SUMMARY OF PREVIOUS PART.

$\boxed{\dim X < +\infty}$

Ⓐ • If L linear $: X \to X$, then
- L is continuous
- L is bdd
- Fredholm alternative holds:

L is onto $\iff$ L is one-to-one

$\Updownarrow$

Either: $\forall f : \exists u \; Lu = f$ OR there is a non-trivial solution: $Lu = 0$

there is mutual correspondance between L and the matrices $A \in \mathbb{K}^{\boxed{\dim X} \times \boxed{\dim X}}$

Ⓑ • All norms are equivalent

X is complete, i.e. Banach with whatever norm

Ⓒ • $\{x_n\}$ bdd in X contains a subsequence converging in X

Ⓐ and Ⓑ does not hold (in general) in X: $\dim X = +\infty$

Ⓒ characterises finite-dimensional spaces.

SEMINORMS & FRÉCHET SPACES

There are function spaces where there is no natural way to introduce a norm. For example $C((a,b))$, $\mathcal{D}(\Omega)$, $L^p_{loc}(\Omega)$, $\mathcal{S}(\mathbb{R}^d)$. We will be however able to introduce on these spaces the sequence of separating seminorms and use them to introduce the metric*) so that the above spaces will be complete (metric spaces), i.e. Fréchet.

$\boxed{\text{Example}}$ (serving as motivation) Consider $X = C((0,1))$.

Since X contains unbounded functions, setting

$$p(f) \overset{df.}{=} \sup_{x\in(0,1)} |f(x)|,$$

we see that $p(f)$ can be $+\infty$ and consequently, $p(f)$ does not generate the norm.

However, for any $\langle a,b\rangle \subset (0,1)$

$$p^{a,b}(f) \overset{df.}{=} \sup_{a\le x\le b} |f(x)|$$

is always finite. We easily observe that $p^{a,b}$ is 1-homogeneous $p^{a,b}(0) = 0$ and $p^{a,b}$ fulfills the triangle inequality, i.e. (N2) and (N3) holds. But there are nontrivial f so that $p^{a,b}(f) = 0$. Draw one. Hence $p^{a,b}$ is not norm, but it is a seminorm. ∎

$\boxed{\text{Def.}}$ Let X be a vector space over $\mathbb{K}$. The mapping $p: X \to \mathbb{R}$ is called a seminorm if

(SN1) $p(x) \geq 0$ for $\forall x \in X$

(SN2) $p(\lambda x) = |\lambda|\, p(x)$ —"— $\forall \lambda \in \mathbb{K}$

(SN3) $p(x+y) \leq p(x) + p(y)$

Since $p(x)$ can be zero for $x \neq 0$, setting $[d(x,y) = p(x-y)]$ we do not obtain a distance on X. We can have $x, y, x\neq y$ and yet $d(x,y) = 0$.

There are cases when we can introduce the distance by means of infinitely (countable) many seminorms.

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*) distance

Def. A sequence $\{p_k\}_{k\in\mathbb{N}}$ of seminorms on X is <u>separating</u> if, for every $x\in X$ with $x\neq 0$, there is $k_0\in\mathbb{N}$ such that $p_{k_0}(x)>0$.

$\boxed{\text{Assertion}}$ (Distance generated by seminorms)
Let $\{p_k\}_{k\in\mathbb{N}}$ be a sequence of separating seminorms.
Then
(d**)
$$\boxed{d(x,y) \overset{\text{df.}}{=} \sum_{k=1}^{\infty} \frac{1}{2^k}\,\frac{p_k(x-y)}{1+p_k(x-y)}}$$
is a distance on X

<u>Proof.</u>
- If $x\neq y$, then $d(x,y)>0$ as there is k_0: $p_k(x-y)>0$
- Also, $d(x,y)=d(y,x)$ due to 1-homogeneity of $p_k(\cdot)$.
- The triangle inequality follows from the fact that $s\mapsto \frac{s}{1+s}$ is increasing and concave, which implies, for $0\le a,b,c$ with $c\le a+b$, that
$$\underbrace{\frac{c}{1+c} \le \frac{a+b}{1+a+b} \le \frac{a}{1+a}+\frac{b}{1+b}}$$
setting $c=p_k(x-z)$, $a=p_k(x-y)$ and $b=p_k(y-z)$, we get the triangle inequality.

$\boxed{\text{Def.}}$ X is <u>Fréchet</u> if X is a <u>complete</u> <u>metric</u> <u>space</u>. with the distance (d**).

<u>Examples</u> ① The (space) $\big(C(\Omega), d(f,g)\big)$, $\Omega\subset\mathbb{R}^d$ <u>open</u>, is <u>Fréchet</u>
(not necessarily bounded)
provided that we set
$$p_k(f) \equiv \max_{x\in A_k}|f(x)| \quad \text{where } A_k := \{x\in\Omega;\ |x|\le k,\ \operatorname{dist}(x,\partial\Omega) < \tfrac{1}{k}\}.$$
and $d(f,g)$ is given by (d**).

