Incompressible Navier–Stokes–Fourier system

Goal: to extend Leray's programme to this system

Governing eqs

$\Omega \subset \mathbb{R}^3$, $(0,T)$

- $\boxed{\operatorname{div} v = 0}$, $\quad \varrho(t,x) = \varrho_* \in (0,\infty)$ everywhere

- $\boxed{\varrho_* \left(\partial_t v + \operatorname{div}(v \otimes v)\right) = \operatorname{div} \mathbb{T}}$, $\quad \boxed{\mathbb{T} = \mathbb{T}^T}$

$E := e + \frac{|\vec{v}|^2}{2}$

- $\boxed{\varrho_* \partial_t \left(e + \frac{|v|^2}{2}\right) + \varrho_* \left(\operatorname{div}\left(\left(e + \frac{|v|^2}{2}\right) v\right)\right) = \operatorname{div}(\mathbb{T} v) - \operatorname{div} \vec{j}_e}$

5 Eqs for $v = (v_1, v_2, v_3)$, p, e

- 2-nd law

$\boxed{\varrho_* (\partial_t \eta + \operatorname{div}(\eta \vec{v})) + \operatorname{div} \vec{j}_\eta = \zeta}$ with $\zeta \geq 0$ entropy production

Const. eqs

$$\boxed{\mathbb{T} = -p\mathbb{I} + 2\tilde{\nu}(e)\,\mathbb{D}v}$$

$$\boxed{\vec{j}_e = -\kappa(e)\nabla e}$$

Balance energy Take eq. for E and $\int_\Omega$: (Gauss

$$\frac{d}{dt}\int_\Omega \varrho_*\left(e + \frac{|v|^2}{2}\right) dx + \cancel{\int_{\partial\Omega} \varrho_* \left(e + \frac{|v|^2}{2}\right) \vec{v}\cdot n \, dS}$$

$$+ \int_{\partial\Omega} \vec{j}_e \cdot n - \mathbb{T}\vec{v}\cdot\vec{n} \, dS = 0$$

$= \gamma_* |v_\tau|^2$

Boundary conditions.

- $\vec{v}\cdot\vec{n} = 0$ on $(0,T)\times\partial\Omega$ — internal flows
- $-(\mathbb{S}n)_\tau = \gamma_* v_\tau$ —"— Navier slip (stick-slip is possible)
- $\vec{j_e}\cdot n = \begin{cases} 0 & \to \text{thermally isolated body} \\ -\gamma_* |v_\tau|^2 & \to \text{energetically conserved bc's} \end{cases}$

Aux calcs: $\mathbb{T}=\mathbb{T}^T$

$$\mathbb{T}v\cdot n \underset{v\cdot n=0}{=} (\mathbb{T}n)_\tau\cdot v_\tau = (\mathbb{S}n)_\tau\cdot v_\tau$$

!! The analysis presented below does not work for no-slip bc's.

Under bc's in yellowed boxes:

$$\boxed{\frac{d}{dt}\int \rho_*\left(e+\frac{|v|^2}{2}\right) = 0} \Rightarrow E(t)=E_0 \quad \forall t>0$$

$$E(t) := \int \rho_*\left(e(t,x)+\frac{|v(t,x)|^2}{2}\right)dx$$

$$E_0 := \int \rho_*\left(e_0+\frac{|v_0|^2}{2}\right)dx\,,$$

where v_0, e_0 are initial data:

IC's

$$v(0,\cdot) = v_0 \quad \text{in } \Omega$$

$$e(0,\cdot) = e_0 \quad \text{in } \Omega$$

- $0 < \nu_* \le \nu(e) \le \nu^* < +\infty$
- $0 < k_* \le k(e) \le k^* < +\infty.$

✓ The simplest model relating internal energy e to temperature θ: $\boxed{e = c_v\theta}$ $(c_v > 0)$

