SEMINORMS & FRÉCHET SPACES

There are function spaces where there is no natural way to introduce a norm. For example $C((a,b))$, $\mathcal{D}(\Omega)$, $L^p_{loc}(\Omega)$, $\mathcal{S}(\mathbb{R}^d)$. We will be however able to introduce on these spaces the sequence of separating seminorms and use them to introduce the metric*) so that the above spaces will be complete (metric spaces), i.e. Fréchet.

$\boxed{\text{Example}}$ (serving as motivation) Consider $X = C((0,1))$. Since X contains unbounded functions, setting

$$p(f) \overset{df.}{=} \sup_{x\in(0,1)} |f(x)|,$$

we see that $p(f)$ can be $+\infty$ and consequently, $p(f)$ does not generate the norm.

However, for any $\langle a,b\rangle \subset (0,1)$

$$p^{a,b}(f) \overset{df.}{=} \sup_{a\le x\le b} |f(x)|$$

is always finite. We easily observe that $p^{a,b}$ is 1-homogeneous $p^{a,b}(0) = 0$ and $p^{a,b}$ fulfills the triangle inequality, i.e. (N2) and (N3) holds. But there are nontrivial f so that $p^{a,b}(f) = 0$. Draw one. Hence $p^{a,b}$ is not norm, but it is a seminorm ∎

$\boxed{\text{Def.}}$ Let X be a vector space over $\mathbb{K}$. The mapping $p : X \to \mathbb{R}$ is called a seminorm if

(SN1) $p(x) \ge 0$ for $\forall x \in X$

(SN2) $p(\lambda x) = |\lambda|\, p(x)$ —"— $\forall \lambda \in \mathbb{K}$

(SN3) $p(x+y) \le p(x) + p(y)$

Since $p(x)$ can be zero for $x \neq 0$, setting $d(x,y) = p(x-y)$ we do not obtain a distance on X. We can have x, y, $x \neq y$ and get $d(x,y) = 0$.

There are cases when we can introduce the distance by means of infinitely many (countable) seminorms.

*) distance

Def. A sequence $\{p_k\}_{k\in\mathbb{N}}$ of seminorms on X is *separating* if, for every $x \in X$ with $x \neq 0$, there is $k_0 \in \mathbb{N}$ such that $p_{k_0}(x) > 0$.

Assertion (Distance generated by seminorms)

Let $\{p_k\}_{k\in\mathbb{N}}$ be a sequence of separating seminorms.

Then
$$(d**)\qquad \boxed{d(x,y) \overset{\text{df.}}{=} \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x-y)}{1+p_k(x-y)}}$$
is a distance on X

Proof. • If $x \neq y$, then $d(x,y) > 0$ as there is k_0: $p_k(x-y) > 0$

• Also, $d(x,y) = d(y,x)$ due to 1-homogeneity of $p_k(\cdot)$.

• The triangle inequality follows from the fact that $s \mapsto \frac{s}{1+s}$ is increasing and concave, which implies, for $0 \le a,b,c$ with $c \le a+b$, that

$$\underbrace{\frac{c}{1+c} \le \frac{a+b}{1+a+b} \le \frac{a}{1+a} + \frac{b}{1+b}}$$

setting $c = p_k(x-z)$, $a = p_k(x-y)$ and $b = p_k(y-z)$, we get the triangle inequality.

Def. X is *Fréchet* if X is a *complete metric space* with the distance $(d**)$.

Examples ① The (space) $(C(\Omega), d(f,g))$, $\Omega \subset \mathbb{R}^d$ *open*, is *Fréchet* provided that we set
(not necessarily bounded)

$$p_k(f) \equiv \max_{x\in A_k} |f(x)| \quad \text{where } A_k := \{x \in \Omega;\ |x| \le k,\ \mathrm{dist}(x,\partial\Omega) < \tfrac{1}{k}\}.$$

and $d(f,g)$ is given by $(d**)$.

