$\boxed{X \text{ Banach over } \mathbb{C}}$, $\boxed{T \in \mathcal{L}(X)}$,

$\rho(T) = \{\lambda \in \mathbb{C} ; \operatorname{Ker}(\lambda I - T) = \{0\} \ \& \ \operatorname{Im}(\lambda I - T) = X\}$ RESOLVENT SET

SPECTRUM

$\sigma(T) = \mathbb{C} \setminus \rho(T)$

$\boxed{\sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)}$

- $\sigma_p(T) = \{\lambda \in \sigma(T) ; \operatorname{Ker}(\lambda I - T) \neq \{0\}\}$
- $\sigma_c(T) = \{\lambda \in \sigma(T) ; \operatorname{Ker}(\lambda I - T) = \{0\} \ \& \ \operatorname{Im}(\lambda I - T) \neq X$ but $\overline{\operatorname{Im}(\lambda I - X)} = X\}$
- $\sigma_r(T) = \{\lambda \in \sigma(T) ; \operatorname{Ker}(\lambda I - T) = \{0\}$, $\overline{\operatorname{Im}(\lambda I - X)} \neq X$

Observations (1) $\lambda \in \rho(T) \iff \lambda I - T$ is bijective

$\iff R(\lambda, T) := (\lambda I - T)^{-1}$ resolvent of T in λ

by open mapping theorem

resolvent function

(2) $\lambda \in \sigma_p(T) \iff \exists \overset{\in X}{x \neq 0} \quad Tx = \lambda x$

eigenvalue eigenvector

(If X is a function space, x is called eigenfunction)

The space $\boxed{\operatorname{Ker}(\lambda I - T)}$ is called $\boxed{\text{the eigenspace of } T \text{ to } \lambda}$

It is T-invariant, i.e.

$T(\operatorname{Ker}(\lambda I - T)) \subset \operatorname{Ker}(\lambda I - T)$

$T(Y) \subset Y$

$\boxed{\text{Theorem 1}}$ ◦ $\varrho(T)$ is open

• $\lambda \longmapsto R(\lambda,T)$ is complex analytic from $\varrho(T)$ ~~to~~ $\mathcal{L}(X)$

holomorphic

(DS) $\forall \lambda_0 \in \varrho(T) \; \exists \; B_{r_0}(\lambda_0) \subset \varrho(T)$:

$$R(\lambda,T) = \sum_{n=0}^{\infty} a_n(T)(\lambda-\lambda_0)^n$$

$\boxed{\text{Proof of Th 1}}$ $\lambda \in \varrho(T)$. Then for $\mu \in \mathbb{C}$

$$(\lambda-\mu)I-T = (\lambda I - T)\left(I - \mu(\lambda I-T)^{-1}\right)$$
$$= (\lambda I - T)(I - \mu R(\lambda,T))$$
$$= (\lambda I - T)\; S(\mu)$$

$S(\mu)$ is invertible, by Theorem 9.1, if $|\mu| \, \|R(\lambda,T)\| < 1$

then $\lambda-\mu \in \varrho(T)$ with

$$R(\lambda-\mu,T) = \{S(\mu)\}^{-1} R(\lambda,T) = \sum_{k=0}^{\infty} \mu^k \{R(\lambda,T)\}^{k+1}$$

von-Neumann series, Theorem 9.1.

Setting $d := \|R(\lambda,T)\|^{-1}$ yields that $B_d(\lambda) \subset \varrho(T)$ i.e. $\operatorname{dist}(\lambda,\sigma(T)) \geq d$.

Theorem

- $\sigma(T)$ is compact and non-empty (if $X \neq \{0\}$)
- $\sup_{\lambda \in \sigma(T)} |\lambda| = \lim_{m\to\infty} \|T^m\|^{\frac{1}{m}} \leq \|T\|$

This value is called spectral radius

Pf Let $\lambda \neq 0$. Then $I - \frac{1}{\lambda}T$ is invertible if $\|\frac{T}{\lambda}\|_{\mathcal{L}(X)} < 1$

i.e. if $\|T\|_{\mathcal{L}(X)} < |\lambda|$. Then

(R) $$R(\lambda;T) = \frac{1}{\lambda}\left(Id - \frac{T}{\lambda}\right)^{-1} = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}$$

This shows that $r := \sup_{\lambda \in \sigma(T)} |\lambda| \leq \|T\|_{\mathcal{L}(X)}$.

