DUAL SPACES. SECOND DUAL. REFLEXIVITY.
WEAK AND WEAK-* CONVERGENCES

Let $(X, \|\cdot\|_X)$ be a Banach space.

Def. The space

$$X^* = X' = \mathcal{L}(X, \mathbb{K}) = \{\phi : X \to \mathbb{K};\ \phi \text{ is linear \& } \underset{\text{bdd}}{\overset{\Updownarrow}{\text{continuous}}}\}$$

is called dual space to X.

Recall that

$$(X^*, \|\phi\|_{X^*}) \quad \text{where } \|\phi\|_{X^*} = \sup_{\substack{x \in X \\ \|x\|_X = 1}} |\phi(x)|$$

is a Banach space.

Notation Often $\phi(x)$ is denoted $\langle \phi, x \rangle = \langle \phi, x \rangle_{X^*, X}$ or $\langle x, \phi \rangle_{X, X^*}$ and $\langle \cdot, \cdot \rangle_{X^*, X}$ is called duality pairing.

Let $x \in X$ be arbitrary. The mapping

J_x: $\boxed{\Phi \in X^* \longmapsto \Phi(x)}$ is a linear bounded*) map of X^* into $\mathbb{K}$.

It means that it belongs to $(X^*)^* =: X^{**}$, which is the second dual to X. Since, by Corollary of H-B theorem there is $\phi \in X^*$ s.t. $\|\phi\|_{X^*} = 1$ and $\phi(x) = \|x\|_X$, then

$$\|J_x\|_{X^{**}} = \|x\|_X$$

Hence, the mapping
$$J : X \to X^{**}, \quad x \longmapsto J_x$$
is isometry and always $J(X) \subseteq X^{**}$.

$\boxed{J \text{ is called canonical embedding}}$

Def The space X is called reflexive if $\boxed{J(X) = X^{**}}$

(It means that one can identified the second dual with X)

*) Towards boundedness: we know $|\phi(x)| \le \|\phi\|_{X^*} \|x\|_X$, which implies that

$$\|J_x\|_{X^{**}} \le \|x\|_X$$

$\boxed{\text{Def.}}$ (1) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ converges weakly to x in X, $\boxed{x_n \rightharpoonup x \text{ in } X}$ if $\phi(x_n) \to \phi(x)$ for all $\phi \in X^*$

(2) A sequence $\{\phi_n\}_{n=1}^{\infty} \subset X^*$ converges weakly-* to ϕ in X^* $\boxed{\phi_n \overset{*}{\rightharpoonup} \phi \text{ in } X^*}$ if $\phi_n(x) \to \phi(x)$ for all $x \in X$.

weakly star

Several observations

$\boxed{\text{i}}$ It holds: If $x_n \to x$ in X (strongly), then $x_n \rightharpoonup x$ in X (weakly)

Pf For arbitrary fixed $\phi \in X'$:

$$|\phi(x_n) - \phi(x)| \underset{\text{lin}}{=} |\phi(x_n - x)| \le \|\phi\|_{X^*} \|x_n - x\|_X$$

Since RHS $\to 0$ as $n \to \infty$, LHS $\to 0$ as $n \to \infty$, which gives the result. ☺

$\boxed{\text{ii}}$ Weak limit is well-defined, i.e., it cannot happen that

(*) $x_n \rightharpoonup x$ in X a $x_n \rightharpoonup y$ in X a $x \neq y$.

Pf If $x \neq y$, then by Corollary of H-B theorem: $\exists \phi \in X^*$: $\phi(x) \neq \phi(y)$, but then $\{\phi(x_n)\}$ cannot converge to both $\phi(x)$ and $\phi(y)$. ↯ ∎

$\boxed{\text{iii}}$ Recalling that $\phi_n \to \phi$ in X^* means that $\boxed{\sup_{\substack{x \in X \\ \|x\|_X \le 1}} |\phi_n(x) - \phi(x)| \to 0 \text{ as } n \to \infty}$ (S)

we see that

weak-star convergence is weaker than the (strong) convergence in the norm of X^*. Weak-* convergence requires that

$\boxed{\forall x \in X \quad |\phi_n(x) - \phi(x)| \to 0 \text{ as } n \to \infty}$ (W*)

$\boxed{\text{iv}}$ If X is reflexive, then weak and weak-* convergences "coincide."

Pf We first observe that if there is a Banach space Y so that $Y^* = X$ we can rewrite weak-* convergence in Y as: $x_n \overset{*}{\rightharpoonup} x$ in $X \overset{\text{df.}}{=} x_n(y) \to x(y)$ for all $y \in Y$

Saying differently, weak-* convergence requires the existence of pre-dual while weak convergence requires the existence of dual.