Indeed If $\{f_j\}_{j\in\mathbb{N}}$ is Cauchy sequence w.r.t. $(d**)$, then

$$\limsup_{n,m\to\infty} p_k(f_n - f_m) = \limsup_{n,m\to\infty} \sup_{x\in A_k} |f_n(x) - f_m(x)| = 0$$

This implies that, for each $x \in \Omega$ ($\Rightarrow x \in A_k$ for certain k), $f_n(x) \to f(x)$ and in fact $f_n \rightrightarrows f$ in A_k. Hence

$$\limsup_{n\to\infty} d(f_n,f) = \lim_{n\to\infty} \sum_{k=1}^{m} 2^{-k} \frac{p_k(f_n-f)}{1+p_k(f_n-f)} + \limsup_{n\to\infty} \sum_{m+1}^{\infty} \cdots$$
$$= 0 + \frac{1}{2^m} \to 0 \text{ as } m \to \infty. \qquad \text{Q.E.D.}$$

Examples (1) For an open set $\Omega \subset \mathbb{R}^d$ consider $C(\Omega)$. It does not have a natural norm. But $C(\Omega)$ can be made a Fréchet space. For every $k \in \mathbb{N}$, consider compact sets

$$A_k \overset{df.}{=} \{ x \in \Omega ;\ |x| \le k,\ \underbrace{d(x, \partial\Omega)}_{\text{jiné } d \text{ než výše.}} \le \tfrac{1}{k} \}$$

Define

$$p_k(f) \overset{df.}{=} \max_{x \in A_k} |f(x)|$$

and $d(\cdot,\cdot)$ as above

Then $\left(C(\Omega), d(\cdot,\cdot) \right)$ is complete metric space. (i.e. Fréchet space)

Indeed, let $\{f_j\}_{j \in \mathbb{N}}$ be a Cauchy sequence w.r.t. $d(\cdot,\cdot)$. Then

$$\limsup_{i,j \to \infty} p_k(f_i - f_j) = \limsup_{i,j \to \infty} \sup_{x \in A_k} |f_i(x) - f_j(x)| = 0$$

Since each $x \in X$ is contained in some A_k, $\{f_j(x)\}_{j \in \mathbb{N}}$ is Cauchy a there is the limit denoted $f(x)$.

Even more, for each compact set $K \subset \Omega$ there is A_k such that $K \subset A_k$ and $f_j \rightrightarrows f$ in A_k, hence f is continuous.

It remains to show that $d(f_j, f) \to 0$ as $j \to +\infty$. However, for any $m \in \mathbb{N}$

$$\limsup_{j \to \infty} d(f_j, f) \le \limsup_{j \to \infty} \sum_{k=1}^{m} \frac{p_k(f_j - f)}{2^k (1 + p_k(f_j - f))} + \limsup_{j \to \infty} \sum_{m+1}^{\infty} \cdots$$

$$\le \quad 0 + 2^{-m} \xrightarrow[\text{as } m \to \infty]{} 0$$

$\square$

(2) $\boxed{L^p_{loc}(\Omega)}$ Again $\Omega \subset \mathbb{R}^d$ open set.
We say Ω' is compactly contained in Ω, if $\boxed{\Omega' \subset\subset \Omega}$ $\quad \Omega' \subset \boxed{\overline{\Omega'} \subset \Omega}$

$$L^p_{loc}(\Omega) \overset{df.}{=} \{ u : \Omega \to \mathbb{R} \text{ measurable};\ \int_{\Omega'} |f(x)|^p dx < \infty \text{ for all } \Omega' \subset\subset \Omega \}$$

The space L^p_{loc} is not endowed with a natural norm.

However, we can set, for each $k \ge 1$:

$$\boxed{p_k(f) \overset{df.}{=} \left(\int_{A_k} |f(x)|^p dx \right)^{1/p}} = \|f\|_{L^p(A_k)} = \|f\|_{p, A_k}$$

where A_k are as above.

Then $\left\{ L^p_{loc}(\Omega);\ d(f,g) := \sum \frac{1}{2^k} \frac{p_k(f-g)}{1 + p_k(f-g)} \right\}$ is Fréchet space.

