Definice — Definice $\mathcal{D}(\Omega)$ a konvergence v $\mathcal{D}(\Omega)$

20.5.2021

Buď $\Omega \subset \mathbb{R}^d$ otevřená, pak

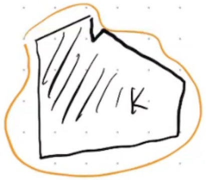
$$\mathcal{D}(\Omega) := \{\varphi \in C^\infty(\Omega);\ \operatorname{supp}\varphi \text{ kompaktní v } \Omega\}$$

$$\operatorname{supp}\varphi := \overline{\{x\in\mathbb{R}^d;\ \varphi(x)\neq 0\}}$$

Dále říkáme, že $\boxed{\varphi_k \to \varphi \text{ v } \mathcal{D}(\Omega)}$, kde $\varphi_k,\varphi \in \mathcal{D}(\Omega)$

právě tehdy: $\exists K \subset \Omega$ kompaktní tak, že

- $\operatorname{supp}\varphi_k \subset K \quad \forall k\in\mathbb{N}$, $\operatorname{supp}\varphi \subset K$
- $\varphi_k \rightrightarrows \varphi$ v K a také $D^\alpha\varphi_k \rightrightarrows D^\alpha\varphi$ v K $\forall$ multiindex $\alpha=(\alpha_1,\dots,\alpha_d)$



Def. (DISTRIBUCE) Distribuce (nebo zobecněná funkce nebo symbolická fce) je funkcionál T na $\mathcal{D}(\Omega)$, který je lineární a spojitý, tj. $T: \mathcal{D}(\Omega) \to \mathbb{R}$ (nebo $\mathbb{C}$) tak, že

a) $T(\alpha_1\phi_1+\alpha_2\phi_2) = \alpha_1 T(\phi_1) + \alpha_2 T(\phi_2)$ $\quad\forall \alpha_1,\alpha_2 \in \mathbb{R}$ (nebo $\mathbb{C}$) $\forall \phi_1,\phi_2 \in \mathcal{D}(\Omega)$

b) $T(\varphi_k) \to 0$ pokud $\varphi_k \to 0$ v $\mathcal{D}(\Omega)$

Pozorování Z vlastností a) a b) plyne tvrzení: pokud $\phi_k \to \phi$ v $\mathcal{D}(\Omega)$, pak $T(\phi_k) \to T(\phi)$

Dk: $\varphi_k := \phi_k - \phi \Rightarrow \varphi_k \to 0$ v $\mathcal{D}(\Omega) \overset{b)}{\Rightarrow} T(\varphi_k) \to 0$

ale $T(\varphi_k) \underset{a)}{=} T(\phi_k) - T(\phi)$ $\Rightarrow T(\phi_k) \to T(\phi)$

$$\mathcal{S} := \{\varphi \in C^\infty(\mathbb{R}^d);\ \sup_{x\in\mathbb{R}^d} |x^\beta D^\alpha \varphi(x)| < \infty\}$$
pro všechny multiindexy α,β

kde $\boxed{\varphi_k \to 0 \text{ v } \mathcal{S}(\mathbb{R}^d)}$ znamená, že $\forall \alpha,\beta$ $\max_{x\in\mathbb{R}^d} |x^\alpha D^\beta \varphi_k(x)| \to 0$
$\Updownarrow$
$\boxed{x^\alpha D^\beta \varphi_k \rightrightarrows 0 \text{ v } \mathbb{R}^d}$

Def $T: \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ (nebo $\mathbb{C}$) je temperovaná distribuce jestliže

- smysluplnost: $\langle T,\varphi\rangle \in \mathbb{R}$ ($\mathbb{C}$) $\quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d)$
- linearita: $\langle T, \alpha\varphi+\beta\psi\rangle = \alpha\langle T,\varphi\rangle + \beta\langle T,\psi\rangle$ $\quad\forall \varphi,\psi \in \mathcal{S}(\mathbb{R}^d)$ $\forall \alpha,\beta\in\mathbb{R}$ (nebo $\mathbb{C}$)
- spojitost: pokud $\boxed{\varphi_k \to 0 \text{ v } \mathcal{S}}$ pak $\langle T,\varphi_k\rangle \to 0$ $(k\to\infty)$

$$\underset{f}{\mathcal{D}(\Omega)} \subset \underset{f}{L^1_{loc}} \subset \underset{T_f}{\mathcal{D}'(\Omega)}$$

$$\underset{f}{\mathcal{D}(\mathbb{R}^d)} \subset \underset{f}{\mathcal{S}(\mathbb{R}^d)} \subset \underset{f\leftrightarrow T_f}{L^1(\mathbb{R}^d)} \subset \underset{T_f}{\mathcal{S}'(\mathbb{R}^d)} \subset \underset{T_f}{\mathcal{D}'(\mathbb{R}^d)}$$

