B. Baire's theorem and Hahn-Banach theorem are two building blocks of FA.

Baire's theorem is used to prove Banach-Steinhaus uniform boundedness principle, Open mapping theorem and Closed Graph theorem.

Baire's theorem will be applied to Banach spaces, but it holds for complete metric spaces. We will keep this generality.

Theorem B.1 (Cantor's intersection theorem).

Let X be a complete metric space.
Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of nonempty closed subsets A_n of X:

(D) $\boxed{A_0 \supset A_1 \supset \dots \supset A_n \supset A_{n+1} \supset \dots \quad \text{and} \quad \operatorname{diam} A_n \to 0 \text{ as } n \to \infty.}$

Then $\exists!\, x \in X$: $\displaystyle\bigcap_{n=0}^{\infty} A_n = \{x\}$

Recall: $\operatorname{diam} A \overset{\text{df}}{=} \sup \{\varrho(x,y);\, x \in A, y \in A\}$.

Pf
- For $\forall n \in \mathbb{N}$, we pick $x_n \in A_n$ (A_n is nonempty) and observe $\varrho(x_n, x_m) \leq \operatorname{diam} A_n$ for all $m \geq n$.
  As $\operatorname{diam} A_n \to 0$ for $n \to \infty$, $\{x_m\}_{n=0}^{\infty}$ is Cauchy sequence.
  Due to completness of X: $\exists x \in X$: $x_n \xrightarrow[n\to\infty]{} x$ in X.
- For any $n \in \mathbb{N}$, $x_m \in A_n$ for all $m \geq n$.
  As A_n are closed, $x_m \to x$ as $m \to \infty$, we have $x \in A_n$.
  Hence $\boxed{x \in \bigcap_{n=1}^{\infty} A_n}$.
- If $y \neq x$ and $y \in \bigcap_{n=1}^{\infty} A_n$, then there is $n_0 \in \mathbb{N}$: $\boxed{d(x,y) > \operatorname{diam} A_{n_0}}$
  But x, y belongs to A_{n_0}: $\boxed{d(x,y) \leq \operatorname{diam} A_{n_0}}$,
  which gives contradiction, i.e. uniqueness of $x \in \bigcap_{n=0}^{\infty} A_n$.

NOTES
- $\boxed{\operatorname{diam} A_n \to 0}$ is essential; see e.g. $X = \mathbb{R}$, $A_n = \langle n, +\infty)$.
- It holds: $(X, \varrho(\cdot,\cdot))$ is complete $\iff$ For any $\{A_n\}_{n=0}^{\infty}$ closed, nonempty satisfying (D) above, then $\bigcap_{n=0}^{\infty} A_n$ is nonempty (in fact, it contains one point)

this property characterizes completness of a metric space

Theorem B.2 (BAIRE'S THEOREM)

Let X be a complete metric space. Then the following two equivalent properties hold:

(a) Let $\{C_n\}_{n=0}^{\infty}$ be a sequence of closed subsets: $\operatorname{int} C_n := \overset{\circ}{C}_n = \phi$, for all $n \in \mathbb{N}$

then $\boxed{\operatorname{int}\left(\bigcup_{n=0}^{\infty} C_n\right) = \phi}$

(b) Let $\{O_n\}_{n=0}^{\infty}$ be a sequence of open subsets: $\overline{O_n} = X$ for all $n \in \mathbb{N}$,

then $\boxed{\overline{\bigcap_{n=0}^{\infty} O_n} = X}$

Recall For $A \subset X$, $\operatorname{int} A = \overset{\circ}{A}$ denotes <u>interior</u> of A, while $\overline{A}$ denotes the closure.

Proof **Step 1** **(a) ⟺ (b)**

(a) ⟹ (b) Observe that $\overline{A} = X - (X - A)^{\circ} \quad \forall A \subset X$.