$S = S(E, \dots)$

✓ $\left.\begin{array}{l} E \\ S \end{array}\right\}$ $S \nearrow$ as $E \nearrow$

$$\frac{1}{\text{temperature}} = \frac{\partial S}{\partial E}$$

classical thermodynamics

$$\boxed{\frac{1}{\theta} := \frac{\partial \eta}{\partial e}} \iff \frac{c_v}{e} = \frac{\partial \eta}{\partial e} = \eta'(e)$$

$\eta = \eta(e)$

$$\boxed{\eta = c_v \ln e + C_*} \Rightarrow$$

$$\vec{j}_\eta = \frac{\vec{j}_e}{\theta} = c_v \frac{\vec{j}_e}{e}$$

2nd law eq.

$$\varrho_* \cancel{\frac{c_v}{e}}\Big(\partial_t e + \underbrace{v_k \frac{\partial e}{\partial x_k}}_{\operatorname{div}(e\vec{v})}\Big) + \underbrace{c_v \operatorname{div}\left(\frac{\vec{j}_e}{e}\right)}_{\parallel} = \zeta$$

$$-\cancel{\frac{c_v}{e}}\operatorname{div}(\kappa(e)\nabla e) + \cancel{\frac{c_v}{e^2}}\kappa(e)\nabla e\cdot\nabla e$$

where $\zeta = \frac{1}{\theta}\xi = \frac{c_v}{e}\xi =$

$\xi = \theta\zeta$

$$= \cancel{\frac{c_v}{e}}\left\{2\nu(e)|\mathbb{D}v|^2 + \frac{\kappa(e)}{e}|\nabla e|^2\right\}$$

$$\boxed{\varrho_*\left(\partial_t e + \operatorname{div}(e\vec{v})\right) - \operatorname{div}(\kappa(e)\nabla e) = 2\nu(e)|\mathbb{D}v|^2}$$

The same eq. is obtained if one multiply bal. of lin. mom. scalarly by $\vec{v}$, and subtract the result from the bal of total energy:

On the level of weak solutions, it is $\boxed{\text{open}}$ how to show the validity of

EQUALITY

$$\rho_*(\partial_t e + \mathrm{div}(e\vec{v})) - \mathrm{div}(\kappa(e)\nabla e) = 2\nu(e)|\mathbb{D}v|^2$$

However, we can add to a closed system

weak sol.

$$\boxed{\mathrm{div}\, v = 0, \quad (\mathrm{BLM}) + (\mathrm{BE})}$$

system of 5 eqs for 5 unknowns

ineq.

$$\rho_*(\partial_t e + \dots) - \mathrm{div}(\kappa(e)\nabla e) \geq 2\nu(e)|\mathbb{D}v|^2$$

suitable weak sols

Why? Because subtract the last ineq. from balance of energy, we get

$$\rho_*\left(\partial_t \frac{|v|^2}{2} + \mathrm{div}\left(\frac{|v|^2}{2}\vec{v}\right)\right) + \mathrm{div}(p\vec{v}) - \mathrm{div}(2\nu(e)\mathbb{D}v\, v) \leq 0$$

and noticing that for $\nu(e) = \nu^*$ we will get Caffarelli, Kohn, Nirenberg ineq. for kinetic energy

Q: When these inequalities can be replaced by equalities?

$$\frac{1}{\theta} := \frac{\partial \eta}{\partial e}$$

$$\boxed{e = c_v \theta} \quad \& \quad \boxed{\eta = \eta(e)}$$

$\Downarrow$

$$\boxed{\frac{c_v}{e} = \eta'(e)}$$

$$\boxed{\eta(e) = c_v \ln e + c_*}$$

$$\varrho_* \left(\frac{\cancel{c_v}}{\cancel{e}} \partial_t e + \frac{c_v}{\cancel{e}} v_k \frac{\partial e}{\partial x_k} \right) - \operatorname{div}\left(\frac{\cancel{c_v} \kappa(e) \nabla e}{e} \right) =$$

$$- \frac{\operatorname{div}(\cancel{c_v} \kappa(e) \nabla e)}{\cancel{e}} + \cancel{c_v} \kappa(e) \frac{\nabla e \cdot \nabla e}{e^2} =$$

