DUAL SPACES. SECOND DUAL. REFLEXIVITY. WEAK AND WEAK-* CONVERGENCES

Let $(X, \|\cdot\|_X)$ be a Banach space.

Def. The space

$$X^* = X' = \mathcal{L}(X, \mathbb{K}) = \{\phi : X \to \mathbb{K} \,;\, \phi \text{ is linear \& continuous}\}$$

bdd $\Updownarrow$

is called <u>dual space</u> to X.

Recall that

$$(X^*, \|\phi\|_{X^*}) \quad \text{where } \|\phi\|_{X^*} = \sup_{\substack{x \in X \\ \|x\|_X = 1}} |\phi(x)|$$

is a Banach space.

Notation Often $\phi(x)$ is denoted $\langle \phi, x\rangle = \langle \phi, x\rangle_{X^*, X}$ or $\langle x, \phi\rangle_{X, X^*}$ and $\langle \cdot, \cdot\rangle_{X^*, X}$ is called <u>duality pairing</u>.

Let $x \in X$ be arbitrary. The mapping

$J_x : \boxed{\Phi \in X^* \longmapsto \Phi(x)}$ is a linear bounded*) map of X^* into $\mathbb{K}$.

It means that it belongs to $(X^*)^* =: X^{**}$, which is the second dual to X. Since, by Corollary of H-B theorem there is $\phi \in X^*$ s.t. $\|\phi\|_{X^*} = 1$ and $\phi(x) = \|x\|_X$, then

$$\|J_x\|_{X^{**}} = \|x\|_X$$

Hence, the mapping

$$J : X \to X^{**}, \quad x \longmapsto J_x$$

is isometry and always $J(X) \subseteq X^{**}$.

$\boxed{J \text{ is called canonical embedding}}$

Def The space X is called <u>reflexive</u> if $\boxed{J(X) = X^{**}}$

(It means that one can identified the second dual with X)

*) Towards boundedness: we know $|\phi(x)| \le \|\phi\|_{X^*} \|x\|_X$, which implies that

$$\|J_x\|_{X^{**}} \le \|x\|_X$$

Def. (1) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ converges weakly to x in X,
$\boxed{x_n \rightharpoonup x \text{ in } X}$ if $\phi(x_n) \to \phi(x)$ for all $\phi \in X^*$

(2) A sequence $\{\phi_n\}_{n=1}^{\infty} \subset X^*$ converges weakly-* to ϕ in X^*
$\boxed{\phi_n \xrightharpoonup{*} \phi \text{ in } X^*}$ if $\phi_n(x) \to \phi(x)$ for all $x \in X$.

weakly star

Several observations

i It holds: If $x_n \to x$ in X (strongly), then $x_n \rightharpoonup x$ in X (weakly)

Pf For arbitrary fixed $\phi \in X'$:
$$|\phi(x_n) - \phi(x)| \underset{\text{lin}}{=} |\phi(x_n - x)| \le \|\phi\|_{X^*} \|x_n - x\|_X$$
Since RHS $\to 0$ as $n \to \infty$, LHS $\to 0$ as $n \to \infty$, which gives the result.

ii Weak limit is well-defined, i.e., it cannot happen that

(*) $x_n \rightharpoonup x$ in X a $x_n \rightharpoonup y$ in X a $x \neq y$.

Pf If $x \neq y$, then by Corollary of H-B theorem: $\exists \phi \in X^*$: $\phi(x) \neq \phi(y)$, but then $\{\phi(x_n)\}$ cannot converge to both $\phi(x)$ and $\phi(y)$. ↯ ∎

iii Recalling that
$\phi_n \to \phi$ in X^* means that
$$\boxed{\sup_{\substack{x \in X \\ \|x\|_X \le 1}} |\phi_n(x) - \phi(x)| \to 0 \text{ as } n \to \infty} \quad (S)$$
we see that
wear-star convergence is weaker than the (strong) convergence in the norm of X^*. Weak-* convergence requires that
$$\boxed{\forall x \in X \quad |\phi_n(x) - \phi(x)| \to 0 \text{ as } n \to \infty} \quad (W^*)$$

iv If X is reflexive, then weak and weak-* convergences "coincide".

Pf We first observe that if there is a Banach space Y so that $Y^* = X$ we can rewrite weak-* convergence in Y as: $x_n \xrightharpoonup{*} x$ in X $\overset{\text{df.}}{=}$ $x_n(y) \to x(y)$ for all $y \in Y$
Saying differently, weak-* convergence requires the existence of pre-dual while weak convergence requires the existence of dual.

If X is reflexive, then $(X^*)^* = J(X)$. and hence $J_{x_n} \xrightharpoonup{*} J_x$ in X^{**} $\Leftrightarrow$ $\boxed{J_{x_n}(\phi) \to J_x(\phi) \; \forall \phi \in X^*}$
$\Leftrightarrow$ $x_n \rightharpoonup x$ in X ∎

[(v)] If $\phi_k \to \phi$ weakly* in X^* (as $k\to\infty$), then

$$\|\Phi\|_{X^*} \le \liminf_{k\to\infty} \|\phi_k\|_{X^*}$$

Pf For all $x \in X$:

$$|\Phi(x)| \underset{k\to\infty}{\leftarrow} |\phi_k(x)| \le \|\phi_k\|_{X^*}\|x\|_X$$

which implies

$$|\phi(x)| \le \liminf_{k\to\infty} \|\phi_k\|_{X^*}\|x\|_X$$

By definition of $\|\phi\|_{X^*}$:

$$\boxed{\|\phi\|_{X^*} = \sup_{\substack{x\in X\\ \|x\|_X=1}} |\phi(x)| \le \liminf \|\phi_k\|_{X^*}}$$

