\Rightarrow$ $\underbrace{\mathcal{F}[\mathcal{F}^{-1}[\delta]]}_{=\delta} = \mathcal{F}[1]$ $\Rightarrow$ $\boxed{\hat{1} = \delta \text{ v } \mathcal{S}'}$

$\delta \underset{\mathcal{F}}{\overset{\mathcal{F}}{\rightleftarrows}} 1$ Fourierův pár

(Př) $T = x^n \in \mathcal{S}'(\mathbb{R})$ x^n roste v ∞ jen jako polynom, tedy $\varphi \in \mathcal{S}$ zkulturní integrand

$\updownarrow$

- $\widehat{D^\alpha f}(s) = (i2\pi s)^\alpha \hat{f}(s)$
- $\widehat{(-i2\pi x)^\alpha f(x)}(s) = D^\alpha \hat{f}(s)$

(Př) $T = \delta^{(n)} \in \mathcal{S}'(\mathbb{R})$ $n \in \mathbb{N}$

$$\langle \hat{T}, \varphi \rangle = \langle \widehat{\delta^{(n)}}, \varphi \rangle = \langle \delta^{(n)}, \hat{\varphi} \rangle = (-1)^n \langle \delta, (\hat{\varphi})^{(n)} \rangle = (-1)^n \langle \delta(s), \frac{d^n}{ds^n} \hat{\varphi}(s) \rangle$$

$$= (-1)^n \langle \delta(s), \widehat{(-2\pi i x)^n \varphi(x)}(s) \rangle = \langle \hat{\delta}(x), \underbrace{(2\pi i x)^n}_{n} \varphi(x) \rangle = \langle (2\pi i x)^n \cdot 1, \varphi(x) \rangle$$

Analogicky:
$$\boxed{\begin{aligned} \widehat{\delta^{(n)}} &= (2\pi i)^n x^n \quad \text{v } \mathcal{S}' \\ \check{\delta}^{(n)} &= (-2\pi i)^n x^n \quad \text{v } \mathcal{S}' \end{aligned}}$$

$$\mathcal{F}^{-1}[\delta^{(n)}] = (-2\pi i)^n x^n \longrightarrow \delta^{(n)} \overset{\text{inv. Four. vzorec}}{=} \mathcal{F}[\mathcal{F}^{-1}[\delta^{(n)}]] = \mathcal{F}[(-2\pi i)^n x^n]$$

$$\widehat{x^n} = \frac{1}{(-2\pi i)^n} \delta^{(n)} \quad \& \quad \widehat{\delta^{(n)}} = (2\pi i)^n x^n$$

(Př) $T = \exp(2\pi i b x)$ $b \in \mathbb{R}$

$T = \delta_b$ ... $\langle \delta_b, \varphi \rangle = \varphi(b)$

$$\langle \hat{T}, \varphi \rangle = \langle \hat{\delta}_b, \varphi \rangle = \langle \delta_b, \hat{\varphi} \rangle = \hat{\varphi}(b) = \int_{\mathbb{R}} \varphi(x) \underline{e^{-2\pi i b x}}\, dx = \langle e^{-2\pi i b x}, \varphi \rangle$$

$$\boxed{\hat{\delta}_b = e^{-2\pi i b x}}$$

$$\check{\delta}_b = e^{2\pi i b x} \quad \Rightarrow \quad \boxed{\delta_b = \widehat{e^{2\pi i b x}}} \quad b \in \mathbb{R} \quad \text{v } \mathcal{S}'$$

Důsledky: $\cos(2\pi b x) = \dfrac{e^{i2\pi b x} + e^{-i2\pi b x}}{2}$

$$\widehat{\cos(2\pi b x)} = \frac{1}{2}(\delta_b + \delta_{-b}) \rightsquigarrow \langle \widehat{\cos(2\pi b x)}, \varphi(x) \rangle = \frac{1}{2}(\varphi(b) + \varphi(-b))$$