$$= \cancel{c_v} \kappa(e) \frac{\nabla e \cdot \nabla e}{e^2} + \cancel{c_v} \frac{2\nu(e) |D(v)|^2}{e}$$

$$\xi = \theta \zeta$$

$$\boxed{\vec{v} \cdot \vec{n} = 0 \quad \text{on } \partial\Omega}$$

$$\int_\Omega \varrho_* \left(e + \frac{|v|^2}{2}\right) dx = \int_{\partial\Omega} \left(\mathbb{T}\vec{v} \cdot \vec{n} - \vec{j}_e \cdot \vec{n} \right) dS$$

$$= \int_{\partial\Omega} (\mathbb{T}\vec{n})_{\tau} \cdot \vec{v}_{\tau} \, dx$$

$$\boxed{\text{Stick}/\text{slip}}$$

$$\frac{1}{2}\partial_t \int_\Omega |v^\varepsilon|^2 + 2\nu_* \int_\Omega |Dv^\varepsilon|^2 = 0$$

$\varepsilon \to 0$

$$\leq 0$$

$$\int_0^T \langle \partial_t v, \varphi \rangle + \text{c.t.} + \nu(Dv, D\varphi) = 0 \quad \forall \varphi$$

The energy eq. would hold if

$\varphi = v$ is admissible test funct

in balance of lin. momentum

Summary

At the level of smooth fcts (C^∞, C^1) but also at the level of weak solutions when v is admissible test f. in BLM, then

$$
(1)\quad \boxed{\begin{aligned}
&\operatorname{div} v = 0\\
&\partial_t v + \operatorname{div}(v \otimes v) = \operatorname{div}(\mathbb{T} v)\\
&\partial_t (E) + \operatorname{div}(E v) = \operatorname{div}(\mathbb{T} v - \vec{j_e})
\end{aligned}} \qquad E := e + \frac{|v|^2}{2}
$$

$$\Updownarrow$$

$$
(2)\quad \boxed{\begin{aligned}
&\operatorname{div} v = 0\\
&\partial_t v + \operatorname{div}(v \otimes v) = \operatorname{div}(\mathbb{T} v)\\
&\partial_t e + \operatorname{div}(e \vec{v}) = -\operatorname{div}(j_e) + \underbrace{\mathbb{T} : \mathbb{D} v}_{\geq 0}
\end{aligned}} \qquad \in L^1(Q_T)
$$

In 3D: this equivalence is open.

- (1) has better structure since all terms in eq. for (E) are in divergence form.
- However, eq. for e has good structure to show that $e \geq 0$ (or $e \geq e_* > 0$)

Formal apriori estimates $\longrightarrow$ both (1) and (2) are in our disposal.

① Energy estimates $\quad E(t) = e + \frac{|u|^2}{2}$

- $\int_\Omega E(t) \overset{=}{\leq} \int_\Omega E_0 \Rightarrow \sqrt{e}, \; v \in L^\infty(0,T; L^2)$

$|u|^2, \; e \in L^\infty(0,T; L^1)$

(assuming $e \geq 0$.)

- $\int_\Omega BLM \cdot \vec{v} \quad \Rightarrow \int_0^T \int_\Omega 2\nu(e) |Du|^2 dx dt < +\infty$

$+ \int_0^T \int_{\partial\Omega} \gamma_* |v_\tau|^2 dx dS < +\infty$

Remark 1) Note that for liquids: $\nu \searrow$ as $e \nearrow$

· Reynold's (1900) $\quad \nu(e) = \mu_* \exp(-m_*/e)$

· Vogel (1922) $\quad \nu(e) = \mu_* \dfrac{A_*}{B_* + e}$

$\boxed{\text{for gas (compressible)}: \quad \nu \nearrow \text{ as } e \nearrow}$