Since $$\lambda^m I - T^m = (\lambda I - T)\underbrace{\sum_{i=0}^{m-1} \lambda^{m-1-i} T^i}_{p_m(T)} = \underbrace{\sum_{i=0}^{m-1} \lambda^{m-1-i} T^i}_{p_m(T)} (\lambda I - T)$$

we conclude that

$$\lambda \in \sigma(T) \Rightarrow \lambda^m \in \sigma(T^m)$$
$$\Rightarrow |\lambda|^m = |\lambda^m| \leq \|T^m\|_{\mathcal{L}(X)}$$
$$\Rightarrow |\lambda| \leq \|T^m\|_{\mathcal{L}(X)}^{\frac{1}{m}}$$

Hence

$$r = \sup_{\lambda \in \sigma(T)} |\lambda| \leq \|T^m\|_{\mathcal{L}(X)}^{\frac{1}{m}}$$

To prove the opposite inequality, we recall that

$$R(\cdot, T) \in H\left(\mathbb{C} \setminus \overline{B_r(0)}\right) \quad [\text{if } \sigma(T) = \emptyset, \text{ then } H(\mathbb{C})]$$

holomorphic

Hence, by Cauchy integral formula

$$J = \frac{1}{2\pi i} \int_{\partial B_s(0)} \lambda^j R(\lambda, T)\, d\lambda \quad \text{is independent of } s \text{ for } j \geq 0 \text{ and } s > r$$

However, if we choose $s > \|T\|_{\mathcal{L}(X)}$, then

$$J \overset{(R)}{=} \frac{1}{2\pi i} \int_{\partial B_s(0)} \sum_{k=0}^{\infty} \lambda^{j-k-1} T^k \, d\lambda \underset{\substack{\lambda = se^{i\theta} \\ d\lambda = sie^{i\theta}}}{=} \frac{1}{2\pi} \sum_{k=0}^{\infty} s^{j-k} \left(\int_0^{2\pi} e^{i\theta(j-k)} d\theta\right) T^k$$

$$= \frac{1}{2\pi} \sum_{\ell=0}^{\infty} s^{j+1} \delta_{j\ell}\, 2\pi\, T^{\ell} = \boxed{T^j} = \int_0^{2\pi} s^{j+1} e^{ij\theta}\, i R(se^{i\theta}, T)\, d\theta$$

Hence, for $j \geq 0$ and $s > r$

$$\|T^j\|_{\mathcal{L}(X)} = \|J\| = \frac{1}{2\pi} \Big\| \int_{\partial B_s(0)} \lambda^j R(\lambda, T) \Big\|_{\mathcal{L}(X)}$$

$$\leq s^{j+1} \sup_{|\lambda|=s} \|R(\lambda, T)\|_{\mathcal{L}(X)},$$

implying

$$\|T^j\|^{\frac{1}{j}} \leq s \left(s \sup_{|\lambda|=s} \|R(\lambda,T)\|_{\mathcal{L}(X)} \right)^{1/j} \longrightarrow s \quad (\text{or } 0)$$

Consequently

$$\boxed{\limsup_{j \to +\infty} \|T^j\|^{\frac{1}{j}} \leq s \qquad \text{for all } s > r}$$

Thus

$$\sup_{|\lambda| \in \sigma(T)} |\lambda| = \lim_{j \to \infty} \|T^j\|^{1/j} \leq \|T\|_{\mathcal{L}(X)}$$

Also, if $\sigma(T) = \phi$, we obtain for $j = 0$ and $s \to 0+$

$$\| \underset{\substack{\uparrow \\ \text{identity}}}{I} \|_{\mathcal{L}(X)} \leq s \sup_{|\lambda|=s} \|R(\lambda, T)\|_{\mathcal{L}(X)} \xrightarrow{s \to 0+} 0$$

using a nice formula

$$T^j = \frac{1}{2\pi i} \int_{\partial B_s(0)} \lambda^j R(\lambda, T)\, d\lambda$$

with $j = 0$

Hence $I = 0$ and $X = \{0\}$

Observations.

▷ If $\dim X < +\infty$, then $\sigma(T) = \sigma_p(T)$ as $\dim X < \infty$

(Pf) $\lambda \in \sigma(T)$, then $\lambda I - T$ is not bijective $\Longrightarrow$ $\lambda I - T$ is not injective, i.e. $\lambda \in \sigma_p(T)$ $\boxtimes$

▷ If $\dim X = +\infty$ and T is compact, then $0 \in \sigma(T)$.

(Pf) If T compact and $0 \in \rho(T)$, then $T^{-1} \in \mathcal{L}(X)$ and $I = T^{-1} \circ T = T \circ T^{-1}$ is compact which is not possible in infinite-dimensional spaces

$\boxed{\text{NORMAL OPERATORS}}$ Let X be a Hilbert space over $\mathbb{K}$. We say that the operator is <u>normal</u> if

$$\boxed{T^*T - TT^* = 0} \quad \text{i.e. } \boxed{T \text{ and its adjoint } T^* \text{ commute}}$$

NOTE ▷ Every self-adjoint operator is normal

▷ If T is normal, then $\lambda I - T$ is normal for any $\lambda \in \mathbb{K}$.

▷ T is normal $\iff \|Tx\|_X = \|T^*x\|_X \quad \forall x \in X$.