(2) Show that $L^p_{loc}(\Omega) := \{ u : \Omega \to \mathbb{R}$ measurable; $\int_{\Omega'} |f(x)|^p dx < +\infty$ for each open $\underbrace{\Omega' \subset\subset \Omega}_{}$ i.e. $\Omega' \subset \overline{\Omega'} \subset \Omega$. $\}$

with
$$p_k(f) \overset{\text{df.}}{=} \left(\int_{A_k} |f|^p dx \right)^{1/p}$$
and $d(f,g)$ given by $(d**)$ is a Fréchet space.

$\boxed{\text{HAHN-BANACH THEOREM}}$ or extension theorems

One of the goals of this section is to show that there are many continuous (bdd) lin. functionals on X.
Towards this goal, we show first Hahn-Banach extension theorem: for a given $f \in V'$, where $V \subset\subset X$, there is $F \in X'$ so that $F = f$ on V and satisfies some other preserving properties.

Consider $p : X \to \mathbb{R}$ satisfying

$$\boxed{(pp) \qquad p(x+y) \le p(x) + p(y), \quad p(\lambda x) = \lambda p(x) \quad \forall x,y \in X,\ \forall \lambda \ge 0.}$$

Example
- $p(x) = \kappa \|x\|_X$ satisfies (pp) for each $\kappa > 0$.
- Every seminorm satisfies (pp).

NOTES
- p fulfilling (pp) is convex.
- p can be negative. (while any seminorm is non-negative).

Example
- Let $(X, \|\cdot\|_X)$ be a normed space and $\Omega \subset X$ be a bdd, open, convex, containing the origin.

Then $\boxed{p(x) \overset{\text{df.}}{=} \inf_{\lambda \ge 0} \{ x \in \lambda\Omega \}}$ satisfies (pp).

i.e.
$$p(x) \overset{\text{def.}}{=} \inf \{ r > 0 ;\ \tfrac{x}{r} \in \Omega \}$$
$\boxed{\text{Minkowski functional}}$

(Pf) follows from
- $\frac{x}{r} \in \Omega \iff \frac{\lambda x}{\lambda r} \in \Omega \quad (\lambda \ge 0)$
- $\frac{x}{r} \in \Omega$ & $\frac{y}{s} \in \Omega \Rightarrow \frac{x+y}{r+s} = \underbrace{\left(\frac{r}{r+s}\right)\frac{x}{r} + \left(\frac{s}{r+s}\right)\frac{y}{s}}_{\text{convex combination of points} \in \Omega} \in \Omega$

Theorem 1.5 (Hahn-Banach)

Let X be a vector space over $\mathbb{R}$ and $p: X \to \mathbb{R}$ satisfies (pp)
Let $V \subset\subset X$ and $f \in V \to \mathbb{R}$ be linear.
so that

(A1) $$f(x) \leq p(x) \quad \forall x \in V.$$

Then $\exists F: X \to \mathbb{R}$ linear such that

(T1) $$F(x) = f(x) \quad \forall x \in V$$

and

(T2) $$-p(-x) \leq F(x) \leq p(x) \quad \forall x \in X.$$

Pf
- If $V = X$, then we are done by observing that for $x \in X$ $f(x) = -f(-x) \underset{(A1)}{\geq} -p(-x)$, which gives (T2).
- If $V \neq X$, then we take any $x_0 \notin V$ and consider the strictly larger subspace $V_0 \overset{df.}{=} \{x + t x_0;\ x \in V, t \in \mathbb{R}\}$

From (A1), for any $x, y \in V$,

$$f(x) + f(y) \underset{\text{linearity}}{=} f(\underbrace{x+y}_{\in V}) \underset{(A1)}{\leq} p(x+y) \leq p(x - x_0) + p(x_0 + y),$$

which implies

$$f(x) - p(x - x_0) \leq p(y + x_0) - f(y) \quad \forall x, y \in V.$$

Set $\beta \overset{df}{=} \sup_{x \in V} \{f(x) - p(x - x_0)\}$, we get

$$\boxed{(*) \quad f(x) - p(x - x_0) \leq \beta \leq p(y + x_0) - f(y) \quad \forall x, y \in V}$$

- $\boxed{\text{Extension of } f \text{ on } V_0}$ set $f(x + t x_0) \overset{df}{=} f(x) + \beta t$