* Feireisl, Novotný – (2011)
* Feireisl (2004)
} focused on thermal phenomena

* Feireisl, Zenpen, Pokorný (2017) – no temperature

plenty of articles

② Minimum principle for e

$$\frac{\partial e}{\partial t} + v_k \frac{\partial e}{\partial x_k} - \operatorname{div}(\kappa(e)\nabla e) = \nu(e)|Dv|^2$$

Goal: Assuming that $e(0,\cdot) = e_0 \geq e_* > 0$ in Ω,
show that $e(t,x) \geq e_*$ in Q_T.

$$(e_* - e_0)^+ = 0 \text{ in } \Omega \Rightarrow (e_* - e)^+ = 0 \text{ in } Q_T$$

$$(*) \quad \frac{\partial (e_* - e)}{\partial t} + v_k \frac{\partial (e_* - e)}{\partial x_k} - \operatorname{div}(\kappa(e)\nabla(e_* - e)) = -\nu(e)|D(v)|^2 \leq 0$$

Multiply $(*)$ by $(e_* - e)^+$, $\int_\Omega$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left[(e_* - e)^+\right]^2 + \underbrace{\int_\Omega v_k \frac{\partial}{\partial x_k} \frac{\left[(e_* - e)^+\right]^2}{2}}_{=0} + \underbrace{\int_\Omega \underbrace{\kappa(e)}_{\geq 0} \underbrace{\nabla(e_* - e) \cdot \nabla (e_* - e)^+}_{|\nabla(e_* - e)^+|^2} dx}_{\geq 0} \leq 0$$

by parts
$\operatorname{div} v = 0$
$v \cdot n = 0$ on $\partial\Omega$.

$$\int_0^t ds \Rightarrow \int_\Omega \left(\left[e_* - e(t,x)\right]^+\right)^2 = \int_\Omega \left(\left[e_* - e_0\right]^+\right)^2 dx = 0$$

$$\Rightarrow \boxed{e(t,x) \geq e_* \text{ a.e. in } Q_T}$$

(3) Goal: to get info on ∇e ? $\mathbb{T} : Du$

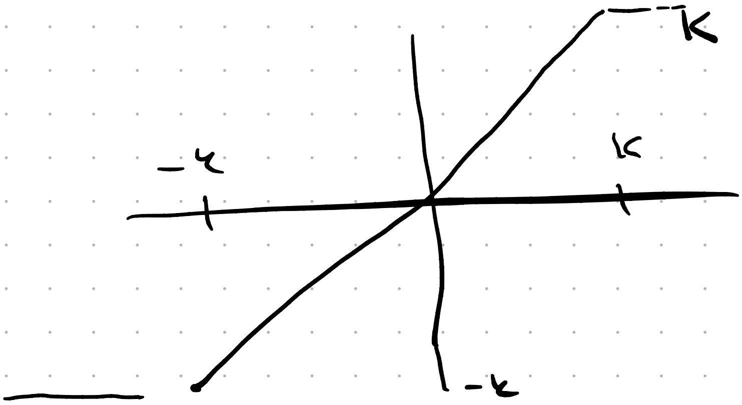
We use

$$\frac{\partial e}{\partial t} + v_k \frac{\partial e}{\partial x_k} - \mathrm{div}\big(\kappa(e)\nabla e\big) = \underbrace{\nu(e)|Du|^2}_{\in L^1}$$

$$\frac{1}{e^\lambda} \overset{?}{\in} L^\infty(Q_T) \qquad \lambda>0$$

$$e \geq e_* \Rightarrow 0 < \frac{1}{e^\lambda} \leq \frac{1}{(e_*)^\lambda}$$

$> \boxed{T_k(e)}$



Or better

$$\frac{1}{[T_k(e)]^\lambda}$$

and after getting the estimate let $k \to \infty$

HW: DO THE FOLLOWING STEPS FOR $\frac{1}{[T_k(e)]^\lambda}$

Gallouët, Boccardo 1991
Boccardo, Murat

$1 > \lambda > 0$

$$\int_\Omega \kappa(e)\nabla e \cdot \nabla \frac{1}{e^\lambda} =$$

$$= -\lambda \int_\Omega \kappa(e)\nabla e \cdot \frac{1}{e^{\lambda+1}}\nabla e = -\lambda \int \kappa(e) \underbrace{\frac{\nabla e}{e^{\frac{\lambda+1}{2}}}}_{} \cdot \frac{\nabla e}{e^{\frac{\lambda+1}{2}}}$$