If X is reflexive, then $(X^*)^* = J(X)$. and hence $Jx_n \overset{*}{\rightharpoonup} Jx$ in $X^{**} \Leftrightarrow \boxed{J_{x_n}(\phi) \to J_x(\phi) \; \forall \phi \in X^*}$

$\Leftrightarrow x_n \rightharpoonup x$ in X ∎

[(v)] If $\phi_k \to \phi$ weakly* in X^* (as $k \to \infty$), then

$$\|\Phi\|_{X^*} \leq \liminf_{k\to\infty} \|\phi_k\|_{X^*}$$

(Pf) For all $x \in X$:

$$|\Phi(x)| \xleftarrow[k\to\infty]{} |\phi_k(x)| \leq \|\phi_k\|_{X^*} \|x\|_X$$

which implies

$$|\phi(x)| \leq \liminf_{k\to\infty} \|\phi_k\|_{X^*} \|x\|_X$$

By definition of $\|\phi\|_{X^*}$:

$$\boxed{\|\phi\|_{X^*} = \sup_{\substack{x\in X\\ \|x\|_X = 1}} |\phi(x)| \leq \liminf \|\phi_k\|_{X^*}}$$

[(vi)] If $x_k \to x$ weakly in X (as $k \to \infty$), then

$$\|x\|_X \leq \liminf_{k\to\infty} \|x_k\|_X$$

(Pf) For $\phi \in X^*$:

$$|\phi(x)| \leftarrow |\phi(x_k)| \leq \|\phi\|_{X^*} \|x_k\|_X$$

which implies

$$|\phi(x)| \leq \|\phi\|_{X^*} \liminf \|x_k\|_X \qquad \forall \phi \in X^*$$

If $x \neq 0$, we pick ϕ, by a corollary of HB theorem, so that $\phi(x) = \|x\|_X$ and $\|\phi\|_{X^*} = 1$. Hence

$$\|x\| \leq \liminf_{k\to\infty} \|x_k\|_X,$$

which holds trivially also for $x = 0$.

[(vii)] Weakly converging sequences and weakly-* converging sequences are bounded.

(Pf) • If $\phi_k \xrightarrow{*} \phi$ in X^*, then $\sup_{k\in\mathbb{N}} |\phi_k(x)| < \infty$ for all $x \in X$

Then by BANACH-STEINHAUS theorem of uniform boundedness (Th. 5.1)

$$\sup_{k\in\mathbb{N}} \|\phi_k\|_{X^*} < +\infty$$

• If $x_k \to x$ weakly in X, then $Jx_k \xrightarrow{*} Jx$ weakly* in X''.
By previous step, $\{Jx_k\}_{k=1}^\infty$ is bdd in X^{**}, but

$\|x_k\|_X = \|Jx_k\|_{X^{**}}$ implies $\{x_k\}_{k=1}^\infty$ is bdd in X.

[(viii)] If $x_k \to x$ strongly in X, $\phi_k \to \phi$ *-weakly in X^*, then

$$\phi_k(x_k) = \langle \phi_k, x_k \rangle_{X^*,X} \longrightarrow \langle \phi, x \rangle_{X^*,X} = \phi(x).$$

(Pf) is based on

$$|\phi_k(x_k) - \phi(x)| = |(\phi_k - \phi)(x)| + |\phi_k(x_k - x)| \leq \ldots + \|\phi_k\|_{X^*} \|x_k - x\|_X \xrightarrow{k\to\infty} 0$$

$\boxed{\text{Theorem (Banach-Alaoglu)}}$ Let X be a Banach space. Then, for every bounded sequence $\{\phi_n\}_{n=1}^\infty \subset X^*$, there is a weak-$*$ converging subsequence.

Proof $\boxed{\text{Under additional assumption that } X \text{ is separable, i.e. } X \text{ contains a countable dense subset.}}$

[0] Let $\{\phi_n\}_{n=1}^\infty \subset X^*$ such that $\boxed{\sup_n \|\phi_n\|_{X^*} \le C < +\infty}$ is given

Let $\{x_k\}_{k=1}^\infty$ be dense subset of X.

[1] To construct $\{\phi_{n_j}\}_{j=1}^\infty \subset \{\phi_n\}$ so that $\phi_{n_j}(x_k) \to \phi(x_k)$ for some $\phi \in X^*$,

we apply Cantor diagonalization procedure:

Since $\{\phi_n(x_1)\}_{n=1}^\infty$ is bounded sequence of numbers, by Bolzano-Weierstrass: $\exists \{\phi_{n_1}\}_{n_1=1}^\infty \subset \{\phi_n\}_{n=1}^\infty$: $\phi_{n_1}(x_1) \to \phi(x_1)$

Since $\{\phi_{n_1}(x_2)\}_{n=1}^\infty$ is bounded, by B-W: $\exists \{\phi_{n_2}\}_{n_2=1}^\infty \subset \{\phi_{n_1}\}$: $\phi_{n_2}(x_2) \to \phi(x_2)$

etc.

$$\phi_{n_n}(x_k) \to \phi(x_k) \quad \forall k \in \mathbb{N}.$$

[2] $\boxed{\phi \text{ is Lipschitz on } \{x_k\}_{k=1}^\infty}$ Indeed

$$\begin{aligned}
|\phi(x_\ell) - \phi(x_k)| &= \lim_{n\to\infty} |\phi_{n_n}(x_\ell) - \phi_{n_n}(x_k)| \\
&\le \limsup_{n\to\infty} \|\phi_{n_n}\|_{X^*} \|x_k - x_\ell\|_X \\
&\le C \|x_k - x_\ell\|_X
\end{aligned}$$

Then ϕ can be extended uniquely to the closure of $\{x_k\}_{k=1}^\infty$, i.e. to the whole X.