We shall show that f satisfies (A1) on V_0, i.e., we want to show that

$$(\bullet) \quad f(x + t x_0) \leq p(x + t x_0) \quad \forall x \in V.$$

Clearly, $(\bullet)$ follows from (A1) for $t = 0$. For $t > 0$, we use $(*)$ with $x = y = \frac{x}{t}$:

$$f\left(\frac{x}{t}\right) - p\left(\frac{x}{t} - x_0\right) \leq \beta \leq p\left(\frac{x}{t} + x_0\right) - f\left(\frac{x}{t}\right)$$

$$\Downarrow$$

$$\boxed{f(x) - p(x - t x_0) \leq t\beta \leq p(x + t x_0) - f(x)}$$

Hence

$$\left.\begin{array}{l} f(x + t x_0) = f(x) + \beta t \leq p(x + t x_0) \\ f(x - t x_0) = f(x) - \beta t \leq p(x + t x_0) \end{array}\right\} \Rightarrow (\bullet).$$

- By the previous step, every $f \in V'$ can be extended to a larger subspace still satisfying (A1).

Let $\mathcal{F}$ be a family of (V, ϕ), where $V \subset\subset X$ and $\phi: V \to \mathbb{R}$, $\phi \in V'$, satisfies $\phi(x) \leq p(x)\ \forall x \in V$.

We can partially order $\mathcal{F}$:

$$\underset{\in \mathcal{F}}{(V_1, \phi_1)} \prec \underset{\in \mathcal{F}}{(V_2, \phi_2)} \overset{\text{def.}}{=} V_1 \subset\subset V_2 \text{ and } \phi_2 = \phi_1 \text{ na } V_1 \quad (\phi_2|_{V_1} = \phi_1).$$

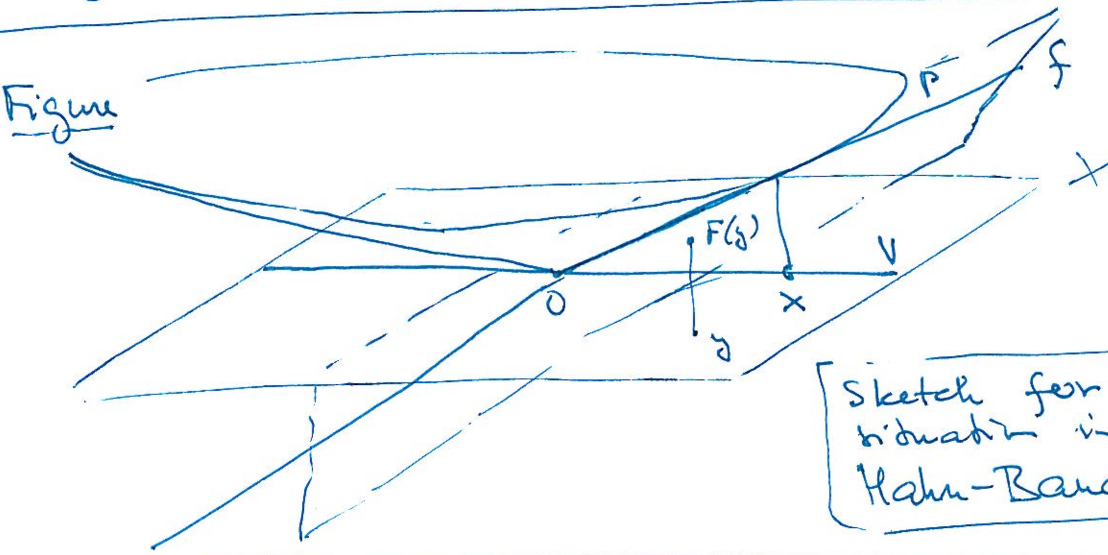
By Hausdorff Maximality principle (equivalent to Axiom of Choice), $(\mathcal{F}, \preceq)$ contains a maximal element: (V_{max}, F).

If $V_{max} \neq X$, then we can extended as above.

Hence $V_{max} = X$ and $F(x) \leq p(x)\ \forall x \in X$.

By linearity: $F(x) = -F(-x) \geq -p(-x)$.

