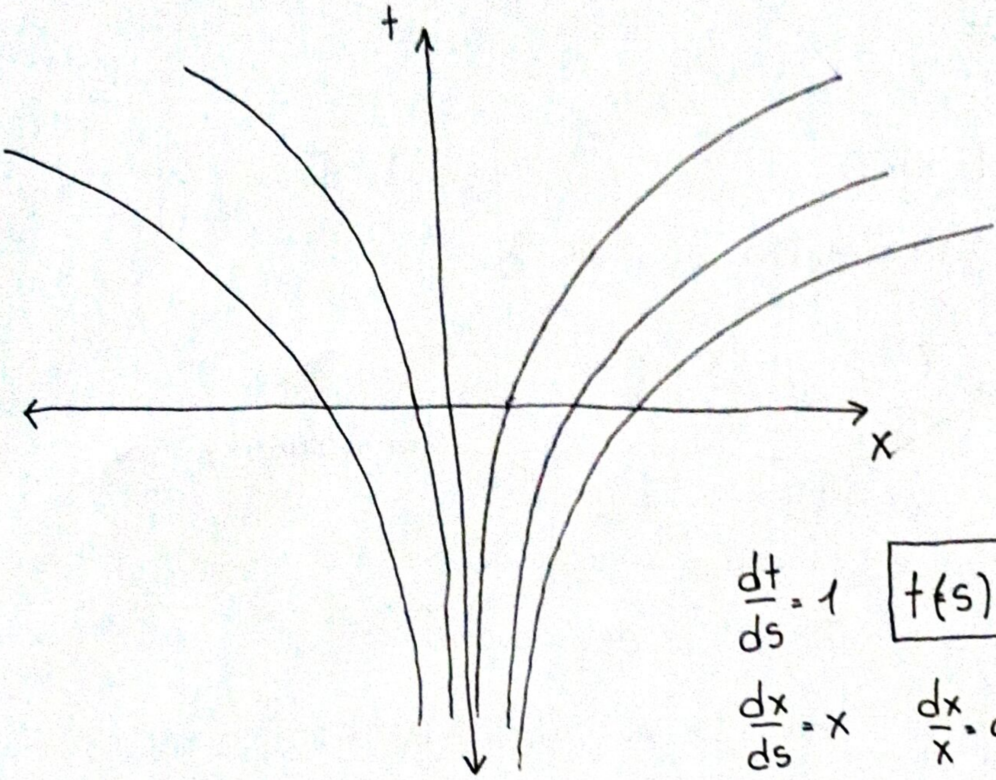
1. $\partial_t u(t,x) + x\,\partial_x u(t,x) = 0 \quad v \ (0,\infty)\times\mathbb{R}$

$u(0,x) = u_0(x) \quad v \ \mathbb{R}$

$\frac{dt}{ds} = 1 \qquad \frac{dt}{dx} = \frac{1}{x} \qquad dt = \frac{dx}{x} \ / \int$

$\frac{dx}{ds} = x$ „inženýrsky" $\qquad t = \ln\frac{x}{x_0} \to \boxed{x = x_0 e^{t}}$

↑ charakteristiky



$\frac{dt}{ds} = 1 \quad \boxed{t(s) = t + s}$

$\frac{dx}{ds} = x \quad \frac{dx}{x} = ds \ / \int \quad \boxed{x(s) = xe^{s}}$

$\boxed{u(t,x)} = u(t(s),x(s))\big|_{s=0} = u(t(s),x(s))\big|_{s=-t} = u(0,x(-t)) =$

$= u_0(x(-t)) = \boxed{u_0(xe^{-t})}$

$u(t,0) = u_0(0)$

$\frac{\partial u(t,x)}{\partial t} = \frac{\partial u_0(xe^{-t})}{\partial t} = u_0'(xe^{-t})(-xe^{-t})$

$\frac{\partial u(t,x)}{\partial x} = \frac{\partial u_0(xe^{-t})}{\partial x} = u_0'(xe^{-t})e^{-t}$

$\partial_t u(t,x) + \underset{x}{\uparrow}\partial_x u(t,x) = -xe^{-t}u_0'(xe^{-t}) + xe^{-t}u_0'(xe^{-t}) = 0$

$$\lim_{t\to+\infty} u_0(\underset{\substack{\downarrow\\ 0}}{xe^{-t}}) = u_0(0)\,, \qquad \lim_{t\to-\infty} u_0(\underset{\substack{\downarrow\\ +\infty}}{xe^{-t}}) = \begin{cases} \lim\limits_{x\to+\infty} u_0(x) & x>0 \\ \lim\limits_{x\to-\infty} u_0(x) & x<0 \end{cases}$$

Řešení je definované na $\mathbb{R}^2$. Řešení musí být konstantní na charakteristikách, které jsou jednoznačně určeny systémem ODR. Řešení je tedy jediné.