Pf $\boxed{\Rightarrow}$ $\|Tx\|_X^2 = (Tx, Tx) = (x, T^*Tx) = (x, TT^*x)$
$= (T^*x, T^*x) = \|T^*x\|_X^2$

$\boxed{\Leftarrow}$ The identity

$$\frac{1}{4}\left(\|a+b\|_X^2 - \|a-b\|_X^2\right) = \mathrm{Re}\,(a,b)_X \quad \forall a,b \in X$$

implies $\mathrm{Re}\,(Tx, Ty)_X = \mathrm{Re}\,(T^*x, T^*y)_X$

Then (replacing y by iy if $\mathbb{K} = \mathbb{C}$)

$$0 = (Tx, Ty)_X - (T^*x, T^*y)_X = (T^*Tx - TT^*x, y) \quad \forall x, y \in X$$

$$\Rightarrow T^*T = TT^*$$

It holds

$\boxed{\text{Theorem 5.1}}$ If X Hilbert over $\mathbb{C}$, $X \neq \{0\}$ and $T \in \mathcal{L}(X)$ normal, then $\sup_{\lambda \in \sigma(T)} |\lambda| = \|T\|_{\mathcal{L}(X)}$

$\boxed{\text{Example}}$ Let $\{e_k\}_{k \in N}$, $N \subset \mathbb{N}$, be an orthonormal system in X and $\lambda_k \in \mathbb{K}$, with $|\lambda_k| \le r < \infty$.

Then $Tx = \sum_{k \in N} \lambda_k (x, e_k)_X e_k$ defines $T \in \mathcal{L}(X)$.

Indeed
$$\|Tx\|_X^2 = \sum_{k \in \mathbb{N}} |\lambda_k|^2 |(x, e_k)|^2 \le r^2 \sum_{k \in N} |(x, e_k)|^2 \le r^2 \|x\|_X^2$$

Since
$$(Tx, y)_X = \sum_{k \in \mathbb{N}} \lambda_k (x, e_k)(e_k, y) = \left(x, \sum_{k \in N} \overline{\lambda_k (e_k, y)_X e_k}\right)_X$$

it follows that
$$T^*x = \sum_{k \in N} \overline{\lambda_k}\, (x, e_k)_X e_k$$

and then
$$T^*Tx = \sum |\lambda_k|^2 (x, e_k)_X e_k \quad . \text{ Hence } T \text{ is } \underline{\text{normal}}$$
$$\parallel$$
$$T^*Tx$$

If N finite, T is also compact.

We will show that if $N = \mathbb{N}$, then

$$\boxed{T \text{ is compact} \iff \lim_{k\to+\infty} \lambda_k = 0}$$

Pf $\boxed{\Leftarrow}$ setting $T_n(x) = \sum_{k\leq n} \lambda_k (x, e_k) e_k$, then $T_n \in \mathcal{L}(X)$ are compact

and $\|Tx - T_n x\|_X^2 \leq \sup_{k>n} |\lambda_k|^2 \|x\|_X^2 \longrightarrow 0$ as $|\lambda_k| \to 0$.

$\boxed{\Rightarrow}$ If $|\lambda_{k_j}| \geq \varepsilon > 0$, then for e_{k_j}

$$\|Te_{k_j} - Te_{k_i}\|_X^2 = \|\lambda_{k_j} e_{k_j} - \lambda_{k_i} e_{k_i}\|_X^2 =$$

$$= (\lambda_{k_j} e_{k_j} - \lambda_{k_i} e_{k_i}, \lambda_{k_j} e_{k_j} - \lambda_{k_i} e_{k_i})$$

$$= |\lambda_{k_j}|^2 + |\lambda_{k_i}|^2 \geq 2\varepsilon > 0 .$$ and we get contradiction to the compactness of T. ▨

$\boxed{\text{Theorem S.2}}$ If $(X, (\cdot,\cdot)_X)$ Hilbert over $\mathbb{C}$ and if $T \in \mathcal{L}(X)$ is compact and normal, then

(1) $\exists \{e_k\}_{k\in N}$ orthonormal in X and $\{\lambda_k\}_{k\in N} \subset \mathbb{C}$ with $N \subset \mathbb{N}$ such that $\lambda_k \neq 0$ and

$$Te_k = \lambda_k e_k \text{ for } k \in \mathbb{N}, \quad \sigma(T)\setminus\{0\} = \{\lambda_k;\, k\in N\}$$

If N infinite, then $\lambda_k \to 0$ as $k \to \infty$.

(2) multiplicity of $m_{\lambda_k} = 1$ for all k (λ_k do not need to be distinct)

(3) $X = \ker T \oplus \overline{\operatorname{span}\{e_k;\, k\in N\}}$

(4) $Tx = \sum_{k\in N} \lambda_k (x, e_k)_X e_k$

NOTES ▷ X Hilbert over $\mathbb{C}$, $T \in \mathcal{L}(X)$

- if T selfadjoint, $T^* = T$, then $\sigma_p(T) \subset [-\|T\|, \|T\|] \subset \mathbb{R}$ and if T is compact, then $\|T\|$ or $-\|T\|$ is eigenvalue
- if T is selfadjoint and positive semidefinite $(Tx, x)_X \geq 0 \;\forall x \in X$ then $\sigma_p(T) \subset [0, \|T\|]$, and if T compact then $\|T\|$ is eigenvalue.