This implies $\boxed{\overline{A} = X} \iff \boxed{\operatorname{int}(X-A) = \phi} \quad (+)$

For open O_n satisfying $\overline{O_n} = X$, we set $C_n = X - O_n$. Then, by above equivalence $(+)$, $\operatorname{int} C_n = \phi$ for all $n \in \mathbb{N} \cup \{0\}$ and by (a) $\operatorname{int}\left(\bigcup_{n=0}^{\infty} C_n\right) = \phi$. But

$$\bigcup_{n=0}^{\infty} C_n = \bigcup_{n=0}^{\infty} (X - O_n) \underset{\text{de Morgan's law}}{=} X - \bigcap_{n=0}^{\infty} O_n$$

By $(+)$ and $\downarrow$: $\boxed{\overline{\bigcap_{n=0}^{\infty} O_n} = X}$, which is b).

(b) ⟹ (a) This is proved similarly by observing $\overline{X - B} = X - \operatorname{int} B$ which implies, for all B,

$$\boxed{\operatorname{int} B = \phi \iff \overline{X-B} = X} \quad (++)$$

Step 2 **Proof of (a)**

Since $\underbrace{A \subset X \text{ has } \operatorname{int} A \neq \phi}$ $\iff \exists\, \underset{\neq\phi}{O} \subset X$ open: $O \subset A$ $\implies \underbrace{\exists\, \underset{\neq\phi}{O} \subset X \text{ open}: O \cap (X-A) = \phi}$

we have $\boxed{\operatorname{int} A = \phi \iff \forall\, \underset{\text{nonempty}}{O \subset X} \text{ open}: O \cap (X-A) \neq \phi}$

Let $\{F_n\}_{n=0}^{\infty}$ be closed subsets of complete metric space $(X, \rho(\cdot,\cdot))$, such that $\operatorname{int} F_n = \emptyset$. We want to prove $\operatorname{int} \bigcup_{n=0}^{\infty} F_n = \emptyset$, i.e.

$$\boxed{\forall\, O \subset X \text{ nonempty, open}: \quad O \cap \left(X - \bigcup_{n=0}^{\infty} F_n\right) \neq \emptyset.}$$

Given a nonempty set $O \subset X$, set $O_0 := O$. Since $\operatorname{int} F_0 = \emptyset$ and O_0 is open: $O_0 \cap (X - F_0)$ is nonempty and open of X.
Hence: $\exists\, O_1 \subset X$ nonempty, open

$$\overline{O_1} \subset O_0 \cap (X - F_0) \quad \text{and} \quad \operatorname{diam} \overline{O_1} < 1$$

Since $\operatorname{int} F_1 = \emptyset$ and O_1 open: $O_1 \cap (X - F_1)$ is nonempty and open of X
Hence: $\exists\, O_2 \subset X$ nonempty, open

$$\overline{O_2} \subset O_1 \cap (X - F_1) \quad \text{and} \quad \operatorname{dia} \overline{O_2} < \frac{1}{2}$$

and so on. We construct $\{O_n\}_{n=0}^{\infty} \subset X$ nonempty, open:

$$\overline{O_{n+1}} \subset O_n \cap (X - F_n) \quad \text{and} \quad \operatorname{diam} \overline{O_{n+1}} < \frac{1}{n+1}$$

The nonempty closed sets $\overline{O_n}$, $n \geq 0$, satisfy the assumption of Cantor's intersection principle (Th. B.1). Hence, $\exists\, x \in X$:

$$\{x\} = \bigcap_{n=0}^{\infty} \overline{O_n}. \quad \text{Also, by construction,}$$

$$x \in \overline{O_1} \subset O \quad \text{and} \quad x \in \overline{O_{n+1}} \subset O_n \quad \text{and} \quad \underline{x \in \overline{O_{n+1}} \subset X - F_n} \ (\forall n)$$

Hence $x \in O$ & $x \in \bigcap_{n=0}^{\infty} (X - F_n) = X - \bigcup_{n=0}^{\infty} F_n$,

i.e. $\quad x \in O \cap \left(X - \bigcup_{n=0}^{\infty} F_n\right)$

which we wish to prove. Q.E.D.

<u>It follows from Baire's theorem (Th. B.2) directly:</u>

$\boxed{\text{Theorem B.3}}$ Let X be a metric space and F_n, $n \geq 0$, be closed subsets of X such that $X = \bigcup_{n=0}^{\infty} F_n$. It holds:

(i) If $\operatorname{int} F_n = \emptyset$ (for all $n \geq 0$), then X is not complete.