D.cv. pečlivě rozmyslete

Tedy definujeme: pro $T \in \mathcal{D}'(\Omega)$

$$(1)\quad \langle \tau_a T, \varphi \rangle = \langle T, \tau_{-a}\varphi \rangle \qquad \forall \varphi \in \mathcal{D}(\Omega),\ a \in \mathbb{R}$$

$$(2)\quad \langle d_\lambda T, \varphi \rangle = \frac{1}{\lambda^d}\langle T, d_{\frac{1}{\lambda}}\varphi \rangle \qquad —\|—,\ \lambda > 0$$

$$(3)\quad \langle D^\alpha T, \varphi \rangle = \langle T, (-1)^{|\alpha|} D^\alpha \varphi \rangle \qquad \forall \varphi \in \mathcal{D}(\Omega),\ \forall \alpha$$

$$(4)\quad \langle mT, \varphi \rangle = \langle T, m\varphi \rangle \qquad \forall \varphi \in \mathcal{D}(\Omega),\ \forall m \in C^\infty.$$

Distributivní počet na $\mathcal{S}$ (resp. pro temperované distribuce)

Vlastnosti (1)–(4) platí i pro temperované distribuce, s důležitou změnou u vlastnosti (4), která platí pro $m \in C^\infty(\mathbb{R}^d)$ takové, že $\exists N \in \mathbb{N}$ $\exists C > 0$ taková, že $|m(x)| \le C|x|^N$ pro $x \to \infty$

$$\langle mT, \varphi \rangle = \langle T, \underbrace{m\varphi}_{\in \mathcal{S}} \rangle$$

$$\boxed{\langle OT, \varphi \rangle = \langle T, S\varphi \rangle}$$

$\boxed{\text{Distributivní počet pro reg. distribuce}}$

- **Posunutí** $\quad a \in \mathbb{R}^d \quad \tau_a f(x) = f(x+a)$

$$\langle \tau_a f, \varphi \rangle = \int_\Omega f(x+a)\varphi(x)\,dx = \int_\Omega f(z)\varphi(z-a)\,dz = \langle f, \tau_{-a}\varphi \rangle$$

$\varphi \equiv 0$ mimo $K \subset \Omega$, $f \equiv 0$ mimo Ω předp.

$z = x + a$, $x = z - a$

- **Škálování** $\quad \lambda > 0 \quad d_\lambda f(x) = f(\lambda x)$

$$\langle d_\lambda f, \varphi \rangle = \int_{\mathbb{R}^d} f(\lambda x)\varphi(x)\,dx = \frac{1}{\lambda^d}\int_{\mathbb{R}^d} f(z)\varphi\left(\frac{z}{\lambda}\right)dz = \frac{1}{\lambda^d}\langle f, d_{\frac{1}{\lambda}}\varphi \rangle$$

$z = \lambda x$, $dx_i = \frac{dz_i}{\lambda}$

- **Derivace**

$$\langle D^\alpha f, \varphi \rangle = \int_\Omega D^\alpha f(x)\varphi(x)\,dx = \int_\Omega \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}\varphi(x)\,dx \overset{\text{per partes}}{=} (-1)^{|\alpha|}\int_\Omega f(x)\, D^\alpha \varphi(x)\,dx = (-1)^{|\alpha|}\langle f, D^\alpha \varphi \rangle$$