<u>Indeed</u> If $\{f_j\}_{j\in\mathbb{N}}$ is Cauchy sequence w.r.t. (d**), then
$$\limsup_{n,m\to\infty} p_k(f_n-f_m) = \limsup_{n,m\to\infty}\ \sup_{x\in A_k} |f_n(x)-f_m(x)| = 0$$
This implies that, for each $x\in\Omega$ ($\Rightarrow x\in A_k$ for certain k), $f_n(x)\to f(x)$ and in fact $f_n \rightrightarrows f$ in A_k. Hence
$$\limsup_{n\to\infty} d(f_n,f) = \lim_{n\to\infty}\sum_{k=1}^{m} 2^{-k}\frac{p_k(f_n-f)}{1+p_k(f_n-f)} + \limsup_{n\to\infty}\sum_{m+1}^{\infty}\cdots$$
$$= 0 + \frac{1}{2^m} \to 0 \text{ as } m\to\infty. \qquad \text{Q.E.D.}$$

(2) Show that $L^p_{loc}(\Omega) := \{ u : \Omega \to \mathbb{R}$ measurable; $\int_{\Omega'} |f(x)|^p dx < +\infty$ for each open $\underbrace{\Omega' \subset\subset \Omega}_{\text{i.e. } \Omega' \subset \overline{\Omega'} \subset \Omega.}$ $\}$

with $p_k(f) \overset{\text{df.}}{=} \left(\int_{A_k} |f|^p dx \right)^{1/p}$

and $d(f,g)$ given by $(d**)$ is a Fréchet space.

$\boxed{\text{HAHN-BANACH THEOREM}}$ or extension theorems

One of the goals of this section is to show that there are many continuous (bdd) lin. functionals on X.

Towards this goal, we show first Hahn-Banach extension theorem: for a given $f \in V'$, where $V \subset\subset X$, there is $F \in X'$ so that $F = f$ on V and satisfies some other preserving properties.

Consider $p : X \to \mathbb{R}$ satisfying

$$\boxed{(pp) \qquad p(x+y) \leq p(x) + p(y), \quad p(\lambda x) = \lambda p(x) \quad \forall x,y \in X \ \forall \lambda \geq 0.}$$

Example
- $p(x) = \kappa \|x\|_X$ satisfies (pp) for each $\kappa > 0$.
- Every seminorm satisfies (pp).

NOTES
- p fulfilling (pp) is convex.
- p can be negative. (while any seminorm is non-negative).

Example
- Let $(X, \|\cdot\|_X)$ be a normed space and $\Omega \subset X$ be a bdd, open, convex, containing the origin.

Then $\boxed{p(x) \overset{\text{df.}}{=} \inf_{\lambda \geq 0} \{ x \in \lambda \Omega \}}$ satisfies (pp).

Theorem 1.5 (Hahn–Banach)

Let X be a vector space over $\mathbb{R}$ and $p: X \to \mathbb{R}$ satisfies (pp).
Let $V \subset\subset X$ and $f \in V \to \mathbb{R}$ be <u>linear</u>.
so that

(A1) $$f(x) \leq p(x) \quad \forall x \in V.$$

Then $\exists F \in X'$ such that

(T1) $$F(x) = f(x) \quad \forall x \in V$$

and

(T2) $$-p(-x) \leq F(x) \leq p(x) \quad \forall x \in X.$$

Pf
- If $V = X$, then we are done by observing that for $x \in X$ $f(x) = -f(-x) \underset{(A1)}{\geq} -p(-x)$, which gives (T2).
- If $V \neq X$, then we take any $x_0 \notin V$ and consider the strictly larger subspace $V_0 \overset{df.}{=} \{x + t x_0 ;\ x \in V, t \in \mathbb{R}\}$

From (A1), for any $x, y \in V$,

$$f(x) + f(y) \underset{\text{linearity}}{=} f(\underbrace{x+y}_{\in V}) \underset{(A1)}{\leq} p(x+y) \leq p(x - x_0) + p(x_0 + y),$$

which implies

$$f(x) - p(x - x_0) \leq p(y + x_0) - f(y) \quad \forall x, y \in V.$$

Set $\beta \overset{df}{=} \sup_{x \in V} \{f(x) - p(x - x_0)\}$, we get

$$(*) \quad \boxed{f(x) - p(x - x_0) \leq \beta \leq p(y + x_0) - f(y) \quad \forall x, y \in V}$$

- $\boxed{\text{Extension of } f \text{ on } V_0}$ Set $f(x + t x_0) \overset{df}{=} f(x) + \beta t$