Figure



Sketch for the situation in the Hahn-Banach Theorem

<u>Def</u> A nonempty set S is partially ordered by a binary relation $\prec$ if, for every $a, b, c \in S$:

(i) $a \prec a$

(ii) $a \prec b$ & $b \prec a \Rightarrow a = b$

(iii) $a \prec b$ & $b \prec c \Rightarrow a \prec c$

A subset $S' \subset S$ of a partially ordered set S is said to be totally ordered if, for every $a, b \in S'$, either $a \prec b$ or $b \prec a$.

We say that S' is maximal (w.r.t. the total ordering) if S' is not strictly contained in any other totally ordered set.

<u>Theorem</u> (Hausdorff maximality principle) If S is partially ordered, then every totally ordered subset $S' \subset S$ is contained in a maximally ordered subset.

$\boxed{\text{Theorem 1.6}}$ (Hahn–Banach for bdd linear functionals)

Let X be a normed space over $\mathbb{K}$. Let $f \in V'$ for $V \subset\subset X$.
Then f can be extended to $F \in X'$ so that

$$\boxed{\|F\|_{X'} = \sup_{\substack{x \in X \\ \|x\|_X = 1}} |F(x)| = \sup_{\substack{x \in V \\ \|x\| = 1}} |f(x)| = \|f\|_{V'}} \quad \& \ F = f \text{ on } V$$

Pf

[i] Let $\mathbb{K} = \mathbb{R}$. Set $K = \|f\|_{V'}$ and $p(x) = K\|x\|_X$. Then, by previous theorem (as $f(x) \le \|f\|_{V'} \|x\|_X = p(x)$), there is $F \in X \to \mathbb{R}$ linear so that

$$|F(x)| \le \|f\|_{V'} \|x\|,$$

which implies $\quad \|F\|_{X'} \le \|f\|_{V'}$.

The opposite inequality is obvious:

$$\|f\|_{V'} = \sup_{\substack{x \in V \\ \|x\| = 1}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x \in V \\ \|x\| = 1}} \frac{|F(x)|}{\|x\|} \le \sup_{\substack{x \in X \\ \|x\| = 1}} \frac{|F(x)|}{\|x\|} = \|F\|_{X'}.$$

[ii] Now $\mathbb{K} = \mathbb{C}$. Yet V and X can be also regarded as vector spaces over $\mathbb{R}$.

The functional $F : X \to \mathbb{C}$ will be constructed separately for real and imaginary part.

For $x \in V$, set $u(x) = \operatorname{Re} f(x)$. This is a real-valued functional on V and $\|u\|_{V'} \le \|f\|_{V'}$. Hence, there is $U \in X'$ such that $\|U\|_{X'} \le \|f\|_{V'}$. Then the map

$$F(x) = U(x) - i\,U(ix)$$

satisfies all requirements. Indeed, for $x \in V$

$$F(x) = \operatorname{Re} f(x) - i \operatorname{Re} f(ix) = \operatorname{Re} f(x) + i \operatorname{Im} f(x) = f(x)$$

HW $\boxed{\operatorname{Im} f(x) = -\operatorname{Re} f(ix)}$

Moreover, let $\alpha \in \mathbb{C}$ be such that $|\alpha| = 1$, $\alpha F(x) = |F(x)|$. Then

$$|F(x)| = \alpha F(x) = U(\alpha x) \le \|f\|_{V'} \|\alpha x\|_X = \|f\|_{V'} \|x\|_X$$

Hence $\quad \|F\|_{X'} \le \|f\|_{V'}$.

As an aplicaton, we show that points in $(X, \|\cdot\|_X)$ can be separated from subspaces by means of linear functionals. (Further generalisat. to separat. of convex sets will follow.) This separation property is often used to show that a given subspace is dense in X.

Theorem 1.7 Let $Y \subset\subset X$ closed, $(X, \|\cdot\|)$ normed and $x_0 \notin Y$.
Then $\exists\, \phi \in X'$: $\phi = 0$ on Y, $\|\phi\|_{X'} = 1$ and $\phi(x_0) = \mathrm{dist}(x_0, Y)$.