- **Násobení skalární fcí** $\quad (m \in C^\infty(\Omega))$

$$\boxed{\langle mf, \varphi \rangle} = \int_\Omega m(x) f(x)\varphi(x)\,dx = \int f(x)\, m(x)\varphi(x)\,dx = \langle f, \underbrace{m\varphi}_{\in \mathcal{D}(\Omega)} \rangle$$

$$\boxed{\left\langle \frac{\partial}{\partial x_k} T_f, \varphi \right\rangle} \overset{(3)}{=} -\left\langle T_f, \frac{\partial \varphi}{\partial x_k} \right\rangle = -\int_\Omega f(x) \frac{\partial \varphi}{\partial x_k}(x)\,dx$$

$$= -\lim_{h\to 0} \int_\Omega f(x) \frac{\varphi(x+he_k)-\varphi(x)}{h}\,dx$$

$$= -\lim_{h\to 0} \frac{1}{h}\left[\int_\Omega f(x)\,\varphi(\underbrace{x+he_k}_{z})\,dx - \int_\Omega f(x)\varphi(x)\,dx \right]$$

$x = z - he_k$ (φ má kpt nosič), h malé

$$= -\lim \frac{1}{h}\left[\int_\Omega f(z-he_k)\varphi(z)\,dz - \int f(x)\varphi(x)\,dx \right]$$

přeznačení $z \to x$

$$= \lim_{h\to 0} \int_\Omega \frac{f(x-he_k)-f(x)}{-h}\varphi(x)\,dx \overset{\frac{\partial f}{\partial x_k} \in L^1_{loc}}{=} \int_\Omega \frac{\partial f}{\partial x_k}(x)\varphi(x)\,dx$$

$$= \boxed{\left\langle T_{\frac{\partial f}{\partial x_k}}, \varphi \right\rangle} \quad \boxtimes$$

Příklady ① Lokálně integrovatelné funkce jsou distribuce

Buď $f \in L^1_{loc}(\Omega) \equiv \{ f \in L^1(K);\ \forall K \subset \Omega \text{ kompaktní} \}$

Def.
$$T(\varphi) := \int_\Omega f(x)\varphi(x)\,dx \qquad \text{pro } \forall \varphi \in \mathcal{D}(\Omega)$$

Zjednodušte $T \in \mathcal{D}'(\mathbb{R})$

$T = x^k D^n \delta(0)$ $\quad k, n \in \mathbb{N}$

Tedy definujeme: pro $T \in \mathcal{D}'(\Omega)$

(1) $\langle \tau_a T, \varphi \rangle = \langle T, \tau_{-a} \varphi \rangle$ $\quad \forall \varphi \in \mathcal{D}(\Omega), a \in \mathbb{R}$

(2) $\langle d_\lambda T, \varphi \rangle = \frac{1}{\lambda^a} \langle T, d_{\frac{1}{\lambda}} \varphi \rangle$ $\quad$ —||— $, \lambda \neq 0$

(3) • $\langle D^\alpha T, \varphi \rangle = \langle T, (-1)^{|\alpha|} D^\alpha \varphi \rangle$ $\quad \forall \varphi \in \mathcal{D}(\Omega), \forall \alpha$

(4) • $\langle m T, \varphi \rangle = \langle T, m \varphi \rangle$ $\quad \forall \varphi \in \mathcal{D}(\Omega), \forall m \in C^\infty$.

$$\langle x^k \delta_0^{(n)}, \varphi \rangle = \langle \delta_0^{(n)}, x^k \varphi \rangle = (-1)^n \langle \delta_0, \underbrace{(x^k \varphi)^{(n)}} \rangle =$$

$$= (-1)^n \langle \delta_0, \sum_{j=0}^{\min(k,n)} \binom{n}{j} \underbrace{(x^k)^{(j)}} \varphi(x)^{(n-j)} \rangle$$

$j \le k$

$$= (-1)^n \langle \delta_0, \sum_{j=0}^{\min(k,n)} \binom{n}{j} k(k-1)\cdots(k-j+1) x^{k-j} \varphi^{(n-j)} \rangle =$$

$$= (-1)^n \sum_{j=0}^{\min(k,n)} k(k-1)\cdots(k-j+1) \left(\underbrace{x^{k-j}} \varphi^{(n-j)} \right) \Big|_{x=0}$$