(ii) If X is complete, there exists $n_0 \geq 0$: $\operatorname{int} F_{n_0} \neq \emptyset$

$\boxed{\text{Proof}}$ $\boxed{\text{Ad (ii)}}$ If X complete & $\operatorname{int} F_n = \emptyset$ $\forall n$, by Theorem B.2(a) $\operatorname{int} X = \emptyset$, but $\overline{X} = X = \operatorname{int} X$ cannot be empty.**

$\boxed{\text{Ad (i)}}$ follows from (ii).

////

**) or by (††) we would have $\overline{\emptyset} = X$, which is ↯

<u>Example</u> • $\mathbb{Q} = \bigcup_{n=0}^{\infty} \{q_n\}$ set of all rational numbers. $(\mathbb{Q}, \varrho(x,y) := |x-y|)$ is a metric space. Each $\{q_n\}_{n\in\mathbb{N}\cup\{0\}}$ is closed and has an empty interior. By the previous theorem $(\mathbb{Q}, \varrho(x,y))$ is not complete.

• It also follows from Theorem B.3 (b) that
  - $\mathbb{R}^2$ cannot be written as $\bigcup_{n=0}^{\infty} (\text{lines})_n$
  - $\mathbb{R}^d$ ——— " ——— as countable union of hypersurfaces (hyperplanes)

• More important applications will follow. One of them is on the next page.

• Another application: Existence of nowhere differentiable continuous functions.

Recall that the functions

- $\underset{\in\mathbb{R}}{x} \mapsto f(x) := \sum_{n=0}^{\infty} \frac{\sin(3^n x)}{2^n}$
- $\underset{\in\mathbb{R}}{x} \mapsto f(x) := \sum_{n=1}^{\infty} \frac{\sin(n^2 \pi x)}{n^2}$

are well-defined continuous, but nowhere differentiable.

Application of Baire's theorem

Def. Let $X \neq \{0\}$ be a vector space. Then a family $\{e_i\}_{i\in I}$ of vectors $e_i \in X$ is called HAMEL BASIS in X if

(1) $\boxed{\{e_i\}_{i\in I} \text{ is linearly independent}}$ in the sense that

$\forall$ finite subfamily $\{e_i\}_{i\in J}$ and any scalar $\{\alpha_j\}_{j\in J} \subset \mathbb{K}$:

if $\sum_{i\in J} \alpha_i e_i = 0$, then $\alpha_i = 0$ for all $i \in J$.

(2) $\boxed{\operatorname{span}\{e_i\}_{i\in I} = X}$ in the sense that $\forall x \in X$

$\exists$ $J(x)$ finite family of indices and $(\alpha_j)_{j\in J(x)} \subset \mathbb{K}$ such that $x = \sum_{j\in J(x)} \alpha_j e_j$

It is possible to prove that each $X \neq \{0\}$ vector space has a Hamel basis (and if they are two of them, they have the same cardinality. The proof uses Axiom of choice.

Theorem B.4 Let $(X, \|\cdot\|_X)$ be a Banach space, $\dim X = \infty$. Then X cannot have countable Hamel basis

In particular, the space of polynomials cannot be equipped with a norm that would make it Banach.

Pf ▶ Assume that a normed space $(X, \|\cdot\|)$ has a countable Hamel basis $\{e_i\}_{i=0}^{\infty}$. Define

$$F_m := \operatorname{span}\{e_i\}_{i=0}^{m} \quad \text{for } \forall n \in \mathbb{N}\cup\{0\}$$

Then $\left[X = \bigcup_{n=0}^{\infty} F_n, \ F_n \text{ are } \underset{\text{(finite-dim space)}}{\text{closed}} \text{ and } \operatorname{int} F_n = \emptyset \right]$

Hence, by Th. B.3 (a), $(X, \|\cdot\|)$ is not complete.