$\sin(2\pi b x) = \dfrac{e^{i2\pi b x} - e^{-i2\pi b x}}{2i}$

$$\widehat{\sin(2\pi b x)} = \frac{1}{2i}(\delta_b - \delta_{-b})$$

Zkusme psát: $\cosh(2\pi kx) = \dfrac{e^{2\pi kx} + e^{-2\pi kx}}{2}$

$$\widehat{\cosh(2\pi kx)} = \frac{1}{2}\left(\delta_{\frac{k}{i}} + \delta_{-\frac{k}{i}}\right) = \frac{1}{2}\left(\delta_{-ik} + \delta_{ik}\right)$$

$$\widehat{\sinh(2\pi kx)} = \frac{1}{2}\left(\delta_{-ik} - \delta_{ik}\right)$$

Q: Umí δ žít v $\mathbb{C}$?

Q: Platí vztahy i pro $k \in \mathbb{C}$?

Platí i pro $T = e^{2\pi i kx}$ s $k \in \mathbb{C}$?

Pozor: $e^{2\pi i kx} \notin \mathcal{S}'$ pro $\operatorname{Im} k \neq 0$ (exponenciální růst)

Ale: $e^{2\pi i kx} \in \mathcal{D}'$ ... zadání kompaktní nosič $\varphi \in \mathcal{D}$

- Platí $e^{2\pi i kx} = \sum_{n=0}^{\infty} \frac{(2\pi i k)^n}{n!} x^n \in \mathcal{D}'(\mathbb{C})$

$$\lim_{N \to \infty} \underbrace{\sum_{n=0}^{N} \frac{(2\pi i k)^n}{n!} x^n}_{\in \mathcal{S}' \text{ ale nekonverguje v } \mathcal{S}'} \text{ v } \mathcal{D}'(\mathbb{C})$$

$$\mathcal{F}\left[\sum_{n=0}^{N} \frac{(2\pi i k)^n}{n!} x^n\right] = \sum_{n=0}^{N} \frac{(2\pi i k)^n}{n!} \mathcal{F}[x^n] = \sum_{n=0}^{N} \frac{(\cancel{2\pi i} k)^n}{n!} \frac{\delta^{(n)}}{(-\cancel{2\pi i})^n}$$

$$= \sum_{n=0}^{N} \frac{(-k)^n}{n!} \delta^{(n)}$$

$$\left\langle \widehat{e^{2\pi i kx}}, \varphi \right\rangle \xleftarrow[?]{N \to \infty} \left\langle \widehat{\sum_{n=0}^{N} \frac{(2\pi i k)^n}{n!} x^n}, \varphi \right\rangle = \left\langle \sum_{n=0}^{N} \frac{(-k)^n}{n!} \delta^{(n)}, \varphi \right\rangle = (-1)^n \left\langle \sum \frac{(-k)^n}{n!} \delta, \varphi^{(n)} \right\rangle$$

$$= \underbrace{\sum_{n=0}^{N} \frac{k^n}{n!} \varphi^{(n)}(0)}_{\text{Taylorův rozvoj } \varphi(x) \text{ kolem nuly}} \xrightarrow[?]{N \to \infty} \varphi(k) = \langle \delta_k, \varphi \rangle$$

$\boxed{\text{Paley - Weiner}}$

(Př) $T = \operatorname{sgn}(x) \in \mathcal{S}'(\mathbb{R})$

- $\operatorname{sgn}(x) \notin L^1(\mathbb{R})$
- zkoumejme $f_k(x) = \operatorname{sgn}(x) e^{-\frac{|x|}{k}}$ $\quad k \in \mathbb{N} \quad \Rightarrow \quad f_k(x) \in L^1(\mathbb{R})$

$$\operatorname{sgn}(x) e^{-\frac{|x|}{k}} \xrightarrow[k\to\infty]{\text{slabě}} \operatorname{sgn}(x)$$

> **Definice** Buď $\Omega \subset \mathbb{R}^d$ otevřená, $\{T_k\}_{k=1}^{\infty} \subset \mathcal{D}'(\Omega)$, $T \in \mathcal{D}'(\Omega)$. Říkáme, že T_k konvergují k T slabě (neboli v $\mathcal{D}'$) pokud
> $$\langle T_k, \varphi\rangle \to \langle T, \varphi\rangle \quad \forall \varphi \in \mathcal{D}(\Omega)$$
> Říkáme, že $\boxed{\sum_{k=1}^{\infty} T_k = T \text{ v } \mathcal{D}'} \Leftrightarrow \sum_{k=1}^{\infty} \langle T_k, \varphi\rangle = \langle T, \varphi\rangle \quad \forall \varphi \in \mathcal{D}(\Omega)$.