$\int_{\mathbb{R}} \delta_0 (\sum \not{x} \quad)$

$$= \begin{cases} 0 & n < k \\ (-1)^n \binom{n}{k} k! \, \varphi^{(n-k)}(0) = & \end{cases}$$

$$= (-1)^n \binom{n}{k} k! \langle \delta_0, \varphi^{(n-k)} \rangle$$

$$= (-1)^n (-1)^{n-k} \binom{n}{k} k! \langle \delta^{(n-k)}, \varphi \rangle$$

$$x^k \delta_0^{(n)} = \begin{cases} (-1)^k \binom{n}{k} k! \, \delta_0^{(n-k)} & n \ge k \\ 0 & n < k \end{cases}$$

speciálně

$k = n \qquad x^n \delta_0^{(n)} = (-1)^n n! \, \delta_0$

$k = 1, n = 1 \qquad x \delta_0' = -\delta$

$k = 1 \; n = 0 \qquad x \delta = 0$

$\langle x^2 \Delta \delta_0, \varphi \rangle$ $\qquad\qquad\qquad\qquad$ $\varphi \in \mathcal{D}(\mathbb{R}^N)$

$\Delta \cdot \quad \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_j} \cdot \right)$

$$= \langle \Delta \delta_0, x^2 \varphi \rangle = \langle \delta_0, \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_j} x^2 \varphi\right) \rangle =$$

$$= \langle \delta_0, \frac{\partial}{\partial x_j}\left(2x_j \varphi + x^2 \frac{\partial \varphi}{\partial x_j} \right) \rangle =$$

$$= \langle \delta_0, 2N\varphi + 4 x_j \frac{\partial \varphi}{\partial x_j} + x^2 \Delta \varphi \rangle =$$

$$= 2N\varphi(0) = 2N \langle \delta_0, \varphi \rangle \to x^2 \Delta \delta_0 = 2N \delta_0$$

Ukažte, že posloupnost

$$f_n(x) = \frac{1}{\pi}\,\frac{n}{n^2x^2+1}$$

konvergují v $\mathcal{D}'(\mathbb{R})$ k δ_0 distribuci.

$$\langle T_{f_n}, \varphi\rangle \xrightarrow[n\to\infty]{?} \langle \delta_0, \varphi\rangle$$

$$\langle T_{f_n}, \varphi\rangle = \int_{-\infty}^{\infty} \frac{1}{\pi}\,\frac{n}{n^2x^2+1}\,\varphi(x)\,dx = \quad \left| nx = y \right|$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi}\,\frac{1}{y^2+1}\,\varphi\left(\frac{y}{n}\right)dy$$

$$\lim_{n\to\infty} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{y^2+1}\,\varphi\left(\frac{y}{n}\right)dy = \int_{-\infty}^{\infty} \frac{1}{\pi}\,\frac{1}{y^2+1}\,\varphi(0)\,dy$$

$\Uparrow$

L^1

$$= \frac{\varphi_0}{\pi}\int_{-\infty}^{\infty} \frac{1}{y^2+1}\,dy = \varphi_0 = \langle \delta_0, \varphi\rangle$$

$f_n = \frac{1}{\pi} \frac{\sin nx}{x}$ $\notin L_1$

postačující podmínky pro koncentraci

$\{f_n\}_{n=1}^{\infty} \subset C(\mathbb{R})$ je taková posloupnost, že

$\forall M > 0 \quad \exists C > 0 \qquad -M \leq a < b \leq M \Rightarrow \left| \int_a^b f_n(x)\,dx \right| \leq C$ (K1)

nechť pro každý omezený interval $(a,b) \subset \mathbb{R}$ platí

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \begin{cases} 0 & \text{pokud } 0 \notin (a,b) \\ 1 & \text{pokud } 0 \in (a,b) \end{cases}$$ (K2)

Pak $T_{f_n} \to \delta_0$

ve smyslu distribucí

Uvažujme T_{F_n} $\quad F_n(x) = \int_{-1}^{x} f_n(t)\,dt$ konverguje k Heaviside. fce

z korektnosti derivací pak plyne tvrzení.