We show that f satisfies (A1) on V_0, i.e., we want to show that

(•) $$f(x + t x_0) \leq p(x + t x_0) \quad \forall x \in V.$$

Clearly, (•) follows from (A1) for <u>$t = 0$</u>. For $t > 0$, we use $(*)$ with $x = y = \frac{x}{t}$:

$$f\left(\frac{x}{t}\right) - p\left(\frac{x}{t} - x_0\right) \leq \beta \leq p\left(\frac{x}{t} + x_0\right) - f\left(\frac{x}{t}\right)$$

$$\Downarrow$$

$$\boxed{f(x) - p(x - t x_0) \leq t\beta \leq p(x + t x_0) - f(x)}$$

Hence

$$\left.\begin{aligned} f(x + t x_0) &= f(x) + \beta t \leq p(x + t x_0) \\ f(x - t x_0) &= f(x) - \beta t \leq p(x + t x_0) \end{aligned}\right\} \Rightarrow (\bullet).$$

- By the previous step, every $f \in V'$ can be extended to a larger subspace still satisfying (A1).

[Let $\mathcal{F}$ be a family of (V, ϕ), where $V \subset\subset X$ and $\phi: V \to \mathbb{R}$, $\phi \in V'$, satisfies $\phi(x) \le p(x)\ \forall x \in V$.

We can partially order $\mathcal{F}$:

$$\underbrace{(V_1, \phi_1)}_{\in \mathcal{F}} \prec \underbrace{(V_2, \phi_2)}_{\in \mathcal{F}} \overset{\text{def.}}{=} V_1 \subset\subset V_2 \text{ and } \phi_2 = \phi_1 \text{ on } V_1$$
$$(\phi_2|_{V_1} = \phi_1).$$

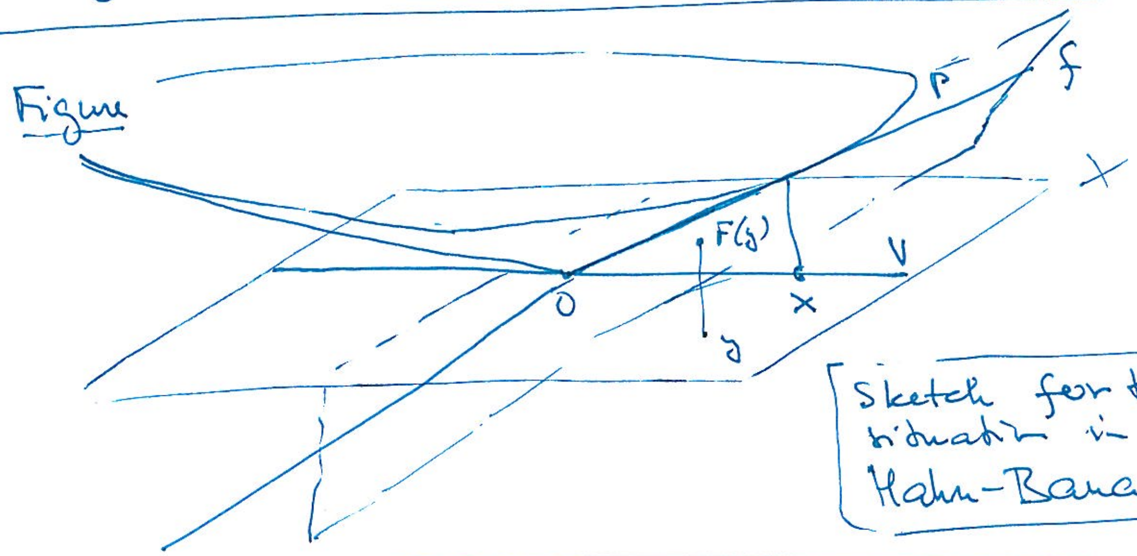
By Hausdorff Maximality principle (equivalent to Axiom of Choice), $(\mathcal{F}, \preceq)$ contains a maximal element: (V_{max}, F).

If $V_{max} \neq X$, then we can extended as above.

Hence $V_{max} = X$ and $F(x) \le p(x)\ \forall x \in X$.

By linearity: $F(x) = -F(-x) \ge -p(-x)$.

Figure



[Sketch for the situation in the Hahn-Banach theorem]

<u>Def</u> A set S is <u>partially ordered</u> by a binary relation $\prec$ if, for every $a, b, c \in S$:

(i) $a \prec a$

(ii) $a \prec b$ & $b \prec a \Rightarrow a = b$

(iii) $a \prec b$ & $b \prec c \Rightarrow a \prec c$

A subset $S' \subset S$ of a partially ordered set S is said to be <u>totally ordered</u> if, for every $a, b \in S'$, either $a \prec b$ or $b \prec a$.

We say that S' is <u>maximal</u> (w.r.t. the total ordering) if S' is <u>not</u> strictly contained in any other totally ordered set.

<u>Theorem</u> (Hausdorff maximality principle) If S is partially ordered, then every totally ordered subset $S' \subset S$ is contained in a maximally ordered subset.