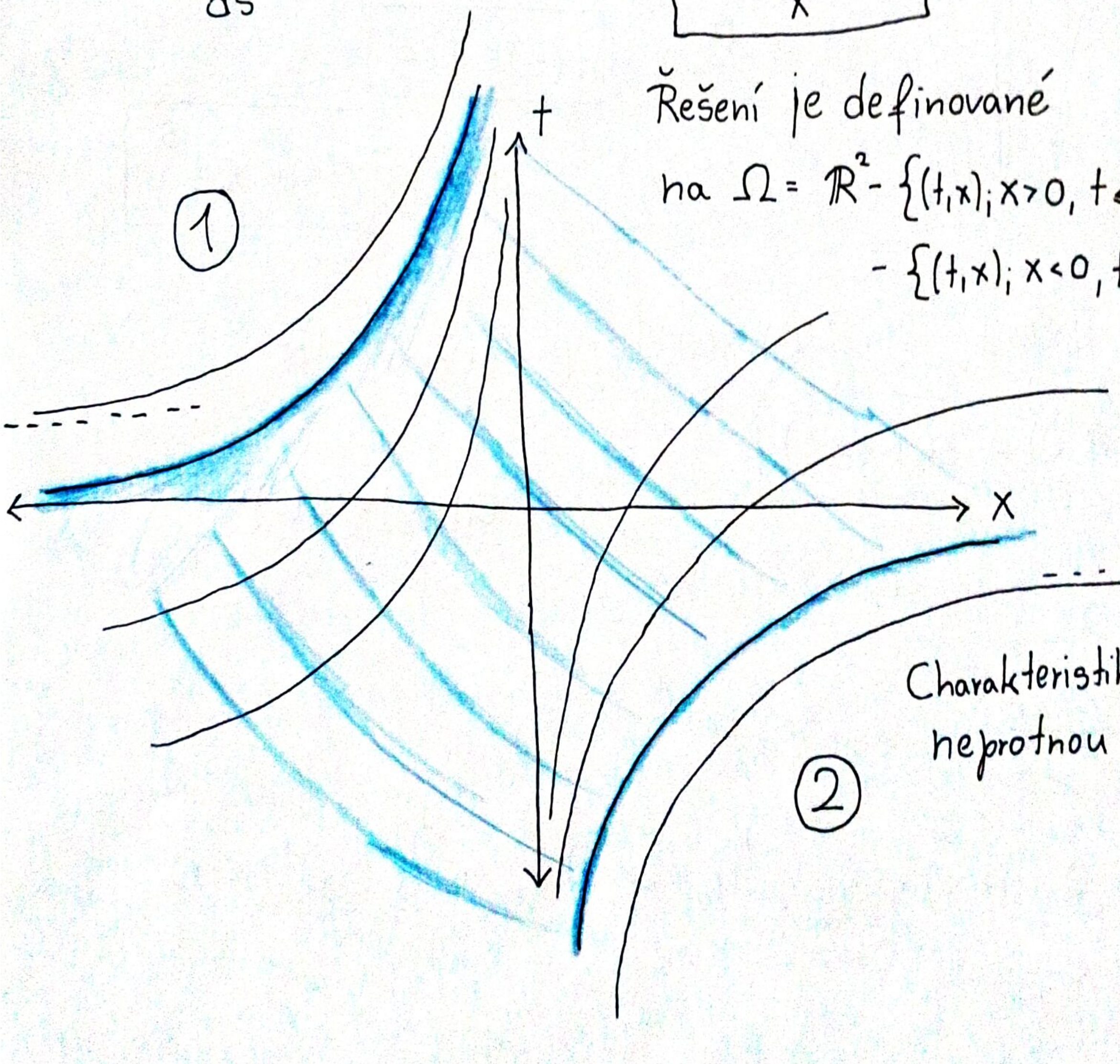
2. $\partial_t u(t,x) + x^2 \partial_x u(t,x) = 0 \qquad$ v $(0,\infty)\times\mathbb{R}$

$u(0,x) = u_0(x) \qquad$ v $\mathbb{R}$

$$\frac{dt}{ds} = 1 \qquad \frac{dt}{dx} = \frac{1}{x^2} \qquad dt = \frac{dx}{x^2} \;\Big|\int$$

$$\frac{dx}{ds} = x^2 \qquad \boxed{t = -\frac{1}{x} + C}$$

charakteristiky



Řešení je definované na $\Omega = \mathbb{R}^2 - \{(t,x);\, x>0,\, t\leq -\frac{1}{x}\} - \{(t,x);\, x<0,\, t\geq -\frac{1}{x}\}$