- pro fixovanou $\underline{\varphi \in \mathcal{D}(\mathbb{R})}$

z předpokladu $|F_n(x)| \leq C$ pro $x \in \operatorname{supp} \varphi$

$$F_n(x) \xrightarrow{n\to\infty} \left.\begin{cases} 0 & x \in (-\infty, 0) \\ 1 & x \in (0, \infty) \end{cases}\right\} = H(x)$$

$$\langle T_{F_n}, \varphi \rangle = \int_{\mathbb{R}} F_n \varphi\,dx = \int_{\operatorname{supp}\varphi} F_n \varphi\,dx \xrightarrow[n\to\infty]{} \int_{\operatorname{supp}\varphi} H\varphi(x)\,dx =$$

$$= \int_{\mathbb{R}} H\varphi(x)\,dx = \langle T_H, \varphi \rangle \Rightarrow T_{f_n} = \quad = DT_{F_n}$$

$$\to DT_H = \delta_0$$

- $f_n = \frac{1}{\pi} \frac{\sin nx}{x}$ + dodefinujeme lim v $x=0$

$$\int_a^b \frac{1}{\pi} \frac{\sin nx}{x} dx = |y = nx| = \frac{1}{\pi} \int_{a \cdot n}^{b \cdot n} \frac{\sin y}{y} dy$$

- pokud $\underline{0 \in (a,b)}$ $\lim_{n\to\infty} \frac{1}{\pi} \int_{a\cdot n}^{b \cdot n} \frac{\sin y}{y} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin y}{y} dy = \underline{1}$

- pokud $0 \notin (a,b)$ $\quad 0 < a < b$

$$\lim_{n\to\infty} \int_a^b f_n(x) dx = \lim_{n\to\infty} \frac{1}{\pi} \int_{an}^{bn} \frac{\sin y}{y} = \lim_{n\to\infty} \frac{1}{\pi} \left(\int_0^{nb} \frac{\sin y}{y} dy - \int_0^{an} \frac{\sin y}{y} dy \right) = 0$$

$a < b < 0$ $\quad$ K2

- K1 $\quad 0 \in (a,b)$

$$\underbrace{\lim_{n\to\infty} \int_a^b f_n(x) dx}_{I_n} = 1$$

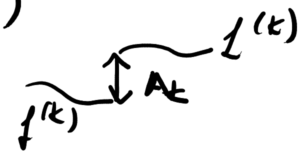
$\exists n_0 \ \forall n > n_0 \quad |I_n - 1| < \varepsilon$

$|I_n| < 2$

pak $C = \max\{2, \max_{k=1,\dots,n_0} I_k\}$

f hladká na $\mathbb{R}\setminus\{0\}$ $\qquad \varphi \in \mathcal{D}(\mathbb{R})$

jednostranné derivace v počátku



$$A_k = \underbrace{f^{(k)}(0^+) - f^{(k)}(0^-)}_{\text{konečné}}$$

T_f regulární distribuce zadanou funkcí f

pak $$D^n T_f = (T_f)^{(n)} = T_{f^{(n)}} + A_{n-1}\delta_0 + A_{n-2}\delta_0^{(1)} + \dots + A_0\,\delta_0^{(n-1)}$$ □

$D^1 T_f$: $$T_{f'}(\varphi) = \int_{\mathbb{R}} f'\varphi\,dx = -\int_{\mathbb{R}} f\varphi'\,dx$$

$$= -\int_{-\infty}^{0} f\varphi'\,dx - \int_{0}^{\infty} f\varphi'\,dx = -[f\varphi]_{-\infty}^{0} + \int_{-\infty}^{0} f'\varphi\,dx$$

$$- [f\varphi]_{0}^{\infty} + \int_{0}^{\infty} f'\varphi\,dx = \varphi(0)\underbrace{\left(f(0+) - f(0-)\right)}_{A_0} +$$

$$+ \int_{-\infty}^{\infty} f'\varphi\,dx = A_0\langle \delta_0, \varphi\rangle + T_{f'}$$

$$\boxed{D T_f = A_0\delta_0 + T_{f'}}$$

ŘEŠENÍ LIN. ODR

L lin. dif. operátor $\quad L = \sum_{i=0}^{n} a_i D^i \quad a_n = 0$

$Lu = f \quad (\Delta)$

fundamentální řešení E_L $\quad \boxed{LE_L = \delta} \quad (*)$

ve smyslu distribucí

pro f, pro které existuje konvoluce $E_L * f$ dává tato konvoluce řešení (Δ)

$$L(E_T * f) = \sum_{i=0}^{n} a_k D^k (E_T * f) = \sum_{i=0}^{n} (a_k D^k E_T) * f =$$

$$= (LE_T) * f = \delta * f = f$$

z linearity $\quad LE_T = \delta_0 \quad L(\underbrace{E_T + G}) = \delta_0$

$LG = 0$

- řešení určeno jednoznačně až na libovolné řešení homogenní rovnice $Lu = 0$
- je-li $\sum_{i=0}^{n} a_i \frac{d^i}{dx^i}$ $a_n \neq 0$ obyč. dif. operátor s konst. koeficienty a_i $i = 0, \dots, n$ pak můžeme řešení hledat následovně