$$\frac{4}{1-\lambda} \nabla e^{\frac{1-\lambda}{2}} \cdot \nabla e^{\frac{1-\lambda}{2}}$$

$$= -\frac{4\lambda}{1-\lambda} \int_\Omega \underbrace{\kappa(e)\left|\nabla e^{\frac{1-\lambda}{2}}\right|^2}_{\geq 0}$$

$$\frac{1}{1-\lambda}\frac{\partial}{\partial x_k} e^{1-\lambda}$$

$$\frac{4\lambda}{1-\lambda}\kappa_* \int_Q \left|\nabla e^{\frac{1-\lambda}{2}}\right|^2 \leq -\int_Q \underbrace{\frac{\partial e}{\partial t}\frac{1}{e^\lambda}}_{-\frac{1}{1-\lambda}\frac{\partial}{\partial t}(e^{1-\lambda})} - \underbrace{\int_Q v_k \frac{\partial e}{\partial x_k}\frac{1}{e^\lambda}}_{=0} + \underbrace{\int_Q \nu(e)|Du|^2 \frac{1}{e^\lambda}}_{<+\infty}$$

$\nu(e)|Du|^2 \in L^1(Q)$, $\frac{1}{e^\lambda} \in L^\infty$

$$\leq \frac{1}{1-\lambda}\int_\Omega [e(T,x)]^{1-\lambda}dx + \frac{1}{1-\lambda}\int_\Omega [e_0]^{1-\lambda}dx$$

bdd due to 1st a priori estimate

Hence $\int_0^T\int_\Omega |\nabla e^{\frac{1-\lambda}{2}}|^2 dx < +\infty \quad \forall \lambda > 0 \ (\lambda < 1)$

we see that $\frac{\nabla\sqrt{e}}{\sqrt{e}}$ is almost in $L^2(Q)$ bdd
is bdd in $L^\infty(L^2)$

One can interpolate information that
$\underbrace{e^{\frac{1}{2}} \in L^\infty(L^2) \text{ and } \nabla e^{\frac{1}{2}-\frac{\lambda}{2}} \in L^2(L^2)}$

$e \in L^{5/3-\varepsilon}(Q_T)$

Hölder

$\nabla e \in L^{5/4-\varepsilon}(Q_T)$

Similarly as $v \in L^{10/3}(Q)$
$\Uparrow$
$v \in L^\infty(L^2) \cap L^2(W^{1,2})$

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(4) Our concept of sol. is based on PDE eqn (1). where eq (1)$_3$, i.e. balance of energy, takes the form

$$\frac{\partial}{\partial t}\left(e+\frac{|v|^2}{2}\right) + v_k\frac{\partial}{\partial x_k}\left(e+\frac{|v|^2}{2}+p\right) - \operatorname{div}(\kappa(e)\nabla e) = \operatorname{div}(2\mu[\mathbb{D}v]v)$$

$= \operatorname{div}(\mathbb{T}v) = -\operatorname{div}(pv) + \operatorname{div}(2\mu[\mathbb{D}v]v)$

$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D}v$

This formulation avoids the pressure p.