Charakteristiky z oblastí 1 a 2 neprotnou osu x.

$$\frac{dt}{ds} = 1 \quad \rightarrow \quad t(s) = t + s$$

$$\frac{dx}{ds} = x^2 \qquad \frac{dx}{x^2} = ds \;/\int \qquad -\frac{1}{x(s)} = s + K \qquad x(s) = \frac{-1}{s+K} = \qquad K = -\frac{1}{x}$$

$$x(0) = x$$

$$x(s) = \frac{-1}{-\frac{1}{x} + s} = \frac{1}{\frac{1 - sx}{x}} = \frac{x}{1 - xs}$$

$$\boxed{u(t,x)} = u(t(s), x(s))\Big|_{s=0} = u(t(s), x(s))\Big|_{s=-t} = u_0(x(-t)) =$$

$$= \boxed{u_0\left(\frac{x}{1+xt}\right)}$$

$$\partial_t u(t,x) = u_0'(\cdot)\left(-\frac{x^2}{(1+xt)^2}\right) \qquad \partial_x u(t,x) = u_0'(\cdot)\left(\frac{1+xt-xt}{(1+xt)^2}\right) = u_0'(\cdot)\frac{1}{(1+xt)^2}$$

$$\partial_t u(t,x) + x^2 \partial_x u(t,x) = 0$$

$\lim_{t \to \pm\infty} u(t,x) = u_0(0)$ pro $x \gtrless 0$ $\qquad$ $\lim_{t \to \pm\infty} u(t,x)$ pro $x \lessgtr 0$ neexistuje protože pro pevné x vždy vylezeme z oblasti, kde je řešení definované

Na množině Ω, kde je řešení definované, je řešení jednoznačné, protože musí být konstantní na charakteristikách, které jsou určeny jednoznačně.

2. $$\partial_t u(t,x) + \vec{b}(t)\cdot \nabla_x u(t,x) = 0 \qquad \text{v } (0,\infty)\times\mathbb{R}^d$$
$$u(0,x) = u_0(x) \qquad \text{v } \mathbb{R}^d$$

$$\hat{u}(t,s) = \int_{\mathbb{R}^d} u(t,x)\, e^{-2\pi i \vec{s}\cdot\vec{x}}\, dx$$

$$\int_{\mathbb{R}^d} \vec{b}(t)\cdot \nabla_x u(t,x)\, e^{-2\pi i \vec{s}\cdot\vec{x}}\, dx = 2\pi i\, \vec{b}(\underset{t}{\cancel{s}})\cdot\vec{s}\, \hat{u}(t,s)$$

$$\partial_t \hat{u}(t,s) + 2\pi i\, \vec{b}(t)\cdot\vec{s}\, \hat{u}(t,s) = 0$$
$$\partial_t \hat{u}(t,s) = -2\pi i\, \vec{b}(t)\cdot\vec{s}\, \hat{u}(t,s) \Rightarrow \boxed{\hat{u}(t,s) = \hat{u}_0(\cancel{t},s)\, e^{-2\pi i \vec{s}\cdot\int_0^t b(t)dt}}$$

$$\int_{\mathbb{R}^d} u(t,x)\, e^{-2\pi i \vec{s}\cdot\vec{x}}\, dx = \hat{u}(t,s) = \left(\int_{\mathbb{R}^d} u_0(x)\, e^{-2\pi i \vec{s}\cdot\vec{x}}\, dx\right) e^{-2\pi i \vec{s}\cdot\int_0^t \vec{b}(z)dz} =$$

$$= \int_{\mathbb{R}^d} u_0(x)\, e^{-2\pi i \vec{s}\cdot\left(\vec{x}+\int_0^t \vec{b}(z)dz\right)}\, dx \underset{\vec{v}=\vec{x}+\int_0^t \vec{b}(z)dz}{=} \int_{\mathbb{R}^d} u_0\left(\vec{v}-\int_0^t \vec{b}(z)dz\right) e^{-2\pi i \vec{s}\cdot\vec{v}}\, dv \underset{\vec{x}=\vec{v}}{=}$$

$$= \int_{\mathbb{R}^d} u_0\left(\vec{x}-\int_0^t b(z)dz\right) e^{-2\pi i \vec{s}\cdot\vec{x}}\, dx \Rightarrow \boxed{u(t,x) = u_0\left(\vec{x}-\int_0^t \vec{b}(s)ds\right)}$$