$$\lim_{k\to\infty} \int_{\mathbb{R}} \operatorname{sgn}(x) e^{-\frac{|x|}{k}} \varphi(x)\,dx \overset{?}{=} \int_{\mathbb{R}} \operatorname{sgn}(x)\varphi(x)\,dx \quad \checkmark$$

Integrovatelná majoranta je $|1 \cdot \varphi(x)|$

- $$\widehat{T_{f_k}} = T_{\widehat{f_k}} = \widehat{\operatorname{sgn}(x) e^{-\frac{|x|}{k}}} \underset{\text{hladce}}{=} \int_{-\infty}^{\infty} \underbrace{\operatorname{sgn}(x) e^{-\frac{|x|}{k}}}_{\text{lichá v } x} \underbrace{e^{-2\pi i x s}}_{\cos + i\sin} dx$$
  ($f_k \in L^1$)

$$= 2i \operatorname{Im} \int_0^{\infty} e^{-\frac{x}{k}} e^{-2\pi i s x}\,dx = 2i \operatorname{Im} \left[\frac{e^{-\frac{x}{k} - 2\pi i s x}}{-\frac{1}{k} - 2\pi i s} \right]_0^{\infty} = 2i \operatorname{Im} \left(\frac{1}{\frac{1}{k} + 2\pi i s} \right)$$

$$= 2i \operatorname{Im} \frac{\frac{1}{k} - 2\pi i s}{\frac{1}{k^2} + 4\pi^2 s^2} = \underline{- \frac{4\pi s i}{\frac{1}{k^2} + 4\pi^2 s^2}}$$

- Spočtěme slabou limitu, resp. ukažme, že $\frac{-4\pi s i}{\frac{1}{k^2} + 4\pi^2 s^2} \xrightarrow{\text{slabě}} -\frac{i}{\pi} T_{v.p.\frac{1}{x}}$ v $\mathcal{S}'$

$$v.p. \int_{-\infty}^{\infty} \left(-\frac{4\pi i s}{\frac{1}{k^2} + 4\pi^2 s^2} + \frac{i}{\pi}\frac{1}{x} \right) \varphi(x)\,dx$$

$$= \frac{i}{\pi}\, v.p. \int_{-\infty}^{\infty} \left(\frac{1}{x} - \frac{4\pi^2 x}{\frac{1}{k^2} + 4\pi^2 x^2} \right) \varphi(x)\,dx = \frac{i}{\pi}\, v.p. \int_{-\infty}^{\infty} \frac{\frac{1}{k^2}}{x\left(\frac{1}{k^2} + 4\pi^2 x^2\right)} \varphi(x)\,dx$$

$$= \underbrace{\frac{i}{\pi} \int_{\mathbb{R}\setminus\langle -1,1\rangle} \frac{\frac{1}{k^2}}{x\left(\frac{1}{k^2} + 4\pi^2 x^2\right)} \varphi(x)\,dx}_{\xrightarrow[k\to\infty]{} 0} + \underbrace{\frac{i}{\pi} \int_{-1}^{1} \frac{\frac{1}{k^2}}{x\left(\frac{1}{k^2} + 4\pi^2 x^2\right)} \big(\varphi(x) - \varphi(0)\big)\,dx}_{\xrightarrow[k\to\infty]{} 0} + \underbrace{v.p. \int_{-1}^{1} \underbrace{\frac{\frac{1}{k^2}}{x(\;)}}_{\text{lichá}} \varphi(0)\,dx}_{=0}$$

$$\underline{= 0}$$

$$\widehat{\mathrm{sgn}(x)} = -\frac{i}{\pi} T_{v.p.\frac{1}{x}} \quad \longleftrightarrow \quad \widehat{T_{v.p.\frac{1}{x}}} = -i\pi \, \mathrm{sgn}(x)$$