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1. Modifikací postupu ze cvičení nalezněte Fourierovou metodou řešení rovnice

$$\Delta u = 0 \qquad \text{v } \Omega \subset \mathbb{R}^2 ,$$

kde (v polárních souřadnicích) $\Omega = \{(r,\varphi), 0 < a < r < \infty, 0 < \varphi < \alpha < 2\pi\}$. Okrajové podmínky jsou

$$\frac{\partial u}{\partial \varphi}(r,0) = \frac{\partial u}{\partial \varphi}(r,\alpha) = 0 , \qquad a < r < \infty ,$$
$$u(a,\varphi) = \cos\left(\frac{2\pi\varphi}{\alpha}\right) , \qquad 0 < \varphi < \alpha .$$

Hledejte pouze "fyzikální" řešení, t.j. řešení omezená pro $r \to \infty$.

2. Pomocí Fourierovy transformace nalezněte fundamementální řešení pro operátor $\Delta^2 + k^4$, t.j. řešení rovnice

$$\Delta\Delta u + k^4 u = \delta , \qquad \text{v } \mathbb{R}^3 .$$

Jako na cvičení se Vám zřejmě může hodit vzoreček pro inverzní Fourierovou transformaci radiální funkce ve třech dimenzích:

$$\mathcal{F}(g(r))(\rho) = \frac{2}{\rho} \lim_{R\to\infty} \int_0^R g(r) r \sin(2\pi r\rho)\, dr . \quad (\bullet)$$

1 Riešenie hľadáme v separovanom tvare

$$u(r,\varphi) \equiv R(r)\Phi(\varphi)$$ [pre netriviálne riešenie $R \neq 0$, $\Phi \neq 0$]

s okrajovými podmienkami

(1) $\dot{\Phi}(0) = \dot{\Phi}(\alpha) = 0$ $\left[\frac{\partial u}{\partial \varphi} = R(r)\dot{\Phi}(\varphi)\right]$

(2) $R(a)\Phi(\varphi) = \cos\left(\frac{2\pi\varphi}{\alpha}\right)$

(3) $R \xrightarrow{r\to\infty} 0$

$$\Delta(\bullet) = \frac{1}{r}\partial_r\left(r\partial_r(\bullet)\right) + \frac{1}{r^2}\partial_\varphi^2(\bullet)$$
$$\Downarrow$$
$$0 \overset{!}{=} r^2\frac{\Delta u}{u} = \frac{r(rR')'}{R}(r) + \frac{\ddot{\Phi}}{\Phi}(\varphi)$$

Takže

$$\frac{r(rR')'}{R}(r) = -\frac{\ddot{\Phi}}{\Phi}(\varphi) \equiv +k^2 \text{ [konšt.]}$$

[aby bolo možné splniť (1), BÚNO $k>0$]

$$r^2R'' + rR' - k^2R = 0$$

Ansatz $R = Cr^\ell$

$$C\left[r^2\ell(\ell-1)r^{\ell-2} + r\ell r^{\ell-1} - k^2r^\ell\right] = 0$$
$$\Rightarrow \ell^2 - \ell + \ell - k^2 = (\ell+k)(\ell-k) = 0$$
$$\overset{(3)}{\Rightarrow} \ell = -k$$

$$\ddot{\Phi} + k^2\Phi = 0$$
$$\Downarrow$$
$$\Phi(\varphi) = A\cos(k\varphi) + B\sin(k\varphi)$$
$$\dot{\Phi}(0) = Bk \overset{(1)}{=} 0 \overset{k\neq 0}{\Longrightarrow} B = 0$$

Takže

$$\dot{\Phi}(\alpha) = -Ak\sin(k\alpha) \overset{(1)}{=} 0$$
$$\Rightarrow k_m \equiv \frac{m\pi}{\alpha} \quad m \in \mathbb{N}$$

Celkovo

$$R_m = C_m r^{-k_m} = C_m r^{-\frac{m\pi}{\alpha}}$$

Celkovo

$$\Phi_m = A_m\cos\left(\frac{m\pi\varphi}{\alpha}\right)$$

$$u(r,\varphi) = \sum_{m\in\mathbb{N}} \tilde{C}_m r^{-\frac{m\pi}{\alpha}} \cos\left(\frac{m\pi\varphi}{\alpha}\right)$$

$$(2) \Rightarrow \sum_{m\in\mathbb{N}} \tilde{C}_m a^{-\frac{m\pi}{\alpha}} \cos\left(\frac{m\pi\varphi}{\alpha}\right) \equiv \cos\left(\frac{2\pi\varphi}{\alpha}\right)$$

$$\Rightarrow \tilde{C}_m \equiv a^{\frac{m\pi}{\alpha}}\delta_{m,2}$$ [vidno automaticky, že naša voľba funguje]