[(vi)] If $x_k \to x$ weakly in X (as $k\to\infty$), then

$$\|x\|_X \le \liminf_{k\to\infty} \|x_k\|_X$$

Pf For $\phi \in X^*$:

$$|\phi(x)| \leftarrow |\phi(x_k)| \le \|\phi\|_{X^*}\|x_k\|_X$$

which implies

$$|\phi(x)| \le \|\phi\|_{X^*} \liminf \|x_k\|_X \qquad \forall \phi\in X^*$$

If $x \neq 0$, we pick ϕ, by a corollary of HB theorem, so that $\phi(x) = \|x\|_X$ and $\|\phi\|_{X^*}=1$. Hence

$$\|x\| \le \liminf_{k\to\infty} \|x_k\|_X,$$

which holds trivially also for $x=0$.

(vii) Weakly converging sequences and weakly-* converging sequences are bounded.

Pf • If $\phi_k \xrightarrow{*} \phi$ in X^*, then $\sup_{k\in\mathbb{N}} |\phi_k(x)| < \infty$ for all $x\in X$

Then by BANACH-STEINHAUS theorem of uniform boundedness

$$\sup_{k\in\mathbb{N}} \|\phi_k\|_{X^*} < +\infty$$

• If $x_k \to x$ weakly in X, then $Jx_k \xrightarrow{*} Jx$ weakly* in X''.
By previous step, $\{Jx_k\}_{k=1}^\infty$ is bdd in X^{**}, but

$$\|x_k\|_X = \|Jx_k\|_{X^{**}} \quad \text{implies } \{x_k\}_{k=1}^\infty \text{ is bdd in } X.$$

(viii) If $x_k \to x$ strongly in X, $\phi_k \to \phi$ *-weakly in X^*, then

$$\phi_k(x_k) = \langle \phi_k, x_k\rangle_{X^*,X} \longrightarrow \langle \phi, x\rangle_{X^*,X} = \phi(x).$$

Pf is based on

$$|\phi_k(x_k) - \phi(x)| = |(\phi_k-\phi)(x)| + |\phi_k(x_k-x)| \le \ldots + \|\phi_k\|_{X^*}\|x_k-x\|_X \xrightarrow{k\to\infty} 0$$

$\boxed{\text{Theorem (Banach-Alaoglu)}}$ Let X be a Banach space. Then, for every bounded sequence $\{\phi_n\}_{n=1}^{\infty} \subset X^*$, there is a weak-$*$ converging subsequence.

Proof $\boxed{\text{Under additional assumption that } X \text{ is } \underline{\text{separable}}, \text{ i.e. } X \text{ contains a countable dense subset.}}$

[0] Let $\{\phi_n\}_{n=1}^{\infty} \subset X^*$ such that $\boxed{\sup_n \|\phi_n\|_{X^*} \leq C < +\infty}$ is given

Let $\{x_k\}_{k=1}^{\infty}$ be dense subset of X.

[1] To construct $\{\phi_{n_j}\}_{j=1}^{\infty} \subset \{\phi_n\}$ so that $\phi_{n_j}(x_k) \to \phi(x_k)$ for some $\phi \in X^*$,

we apply Cantor diagonalization procedure:

Since $\{\phi_n(x_1)\}_{n=1}^{\infty}$ is bounded sequence of numbers, by Bolzano-Weierstrass: $\exists \{\phi_{n_1}\}_{n_1=1}^{\infty} \subset \{\phi_n\}_{n=1}^{\infty}$: $\phi_{n_1}(x_1) \longrightarrow \phi(x_1)$

Since $\{\phi_{n_1}(x_2)\}_{n=1}^{\infty}$ is bounded, by B-W: $\exists \{\phi_{n_2}\}_{n_2=1}^{\infty} \subset \{\phi_{n_1}\}$: $\phi_{n_2}(x_2) \longrightarrow \phi(x_2)$

etc.

$$\phi_{n_n}(x_k) \longrightarrow \phi(x_k) \quad \forall\, k \in \mathbb{N}.$$

[2] $\boxed{\phi \text{ is Lipschitz on } \{x_k\}_{k=1}^{\infty}}$ Indeed

$$\begin{aligned} |\phi(x_\ell) - \phi(x_k)| &= \lim_{n\to\infty} |\phi_{n_n}(x_\ell) - \phi_{n_n}(x_k)| \\ &\leq \limsup_{n\to\infty} \|\phi_{n_n}\|_{X^*} \|x_k - x_\ell\|_X \\ &\leq C\, \|x_k - x_\ell\|_X \end{aligned}$$

Then ϕ can be extended uniquely to the closure of $\{x_k\}_{k=1}^{\infty}$, i.e. to the whole X.

[3] It remains to show that for all $x \in X$: $\phi_{n_n}(x) \longrightarrow \phi(x)$ $\forall x \in X$. i.e. $\phi_{n_n} \xrightarrow{*} \phi$ $*$-weakly.

However $\|x_\ell - x\|_X < \varepsilon$

$$\begin{aligned} &\limsup_{n\to\infty} |\phi_{n_n}(x) - \phi(x)| \\ &\leq \limsup_{n\to\infty} |\phi_{n_n}(x) - \phi_{n_n}(x_\ell)| \\ &\quad + \limsup_{n\to\infty} |\phi_{n_n}(x_\ell) - \phi(x_\ell)| + |\phi(x_\ell) - \phi(x)| \\ &\leq C\|x_\ell - x\|_X + 0 + C\|x_\ell - x\|_X \leq 2C\varepsilon. \end{aligned}$$