$(\Leftrightarrow)$ $\exists\, \phi \in X'$: $\phi = 0$ on Y; $\|\phi\|_{X'} = \frac{1}{\mathrm{dist}(x_0, Y)}$ and $\phi(x_0) = 1$.

Pf On $Y_0 = \mathrm{span}(Y \cup \{x_0\}) = Y \oplus \mathrm{span}\, x_0 = \{y + \alpha x_0;\ y \in Y, \alpha \in \mathbb{K}\}$
we set $\phi_0(y + \alpha x_0) = \alpha\, \mathrm{dist}(x_0, Y)$ $\quad \forall y \in Y\ \forall \alpha \in \mathbb{K}$

Then $\phi_0 : Y_0 \to \mathbb{K}$ is linear and $\phi_0(y) = 0\ \forall y \in Y$.

Once we show that $\phi_0 \in Y_0^*$ and $\|\phi_0\|_{Y_0^*} = 1$, then H-B gives the result.

Let $y \in Y$ and $\alpha \neq 0$. Then $\mathrm{dist}(x_0, Y) \leq \|x_0 - \frac{-y}{\alpha}\|_X$

$$\Rightarrow |\phi_0(y + \alpha x_0)| \leq |\alpha|\, \|x_0 - \tfrac{-y}{\alpha}\|_X = \|\alpha x_0 + y\|_X$$

$$\Rightarrow \phi_0 \in Y_0^* \text{ with } \|\phi_0\|_{Y_0^*} \leq 1.$$

As Y is closed, $\mathrm{dist}(x_0, Y) > 0$ and for $\forall \varepsilon > 0\ \exists\, y_\varepsilon \in Y$
s.t. $\|x_0 - y_\varepsilon\|_X \leq (1+\varepsilon)\, \mathrm{dist}(x_0, Y)$

Then $\phi_0(x_0 - y_\varepsilon) = \mathrm{dist}(x_0, Y) \geq \frac{1}{1+\varepsilon} \|x_0 - y_\varepsilon\|_X$

$\Rightarrow$ $\|\phi_0\|_{Y_0'} \geq \frac{1}{1+\varepsilon} \to 1$ as $\varepsilon \to 0$.

$(x_0 - y_\varepsilon \neq 0)$

Corollaries Let $(X, \|\cdot\|_X)$ and $x_0 \in X$. Then

!

(1) If $x_0 \neq 0$, then $\exists\, \phi_0 \in X^*$: $\|\phi_0\|_{X^*} = 1$ and $\phi_0(x_0) = \|x_0\|_X$
Pf follows from Theorem 1.7. with $Y = \{0\}$.

(2) If $\phi(x_0) = 0\ \forall \phi \in X^*$, then $x_0 = 0$. Pf follows by contradiction from (1)

(3) Setting $J_{x_0} : \phi \in X' \mapsto \phi(x_0)$, then $J_{x_0} \in \mathcal{L}(X^*; \mathbb{K}) = X^{**}$ and $\|J_{x_0}\|_{X^{**}} = \|x_0\|_X$

Pf Clearly $|\langle J_{x_0}, \phi\rangle| = |J_{x_0}(\phi)| = |\phi(x_0)| \leq \|x_0\|_{X^{**}} \|\phi\|_{X'}$ $\Rightarrow$ By (1); $|J_{x_0}(\phi_0)| = \|x_0\|_X$ $\Rightarrow \|J_{x_0}\|_{X^{**}} = \|x_0\|_X$

$\boxed{\text{Theorem 1.8}}$ (Separation of convex sets)

Let $(X, \|\cdot\|_X)$ be normed and A, B nonempty, disjoint, convex subsets of X.

(1) If A is open, then $\exists\ \phi \in X^*$ (mod $\mathbb{R}$) and $c \in \mathbb{R}$ such that $\phi(a) < c \le \phi(b) \quad \forall a \in A \ \forall b \in B$

(2) If A is compact and B closed, then $\exists\ \phi \in X^*$ (mod $\mathbb{R}$) and $c_1, c_2 \in \mathbb{R}$:

$$\phi(a) \le c_1 < c_2 \le \phi(b)$$

Pf Uses Minkowski functional. Uses: Theorem 1.7 as subpart. as a tool.