Jediné „fyzikálne" riešenie je teda

$$u(r,\varphi) = \left(\frac{a}{r}\right)^{\frac{2\pi}{\alpha}} \cos\left(\frac{2\pi\varphi}{\alpha}\right)$$

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$$\Delta\Delta u + k^4 u = \delta$$

$$\mathcal{F}\big\downarrow$$

$$(2\pi i|\xi|)^4\hat u + k^4\hat u = 1$$

$$\Big\Downarrow\ |\xi|\equiv\varrho$$

$$\boxed{\hat u = \frac{1}{(2\pi\varrho)^4+k^4}}$$

$$u = \mathcal{F}^{-1}[\mathcal{F}[u]] \overset{(*)}{=} \frac{2}{r}\int_0^\infty \underbrace{\hat u(\varrho)}_{\text{párna}}\underbrace{\varrho \sin(2\pi r\varrho)}_{\text{párna}}\,d\varrho =$$

[$\mathcal{F}[u]\equiv\hat u$ radiálne symetrická
$\mathcal{F}^{-1}[\hat u](x) =$
$\Rightarrow = \mathcal{F}^{-1}[\hat u](-x) =$
$= \mathcal{F}[\hat u](x)$]

$$= \frac{1}{r}\int_{-\infty}^{\infty} \hat u(\varrho)\varrho \sin(2\pi r\varrho)\,d\varrho =$$

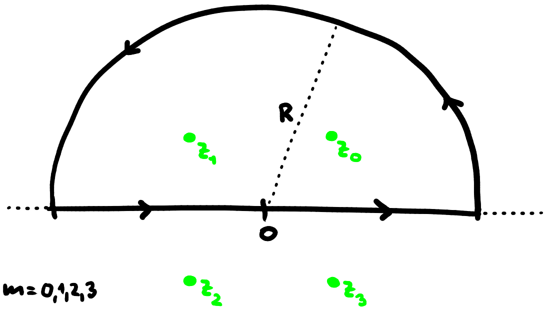
$$= \frac{1}{r}\,\mathrm{Im}\underbrace{\int_{-\infty}^{\infty} \hat u(\varrho)\varrho\, e^{i2\pi r\varrho}\,d\varrho}_{J(r)} \qquad (\bullet)$$

Holomorfné predĺženie

$$f(z) \equiv \frac{z\,e^{i2\pi r z}}{(2\pi z)^4 + k^4}$$

póly $(2\pi z)^4 = -k^4$

$$\Rightarrow z_m = \frac{k}{2\pi}e^{i\pi\left(\frac14+\frac m2\right)} \qquad m=0,1,2,3$$



Integračná krivka

$\Gamma_R = \gamma_1 \oplus \gamma_2$

$\gamma_1:\ z = t \qquad t\in[-R,R]$

$\gamma_2:\ z = Re^{it} \qquad t\in[0,\pi]$

Reziduá

$$\mathrm{Res}_{z_m} f(z) = \frac{z\,e^{i2\pi r z}}{[(2\pi z)^4+k^4]'}\Bigg|_{z=z_m} = \frac{e^{i2\pi r z_m}}{4(2\pi)^2(2\pi z_m)^2} = \frac{e^{ikr\left[\cos\left(\pi\left(\frac14+\frac m2\right)\right)+i\sin\left(\pi\left(\frac14+\frac m2\right)\right)\right]}}{4(2\pi)^2k^2(-1)^m i}$$

[jednoduché póly
čitateľ holomorfný]

$$\mathrm{Res}_{z_\pm} f(z) = \frac{e^{-\frac{kr}{\sqrt2}}e^{\pm\frac{ikr}{\sqrt2}}}{4(2\pi)^2k^2(-1)^m i}$$

$+ \leftrightarrow m=0$
$- \leftrightarrow m=1$

Reziduová veta

$$\int_{\Gamma_R} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz = 2\pi i\sum_{z_m\in \mathrm{Int}\,\Gamma_R}\mathrm{Res}_{z_m} f(z) = \frac{e^{-\frac{kr}{\sqrt2}}}{8\pi k^2}\left(e^{\frac{ikr}{\sqrt2}} - e^{-\frac{ikr}{\sqrt2}}\right) =$$

$z = z_\pm = z_{0/1}$

$$\int_{\gamma_1} f(z)\,dz \xrightarrow{R\to\infty} J(r) \qquad \int_{\gamma_2} f(z)\,dz \xrightarrow{R\to\infty} 0$$

[Jordanovo lemma
s $\alpha>0$
$M_R \sim \frac{1}{R^3}$]

$$= i\,\frac{e^{-\frac{kr}{\sqrt2}}}{4\pi k^2}\sin\left(\frac{kr}{\sqrt2}\right)$$

$$\overset{(\bullet)}{\Longrightarrow}\quad \boxed{u(x) = \frac{e^{-\frac{k|x|}{\sqrt2}}}{4\pi k^2|x|}\sin\left(\frac{k|x|}{\sqrt2}\right)}$$