We need uniform estimates on p

From BLM, applying div:

$$-\Delta p = \operatorname{div}\operatorname{div}(v\otimes v - \nu(e)\mathbb{D}v)$$

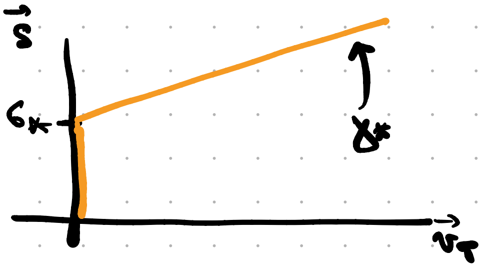
$\boxed{p}$ should behave as $\underbrace{v\otimes v}_{\in L^{5/3}(Q_T)} - \underbrace{\nu(e)\mathbb{D}v}_{\in L^2(Q_T)}$, $\nu(e) \le \nu^* \in L^\infty$, $\mathbb{D}v \in L^2$

$\Rightarrow \boxed{p \in L^{5/3}(Q_T)}$

BUT THIS IS INDEED POSSIBLE TO SHOW EXCEPT NO-SLIP BC.

In order to have the result for almost no-slip bc's, we consider $\boxed{\text{stick-slip bc}}$:

$$\vec{s} = (-Sn)_\tau$$



$$\boxed{\gamma_* v_\tau = \frac{(|\vec{s}| - \sigma_*)^+ s}{|\vec{s}|}} \iff \vec{h}(\vec{s}, \vec{v}_\tau) = \vec{0} \text{ on } (0,T)\times\partial\Omega$$

$\boxed{\text{It holds:}}$ For any data (reasonable) $\exists$ (suitable) weak solution to NSF system with stick-slip bc's in the sense of weak form. of (1).

Construction of approximation. Problem (P) is approximated by (P_m), where (P_m):

$$+\frac{1}{2}\operatorname{div}(G_m(|v|^2)v)v$$

- $-\varepsilon p +$ $\operatorname{div} v = 0$
- $\frac{\partial v}{\partial t} + \operatorname{div}(\underbrace{G_m(|v|^2)}_{\in L^\infty(Q)} v\otimes v) - 2\operatorname{div}(\nu(e) Dv) = -\nabla p \qquad \cdot v$
- $\frac{\partial e}{\partial t} + \operatorname{div}(ev) - \operatorname{div}(\kappa(e)\nabla e) \overset{\nabla v \in L^2}{=} 2\nu(e)|Dv|^2$

$$\operatorname{div}(\dots\, v) + G_m(|v|^2)\underbrace{v_i v_j \frac{\partial v_j}{\partial x_i}}_{v_i \frac{\partial}{\partial x_i}\frac{|v|^2}{2}} = \operatorname{div}(\dots)\, \pm - \frac{|v|^2}{2}\operatorname{div}(G_m(|v|^2)v) + \frac{|v|^2}{2}\operatorname{div}(G_m(|v|^2)v)$$

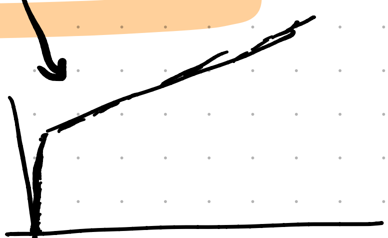
$$+\frac{1}{2}\varepsilon p^2$$

IC

- $v(0,\cdot) = v_0$, $e(0,\cdot) = e_0$
- $\vec{v}\cdot\vec{n} = 0$ + $\gamma_* v_\tau = \frac{(|s| - \sigma_*)^+}{|s|}\vec{s} + \frac{1}{m}\vec{s}$ on $(0,T)\times\partial\Omega$

$$\Downarrow$$

$$\boxed{\vec{s} = \tilde{\vec{s}}_m(v_\tau)\, v_\tau}$$



$\boxed{\text{HW: Write}}$

- $q\cdot n = 0$ on $(0,T)\times\partial\Omega$.