Ⓐ slepování

$\underline{y^+}$ a $\underline{y^-}$ jsou řešením $Lu=0$ (v klasickém smyslu)

$$\frac{d^k y^+}{dx^k}(0) = \frac{d^k}{dx^k} y^-(0) \qquad k=0,1,\dots,n-2$$

$$\frac{d^{n-1} y^+}{dx^{n-1}}(0) - \frac{d^{n-1} y^-}{dx^{n-1}}(0) = \frac{1}{a_n}$$

motivace viz $\boxed{\square}$ skok v $n-1$ derivaci generuje δ-fci na pravé straně

Potom je

$$E_L = \begin{cases} y^+(x) & x>0 \\ y^-(x) & x<0 \end{cases} \in \mathcal{D}'(\mathbb{R})$$

· Speciálně můžeme volit např.: $y^- \equiv 0$

a y^+ splňující $(y^+)^{(k)}(0)=0 \quad k=0,\dots,n-2$

$(y^+)^{(n-1)}(0) = \frac{1}{a_n}$

$\Rightarrow$ řešení z $\mathcal{D}'(\mathbb{R})$ + přidáním vhodného řešení homogenní rovnice můžeme najít fundamentální řešení s dalšími (vhodnými) vlastnostmi např. $\in \mathcal{S}'(\mathbb{R})$, sudost, lichost

(B) Fourierova transformace $\boxed{\text{více později}}$

→ fundamentální řešení z $\mathcal{S}'(\mathbb{R})$

$\hat{E}_L = \mathcal{F}(E_L)$ (0)

$P(\xi)\,\hat{E}_L = 1$ $\qquad P(\xi) = \sum_{k=0}^{m} (2\pi i \xi)^k a_k$

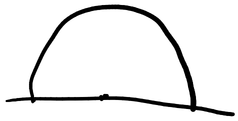
Je-li $P(\xi) \neq 0$ $\forall \xi \in \mathbb{R}$ pak řešením (0) $\left(\frac{1}{P(\xi)} \in \mathcal{S}'\right)$

je $\mathcal{F}^{-1}\left(\frac{1}{P(\xi)}\right)(x) = \int_{\mathbb{R}} \frac{e^{2\pi i \xi x}}{P(\xi)}\, d\xi$ ←

(st $P=1$ int pouze ve smyslu hlavní hodnoty $n \to \infty$)

$= \pm 2\pi i \sum \mathrm{Res}_{\xi_j} \frac{e^{2\pi i \xi x}}{P(\xi)}$

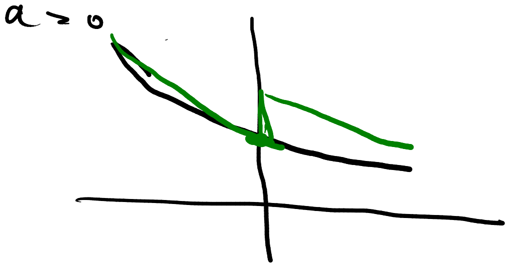
pro $P(\xi) = 0$?



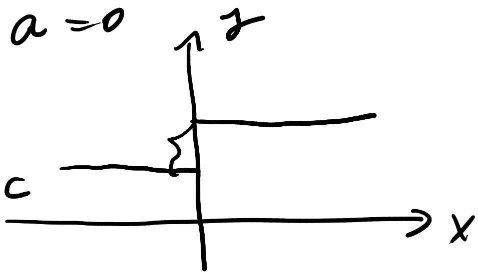
$y' + ay = \delta_0 \qquad a \in \mathbb{R}$