[3] It remains to show that for all $x \in X$: $\phi_{n_n}(x) \to \phi(x)$ $\forall x \in X$. i.e. $\phi_{n_n} \xrightarrow{*} \phi$ $*$-weakly.

However $\qquad \|x_k - x\|_X < \varepsilon$

$$\begin{aligned}
&\limsup_{n\to\infty} |\phi_{n_n}(x) - \phi(x)| \\
&\le \limsup_{n\to\infty} |\phi_{n_n}(x) - \phi_{n_n}(x_k)| \\
&\quad + \limsup_{n\to\infty} |\phi_{n_n}(x_k) - \phi(x_k)| + |\phi(x_k) - \phi(x)| \\
&\le C\|x_k - x\|_X + 0 + C\|x_k - x\|_X \le 2C\varepsilon.
\end{aligned}$$

$\boxed{\text{Theorem}}$ Let X be a reflexive Banach space. Then the closed unit ball $\overline{B_1(0)} \subset X$ is weakly sequentially compact, i.e.

$\forall \{x_n\}_{n=1}^{\infty} \subset \overline{B_1(0)} : \exists \{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ and $x \in \overline{B_1(0)}$: $x_{n_k} \xrightarrow[k\to\infty]{} x$ weakly in X

NOTE: Clearly, the statement of Theorem holds for any $\overline{B_R(x)}$.

Pf Let $\{x_k\}_{k\in\mathbb{N}} \subset \overline{B_1(0)} \subset X$. Set $Y :\overset{\text{df}}{=} \overline{\operatorname{span}\{x_k ; k\in\mathbb{N}\}}$

Then Y as a closed subspace of X is also reflexive, and Y is separable (by its definition). Then $Y^{**} = J(Y)$ is separable and hence Y^* is separable; see Lemma below.

$\subset Y^{**}$

By previous theorem, applied to Y^* and to the sequence $\{Jx_k\}_{k\in\mathbb{N}}$, there is $y'' \in Y^{**}$ such that for a subsequence

$$\langle \Phi, Jx_k \rangle_{Y^*} \to \langle \phi, y'' \rangle \qquad \forall \phi \in Y^*$$

Setting $x := J^{-1} y'' \in Y$, it follows that

$$\phi(x_k) = \langle \phi, x_k \rangle_Y = \langle \Phi, Jx_k \rangle_{Y^*} \xrightarrow[(k\to\infty)]{} \langle \phi, y'' \rangle_{Y^*} = \langle \Phi, Jx \rangle_Y = \phi(x)$$

valid for all $\boxed{\phi \in Y^*}$

Since $\phi \in X^*$ fulfills $\phi|_Y \in Y^*$, we get

$$\boxed{\Phi(x_k) \to \phi(x) \quad \text{as } k \to +\infty \quad \forall \phi \in X^*}$$

$\boxed{\text{Lemma}}$ Let X be a Banach space. Then:

if X^* is separable, then X is separable.

NOTE $L^1(\Omega)$ is separable, but its dual $L^\infty(\Omega) = (L^1(\Omega))^*$ is not. This shows that the opposite implication does not hold.

Pf
- Let $\{\phi_n, n\in\mathbb{N}\}$ be dense in X^*. Choose $x_n \in X$ such that $\|x_n\|_X = 1$ and $|\phi_n(x_n)| \geq \frac{1}{2}\|\phi_n\|$.
- Define Y = closure of $\operatorname{span}\{x_n ; n\in\mathbb{N}\}$. Now, if $\phi \in X^*$ and $\phi \equiv 0$ on Y then

$$\|\phi - \phi_n\|_{X^*} \geq |(\phi - \phi_n)(x_n)| = |\phi(x_n)| \geq \tfrac{1}{2}\|\phi_n\|_{X^*}$$
$$\geq \tfrac{1}{2}\left(\|\phi\|_{X^*} - \|\phi - \phi_n\|_{X^*}\right)$$

$\Rightarrow \|\phi\|_{X^*} \leq 3 \inf_n \|\phi - \phi_n\|_{X^*} = 0$ as ϕ_n is dense subset in X^*.

$\Rightarrow \phi \equiv 0 \Rightarrow Y = X$ (by Hahn-Banach theorem). ▨