Lecture, Jan 4, 2021 = 43x47

$$-\overset{\frac{1}{m}}{\varepsilon} p + \operatorname{div} v = 0$$

$$\partial_t v + \operatorname{div}\left(G_m(\cdot)\, v\otimes v\right) + \frac{1}{2}\operatorname{div}\left(G_m(\cdot) v\right) v + \operatorname{div}\left(2\mu(e) D v\right) = -\nabla p \quad \Big|\ \underline{w}$$

How to get uniform estimates for the pressure:

$(v, p, e) = (v^m, p^m, e^m)$

$$\int_\Omega p \operatorname{div} w = \dots \int \partial_t v \cdot w$$

Take
$$\boxed{\begin{aligned} \operatorname{div} w &= p \\ w &= 0 \text{ on } \partial\Omega \end{aligned}}$$
NO-SLIP

However for stick-slip and other slipping bc's we can look for $\boxed{w = \nabla z}$

$$\boxed{\begin{aligned} -\Delta z &= p \\ \frac{\partial z}{\partial n} &= \nabla z \cdot n = w \cdot n = 0 \end{aligned}}$$

$$\|z\|_{2,2} \le C \|p\|_2$$

$(z, -\Delta \psi) = (p, \psi)$

$$\boxed{\begin{aligned} -\Delta \psi &= \partial_t p \\ \psi|_{\partial\Omega} &= 0 \end{aligned}} \Leftarrow \psi = (-\Delta_D)^{-1} p$$

Homogeneous Neumann problem

$$\int_\Omega |p^m|^2 = \int |p|^2 = -\int \partial_t v \cdot \nabla z = -\int_\Omega \partial_t \operatorname{div} v \cdot z$$

$$= -\varepsilon \int_\Omega \partial_t p\, z = -\varepsilon \int p (-\Delta_D)^{-1} \partial_t p \le 0$$

$$\int |p^m|^2 = \int |p|^2 = -\int \partial_t v \cdot \nabla z - \dots$$

$$\overset{\text{Helmholtz on } v}{=} -\int \left(\partial_t v^{div} + \partial_t \nabla g^v\right) \cdot \nabla z = -\int \partial_t \nabla g^v \cdot \nabla z$$

$$= -\frac{1}{\varepsilon} \int_\Omega \partial_t |\nabla g^v|^2 \le 0$$

$$\boxed{v = v^{div} + \nabla g^v}$$

$$\boxed{\begin{aligned} -\Delta g^v &= \operatorname{div} v = \varepsilon p \\ \frac{\partial g^v}{\partial n} &= 0 \end{aligned}}$$

$$(\nabla g^v, \nabla z) = (\varepsilon p, z)$$

$$z = \frac{g^v}{\varepsilon}$$

On RHS we also have e.g.

$$\Big| -\int_\Omega \underbrace{\nu(e)}_{L^\infty} \underbrace{Du}_{L^2} \cdot \underbrace{\nabla^2 z}_{} \Big| \overset{*}{\leq} \|Du\|_2 \|z\|_{2,2}$$

$$\underset{\nearrow}{\leq} c\|Du\|_2 \|p\|_2$$

$$\boxed{\|z\|_{2,2} \leq c\|p\|_2}$$

$$\|p\|_2^2 \leq \underset{\leq 0}{☺} + \underbrace{c\|Du\|_2 \|p\|_2}_{\text{Young } c\|Du\|_2^2 + \frac{1}{4}\|p\|_2^2}$$

Having n-approximation, we collect the uniform estimates

From these estimates: there is subsequence (v^n, p^n, e^n) such that

$v^n \rightharpoonup v \quad L^\infty(L^2) \cap L^2(W^{1,2}_{n,div})$

$\partial_t v^n \rightharpoonup \partial_t v \quad (L^4(W^{1,2}_{n,div}))^*$

strong $\quad v^n \to v \quad L^q(L^q) \quad \forall\, q \in [1, \frac{10}{3})$

strong $\quad \boxed{v^n \to v \quad L^q(0,T; L^q(\partial\Omega)) \quad \underline{\forall\, q} \in [1, \frac{8}{3})}$

