BASES IN VECTOR SPACES

Three concepts : Hamel basis
Schauder basis
orthonormal (orthogonal) basis

Definitions (1) A set $\mathcal{A} \subset X$, where X is a vector space, is called Hamel basis if
- $\mathcal{A}$ is linearly independent (i.e. any finite subset of $\mathcal{A}$ is linearly independent)
- span $\mathcal{A} = X$

Note: Every nontrivial vector space X has a Hamel basis
$\Updownarrow$
$X \neq \{0\}$

(Pf): by Hausdorff maximal principle *)

- Consider $\mathcal{B}$ ... a collection of all linearly independent subsets of X.
- On $\mathcal{B}$, define $\mathcal{A}_1 < \mathcal{A}_2 \iff \mathcal{A}_1 \subset \mathcal{A}_2$ whenever $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{B}$
- Pick $\mathcal{A} \in \mathcal{B}$ (note that $\mathcal{B} \neq \emptyset$ since $X \neq \{0\}$)
and let $\mathcal{M}$ be a maximal element of $\mathcal{B}$, $\mathcal{A} \subset \mathcal{M}$
[If $\mathfrak{A}$ is a chain in $\mathcal{B}$, then $\sup \mathfrak{A} = \bigcup_{A \in \mathfrak{A}} A$)

Then span $\mathcal{M} = X$.

Any (algebraic = Hamel) basis in infinite-dimensional space has to be uncountable. [Such a representation is of little use.]!

This difficulty leads to an idea to represent each point $x \in X$ as infinite sum.

Def. (2) A sequence $\{e_n\}_{n=1}^{\infty} \subset X$ is called a Schauder basis of X if $\forall x \in X \ \exists$ (uniquely determined) $\xi_n \in \mathbb{K}$:

$$x = \sum_{n=1}^{\infty} \xi_n e_n . \qquad (\circledcirc)$$

Difficulties with this definition: • It is not difficult to see that if X has Schauder basis, then X is separable. However, there are separable Banach spaces which do not posses a Schauder basis. Moreover, there are several (non-equivalent) ways how to understand the convergence in $(\circledcirc)$. These difficulties do not occur in Hilbert spaces.

*) **ZORN's LEMMA (Hausdorff principle)** Let $(\mathcal{B}, <)$ be an ordered set in which every chain has the lowest upper bound. Then for every $\mathcal{A} \in \mathcal{B}$ there is a maximal $m \in \mathcal{A}$: $a < m$.

$\boxed{\text{Def. } ③}$ Let $(H,(\cdot,\cdot))$ be a pre-Hilbert space. A sequence $\{e_n\}_{n=1}^{\infty} \subset H$ is an orthonormal basis $\overset{\text{def.}}{\equiv}$ ⓘ $\{e_n\}_{n=1}^{\infty}$ is orthonormal (i.e. $\|e_n\|_H = 1$; $(e_k, e_\ell)_H = \delta_{k\ell}$)

ⓘⓘ $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis.

$\boxed{\text{Assertion}}$ Let $(H,(\cdot,\cdot)_H)$ be a pre-Hilbert and H is separable (it has a countable dense set)

Then there is an orthonormal basis in H.

Proof. Let $\{v_1, \dots, v_n, \dots\}$ be dense set in H.

Put $V_n = \operatorname{span}\{v_1, \dots, v_n\}$ → We may assume that they are linearly independent.

Then $\dim V_n = n$.

By Gram-Schmidt orthonormalization process there is $e_1, \dots, e_n$ so that
- $\operatorname{span}\{v_1, \dots, v_n\} = \operatorname{span}\{e_1, \dots, e_n\}$
- $\{e_k\}_{k=1}^{n}$ are orthonormal.

$\boxed{\text{Let } x \in H}$ and by density there are $v_n \in V_n$: $v_n \xrightarrow[n\to\infty]{} x$

Since $\|x - v_n\|_H^2 = \|x - \sum_{j=1}^{n} \alpha_j e_j\|^2$

$$= \left(x - \sum \alpha_j e_j, x - \sum \alpha_j e_j\right)_H$$

$$= \|x\|_H^2 - \sum_1^n \alpha_j (e_j, x)_H - \sum_{j=1}^{n} \overline{\alpha_j}(x, e_j) + \sum_{j=1}^{n} |(x,e_j)|^2$$

$$= \|x\|_H^2 + \sum_{j=1}^{n} |\alpha_j - (x,e_j)|^2 - \sum_{j=1}^{n} |(x,e_j)|^2$$

$$\geq \|x\|_H^2 - \sum_{j=1}^{n} |(x,e_j)|^2 = \left\|x - \sum_{j=1}^{n} (x,e_j) e_j\right\|_H^2$$

From underlined wawed ineq.:

$$\lim_{n\to\infty} \left\|x - \sum_{j=1}^{n} (x,e_j) e_j\right\|_H = 0$$

Hence

$$\boxed{x = \sum_{j=1}^{n} (x,e_j) e_j}$$

☺

Gram - Schmidt orthogonalization process/algorithm

Given $\{v_1, \dots, v_n\}$ linearly independent.

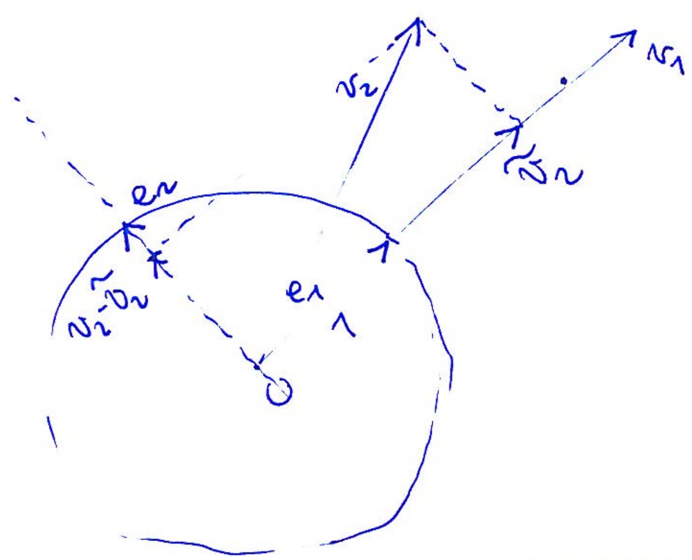
(i) $e_1 = \dfrac{v_1}{\|v_1\|_H}$

(ii) If $\{e_1, \dots, e_{n-1}\}$ constructed, then set $\tilde{v}_n$ as orthogonal projection of v_n onto $\operatorname{span}\{e_1, \dots, e_{n-1}\} = \operatorname{span}\{v_1, \dots, v_{n-1}\}$.

Then
$$e_n := \frac{v_n - \tilde{v}_n}{\|v_n - \tilde{v}_n\|_H} = \frac{v_n - \sum_{k=1}^{n-1} (v_n, e_k) e_k}{\left\| v_n - \sum_{k=1}^{n-1} (v_n, e_k) e_k \right\|}$$

Note: $v_n \neq \tilde{v}_n$ as $\tilde{v}_n \in \operatorname{span}\{v_1, \dots, v_{n-1}\}$ and $v_n \notin$ — " — .

Fig.



See also section 7.1 Abstract Fourier series in (Kapitola = Chapter 14)