- $e^{\lambda x} \to \lambda + a = 0 \to \lambda = -a \qquad y_H = Ce^{-ax}$

- $y^+ = c^+ e^{-ax}$
  $y^- = c^- e^{-ax}$ $\quad \in \mathcal{D}'$



- lepení



- $y^+(0) - y^-(0) = 1$

$$c^+ - c^- = 1$$

$$c^+ = 1 + c^-$$

$$y = \begin{cases} (1+c^-)e^{-ax} & x \geq 0 \\ c^- e^{-ax} & x < 0 \end{cases} \quad \in \mathcal{D}'(\mathbb{R})$$

$\exists\, y \in \mathcal{S}'(\mathbb{R})$

$a > 0 \qquad c^- \neq 0$ kvůli chování u $-\infty$ $\quad \in \mathcal{D}'(\mathbb{R})$

$$c^- = 0 \qquad y = \left.\begin{cases} e^{-ax} & x \geq 0 \\ 0 & x < 0 \end{cases}\right\} \in \mathcal{S}'(\mathbb{R})$$

$a < 0 \qquad$ problém chování $+\infty$

$$(1+c^-) \neq 0 \qquad y \in \mathcal{D}'(\mathbb{R})$$

$$c^- + 1 = 0 \qquad y = \left.\begin{cases} 0 & x > 0 \\ -e^{-ax} & x < 0 \end{cases}\right\} \in \mathcal{S}'(\mathbb{R})$$

$a = 0 \qquad y = C + H(x) \in \mathcal{S}'(\mathbb{R})$

- řešení pomocí F.T. $\qquad y \in \mathcal{S}'$

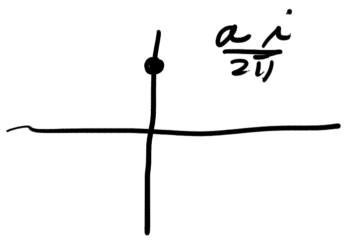
$$y' + ay = \delta \qquad / \, FT \qquad \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x}\, dx$$

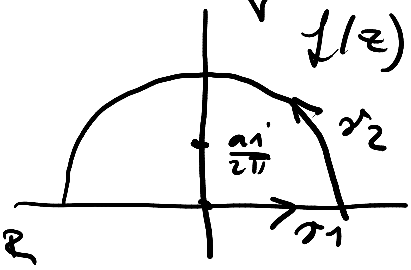
$$(2\pi i \xi + a)\, \hat{y} = 1$$

$$\hat{y} = \frac{1}{2\pi i \xi + a} = \frac{1}{P(\xi)}$$

$$y = \mathcal{F}^{-1}(\hat{y}) = \lim_{R \to \infty} \int_{-R}^{R} \underbrace{\frac{1}{2\pi i \xi + a}\, e^{+2\pi i \xi x}}_{f(z)}\, d\xi$$

$\underline{a > 0}$





- $\underline{x > 0}$

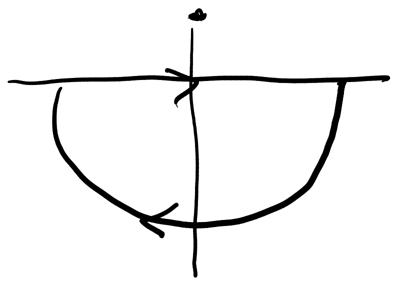
$$\lim_{R \to \infty} \int_{-R}^{R} f(z)\, dz + \underbrace{\int_{\gamma_2} f(z)\, dz}_{\to 0 \text{ Jordan}}$$

$$f(z) = \frac{e^{2\pi i x z}}{2\pi i z + a}$$

$$= 2\pi i \operatorname{Res}_{\frac{ai}{2\pi}} f(z)$$

$$= 2\pi i \frac{e^{2\pi i \frac{ai}{2\pi}}}{2\pi i} = e^{-ax}$$

$x < 0$



$y = 0$

$$a > 0 \quad = \left\{ \begin{array}{ll} e^{-ax} & x \geq 0 \\ 0 & x < 0 \end{array} \right\} \in \mathcal{S}'(\mathbb{R})$$