$e^n \rightharpoonup e \quad L^{\frac{5}{4}-\varepsilon}(W^{1,\frac{5}{4}-\varepsilon})$

strong $\quad e^n \to e \quad L^q(L^q) \quad q \in [1, \frac{5}{3})$

$\partial_t e^n = \nu(e)|Dv|^2 - \operatorname{div}(ev) - \kappa(e)\nabla e$

$\Downarrow \qquad \underline{L^1}$

$s^n \rightharpoonup s \quad L^2(L^2(\partial\Omega))$

$\partial_t e^n$ bdd in $\underline{L^1(0,T; (W^{1,q}(\Omega))^*)}$

$q > 10 > 3$

$$\boxed{e^n \geq e_*}$$

Recall strict-slip $\quad \nu_* = \underbrace{\frac{1}{\gamma^*}\frac{(|s|-\sigma_*)^+}{|s|}s}_{f(s)} + \frac{1}{n\gamma^*}s$

$\int s^n \nu_*^n \leq C$

$\Downarrow$

$\int |s^n|^2 \leq C$

About traces.

- $W^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega)$
- $W^{1,2}(\Omega) \overset{on}{\hookrightarrow} W^{1/2,2}(\partial\Omega)$

$\underbrace{1 \geq s > \frac{1}{2}}$
- $\underline{W^{s,2}(\Omega)} \overset{on}{\hookrightarrow} W^{s-1/2,2}(\partial\Omega)$

$$\|v\|_{s,2} \leq \underline{\|v\|_{1,2}^{\lambda = s} \, \|v\|_{0,2}^{(1-\lambda) = 1-s}} \qquad \boxed{\lambda = s}$$

Taking $2/s$ power

$$\|v\|_{s,2}^{2/s} \leq \|v\|_{1,2}^{2} \, \|v\|_{2}^{(1-s)\frac{2}{s}}$$

$L^\infty(0,T;L^2)$ $\qquad L^2(W^{1,2})$

$$\boxed{\int_0^T \|v\|_{s,2}^{2/s} < +\infty}$$

$$W^{s,2}(\Omega) \hookrightarrow W^{s-1/2,2}(\partial\Omega) \hookrightarrow L^q(\partial\Omega)$$

$$q^{-1} = \frac{3-2s}{4}$$

$$\Downarrow$$

$$\int_0^T \|v\|_{\frac{4}{3-2s}}^{2/s} < +\infty$$

When $\frac{2}{s} = \frac{4}{3-2s} \Rightarrow s = \frac{3}{4}$

$$\Rightarrow \boxed{\int_0^T \|v\|_{8/3}^{8/3} < +\infty}$$

Concl: $L^2(W^{1,2}) \cap L^\infty(L^2)$

Take weak form for v^n, use $v^n u$ as test function
and Add weak form for e^n with test function u

$$E^n = \frac{|v^n|^2}{2} + e^n$$

$$\partial_t E^n + \operatorname{div}(E^n v^n) + \operatorname{div}(p^n v^n) + \operatorname{div}(2\mu(e^n) D v^n \, v^n) - \operatorname{div}(\kappa(e^n)\nabla e^n)$$

$$\operatorname{div}\left(\left(\frac{|v^n|^2}{2} - |v^n|^2 G_n(|v^n|^2) + \frac{1}{2}\int_0^{|v^n|^2} G_n(s)\,ds\right) v^n\right)$$

In the weak form also the term $\int_{\partial\Omega} s^n \cdot v^n$ appears

$\boxed{\text{Now : } n \to \infty}$

$\sim |v^n|^3$ $\qquad v^n$ bdd $\qquad L^{10/3}$

$$\operatorname{div}(\nabla K(e^n)) = \Delta K(e^n)$$

It remains : $\boxed{v_\tau = f(s)}$

$$0 \le \int_{\partial\Omega} (f(s^n) - f(z)) \cdot (s^n - z)$$

$$f(s^n) = v_\tau^n - \frac{1}{n\gamma_n} s^n$$

$\lim_{n\to\infty}$

$$\boxed{0 \le \int_{\partial\Omega} (v_\tau - f(z))\cdot(s - z)\, dS}$$

and then use Minty's idea

$z = s \pm \varepsilon \varphi$

$$\Rightarrow \boxed{v_\tau = f(s)}$$