[Note that if $\operatorname{int} F_n \neq \emptyset$ for some $m \geq 0$, then there exists $x = \sum_{j=0}^{m} x_j e_j$ and $r > 0$ s.t. $B_r(x) \subset F_m$. Then the point $y := \frac{r}{\|e_{m+1}\|} e_{m+1} + x \in \overline{B_r(x)}$, but y cannot be in F_m as $\{e_i\}_{i=0}^{m+1}$ are independent ↯]

▶ In the space of polynomials of d-variables,

$$x \in \mathbb{R}^d \longmapsto x_1^{k_1} \cdots x_d^{k_d}, \quad k_i \in \mathbb{N}\cup\{0\},\ i \in \{1,\dots,d\}$$

form a Hamel basis that is countable. The space of polynomials thus cannot be Banach. ▨

$\boxed{\text{BAIRE CATEGORY THEOREM}}$ Let X be metric. Consider $P(X)$, a set of all subsets. Within $P(X)$ we would like to specify a family of „large sets" and a family of „small sets" with the following properties:

(i) $S \subset X$ is large $\iff$ $X \setminus S$ is small

(ii) $\bigcap_{n=1}^{\infty} S_m$ is large if S_m are large for the

(iii) large set is nonempty.

If μ is a probability measure defined on X, then we can say

- S is large $\iff P(S) = 1$
- S is small $\iff P(S) = 0$

Then (i) - (iii) holds.

If the metric space X is complete, one can still introduce concepts of „large" & „small" sets using exclusively the topological structure. Precisely,

- a set $S \subseteq X$ is of 2nd category (i.e. topologically large) if $S = \bigcap_{n=1}^{\infty} O_m$, where O_m open dense sets in X
- a set $S \subset X$ is of 1st category*) (i.e. topologically small) if $S = \bigcup_{n=1}^{\infty} C_m$, where C_m are closed with empty interior

It follows from these definitions that (i) holds. The
(and de Morgan laws)
properties of (ii) and (iii) follows from BAIRE's Theorem.

$\boxed{\text{Theorem (BAIRE)}}$ Let $O_m, m \in \mathbb{N}$, be open dense subsets of $(X, \rho(\cdot,\cdot))$ complete. Then $S := \bigcap_{n=1}^{\infty} O_m$ is nonempty dense subset of X

*) Sometimes, a set of 1st category is called meager set [czech: řídká množina]

MEAGER

Proof (of Baire's theorem) Let $\Omega \subset X$ be any open set.

(*) We need to show that $\underbrace{\left(\bigcap_{n=1}^{\infty} \mathcal{O}_n\right) \cap \Omega}_{} \neq \emptyset$

Let $x_0 \in \Omega$ and $r_0 < 1$ be such that $B_{3r_0}(x_0) \subset \Omega$.

$\boxed{n=1}$ For $B_{r_0}(x_0) \cap \mathcal{O}_1$, by density of $\mathcal{O}_1$, there is x_1 and $r_1 > 0$

dense in X

such that $B_{3r_1}(x_1) \subset B_{r_0}(x_0) \cap \mathcal{O}_1$

Iteratively (inductively):

$\boxed{n \in \mathbb{N} \text{ general}}$ For $B_{r_{n-1}}(x_{n-1}) \cap \mathcal{O}_n$, by density of $\mathcal{O}_n$,

there is x_n and $r_n > 0$ such that

$$B_{3r_n}(x_n) \subset B_{r_{n-1}}(x_{n-1}) \cap \mathcal{O}_n$$

By construction $\{x_n\}_{n=1}^{\infty}$ is cauchy: $\varrho(x_n, x_{n-1}) < r_{n-1} < \frac{1}{3^{n-1}} r_0$

Hence: there is x^* : $\boxed{x_n \xrightarrow{n\to\infty} x^* \text{ in } X}$

NOTE
$(X, \varrho(\cdot,\cdot))$ is complete

and

$$\varrho(x^*, x_n) \leq \sum_{j=n}^{\infty} \varrho(x_{j+1}, x_j)$$

$$\leq \sum_{j=n}^{\infty} r_j \leq \sum_{j=n}^{\infty} \frac{r_n}{3^{j-n}} = \frac{3}{2} r_n$$

Therefore $\boxed{x^* \in B_{3r_n}(x_n) \subset \mathcal{O}_n \quad \forall n \in \mathbb{N}}$

For $n=0$, the same argument implies $x^* \in B_{3r_0}(x_0) \subset \Omega$.

Hence $x^* \in \left(\bigcap_{n=1}^{\infty} \mathcal{O}_n\right) \cap \Omega$.

∎