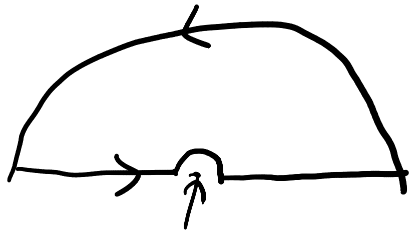
$a < 0$

$x > 0$

$x < 0$

$\frac{ai}{2\pi}$

$a = 0$



$$y = \begin{cases} \frac{1}{2} & x \geq 0 \\ -\frac{1}{2} & x < 0 \end{cases} \in y'(a)$$

$-y^{(4)} + k^2 y'' = \delta_0 \qquad \underline{k>0} \qquad \mathcal{D}', \ \mathcal{S}'$

homogenní rovnice $\quad -y^{(4)} + k^2 y'' = 0$

chor. $\quad -x^4 + k^2 x^2 = 0$

$x_{1,2} = 0$

$x_{3,4} = \pm k$

fund. systém $\quad \{1, x, e^{kx}, e^{-kx}\}$

$y^+ = c_1^+ + c_2^+ x + c_3^+ e^{kx} + c_4^+ e^{-kx} \qquad x>0$

$y^- = c_1^- + c_2^- x + c_3^- e^{kx} + c_4^- e^{-kx} \qquad x<0$

0. derivace v $x=0 \quad y^+(0) - y^-(0) = 0 \qquad \Delta c_i \equiv c_i^+ - c_i^- \quad i=1..4$

$c_1^+ + c_2^+ \cdot 0 + c_3^+ + c_4^+ -$

$- c_1^- \quad 0 - c_3^- - c_4^- = 0$

$$\boxed{\Delta c_1 + \Delta c_3 + \Delta c_4 = 0}$$

1. derivace $\quad (y^+)'|_0 - (y^-)'|_0 = 0$

$c_2^+ + k c_3^+ - k c_4^+ - (c_2^- + k c_3^- - k c_4^-)$

$$\boxed{\Delta c_2 + k \Delta c_3 - k \Delta c_4 = 0}$$

2. derivace $\quad (y^+)''|_0 - (y^-)''|_0 = 0$

$k^2 c_3^+ + k^2 c_4^+ - (k^2 c_3^- + k^2 c_4^-) = 0$

$k^2 \Delta c_3 + k^2 \Delta c_4 = 0 \longrightarrow \boxed{\Delta c_3 = -\Delta c_4}$

3. derivace SKOK!

$$(y^+)^{(3)}\Big|_0 - (y^-)^{(3)}\Big|_0 = -1$$

$$k^3c_3^+ - k^3c_4^+ - (k^3c_3^- - k^3 c_4^-) = -1$$

$$k^3 \Delta c_3 - k^3 \Delta c_4 = -1$$

$$-\Delta c_4 = \Delta c_3 = -\frac{1}{2k^3}$$

$c_1^+ - c_1^- = 0$ $\qquad c_2^+ - c_2^-$

$\Delta c_1 = 0$ $\qquad \Delta c_2 = \frac{1}{k^2}$ $\qquad \Delta c_3 = -\frac{1}{2k^3}$ $\qquad \Delta c_4 = \frac{1}{2k^3}$

$$\Delta c_2 = k(\Delta c_4 - \Delta c_3) = k\frac{2}{2k^3} = \frac{1}{k^2}$$

$$y = \begin{cases} c_1^+ + c_2^+ x + c_3^+ e^{+kx} + c_4^+ e^{-kx} & x \geq 0 \\ c_1^+ + \left(c_2^+ + \frac{1}{k^2}\right)x + \left(c_3^+ - \frac{1}{2k^3}\right)e^{kx} + \\ \quad + \left(c_4^+ + \frac{1}{2k^3}\right)e^{-kx} & x < 0 \end{cases}$$

$y \in \mathcal{D}'(\mathbb{R})$

$\rightarrow \tilde{y} \in \mathcal{S}'(\mathbb{R})$

$c_3^+ = 0$ $\qquad c_4^+ + \frac{1}{2k^3} = 0$

$$\tilde{y} \in \mathcal{S}' \Rightarrow \tilde{y} = \begin{cases} c_1^+ + c_2^+ x - \frac{1}{2k^3} e^{-kx} & x > 0 \\ c_1^+ + \left(c_2^+ + \frac{1}{k^2}\right)x - \frac{1}{2k^3} e^{+kx} & x < 0 \end{cases}$$