8. Fredholm theory / Fredholm alternative

From LA (please check your notes)

$\forall b \in \mathbb{R}^N \ \exists!\ x \in \mathbb{R}^N : Ax = b$ $\Longleftrightarrow$ the problem $Ax=0$ has only trivial solution

$\Updownarrow$ $\Updownarrow$

the corresponding lin. operator is onto $\Longleftrightarrow$ the corresponding lin. operator is injective

This equivalence fails in infinite dimensional spaces in general. But it holds for special class of operators.

Theorem 8.1 (Fredholm) Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space over $\mathbb{R}$. Let $K : H \to H$ be compact linear ($\Rightarrow$ compact, linear, bdd).
$\Downarrow$ K^* is compact (linear, bdd)

Then

(1) $Ker(I-K)$ is finite-dimensional $\Rightarrow$ $\dim(I-K^*)$ is finite dimensional

(2) $Im(I-K)$ is closed $\Rightarrow$ $Im(I-K^*)$ is closed

(3) $Im(I-K) = [Ker(I-K^*)]^{\perp}$ $\Rightarrow$ $Im(I-K^*) = [Ker(I-K)]^{\perp}$

(4) $Ker(I-K) = \{0\} \Longleftrightarrow Im(I-K) = H$

(5) $Ker(I-K)$ and $Ker(I-K^*)$ have the same dimension.

NOTE It follows from (4) that

EITHER for every $f \in H$ $\exists u \in H$ solving $u - Ku = f$
[the operator $I-K$ is injective and onto]

OR the problem to find $u \in H$: $u - Ku = 0$ has a nontrivial solution. In this case, the problem $u - Ku = f$ has solution if and only if $f \in [Ker(I-K^*)]^{\perp}$, which means
$(f, u)_H = 0$ for all u satisfying/solving $u - K^*u = 0$.

This dichotomy EITHER/OR is called <u>Fredholm alternative</u>.

Proof

$\boxed{\text{Ad (1)}}$ If $\mathrm{Ker}(I-K)$ is infinite-dimensional, one can find an orthonormal set $\{e_m\}_{m=1}^{\infty}$ in $\mathrm{Ker}(I-K)$. Then

$$e_m = Ke_m \quad \text{and} \quad \|e_m - e_n\|_H^2 = \|e_m\|_H^2 + \|e_n\|_H^2 = 2.$$

Hence $\|Ke_n - Ke_m\|_H = \|e_n - e_m\|_H = \sqrt{2}$ and we are getting an contradiction to the compactness of K (no way to extract a subsequence $\{e_{n_j}\}$ so that $\{Ke_{n_j}\}_{j=1}^{\infty}$ converges).

$\boxed{\text{Ad (2)}}$ Step 1 We first show that ...

$$(8.1) \qquad \exists \beta > 0 \quad \|u - Ku\|_H \geq \beta \|u\|_H \quad \text{for all } u \in [\mathrm{Ker}(I-K)]^{\perp}$$

$\boxed{\text{Pf of (8.1)}}$ By contradiction, assume that (for all $m \in \mathbb{N}$)

there is $\tilde{u}_m \in [\mathrm{Ker}(I-K)]^{\perp}$: $\|\tilde{u}_m - K\tilde{u}_m\|_H < \frac{1}{m}\|\tilde{u}_m\|_H$ ...

Setting $u_m = \dfrac{\tilde{u}_m}{\|\tilde{u}_m\|_H}$ we get:

there is $u_m \in [\mathrm{Ker}(I-K)]^{\perp}$: $\|u_m - Ku_m\|_H < \frac{1}{m}$ & $\|u_m\|_H = 1$

Since $\{u_m\}_{m=1}^{\infty}$ is bdd, there is $\{u_{m_\ell}\}_{\ell=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$ and $u \in H$

$$u_{m_\ell} \rightarrow u \quad \text{weakly in } H$$

As K is compact,

$$Ku_{m_\ell} \rightarrow Ku \quad \text{strongly in } H.$$

Thus

$$\|u_{m_\ell} - Ku\|_H \leq \|u_{m_\ell} - Ku_{m_\ell}\|_H + \|Ku_{m_\ell} - Ku\|_H \rightarrow 0 \text{ as } \ell \to \infty$$

This yields

$$u_{m_\ell} \rightarrow Ku \quad \text{strongly in } H$$

As

$$u_{m_\ell} \rightarrow u \quad \text{weakly in } H$$

we conclude

$$u_{m_\ell} \rightarrow u \quad \text{strongly in } H$$

Hence

$\|u\|_H = 1$ and $u - Ku = 0$, which implies $u \in \mathrm{Ker}(I-K)$

On the other hand, as $u_{m_\ell} \in [\mathrm{Ker}(I-K)]^{\perp}$, we get $u \in [\mathrm{Ker}(I-K)]^{\perp}$,

which is a contradiction ↯.

Step 2 $\boxed{\text{Im}(I-K) \text{ is closed}}$

Consider $\{v_n\} \in \text{Im}(I-K)$ so that $v_n \xrightarrow{n\to\infty} v$ in H. Aim is to find $\boxed{u \in H: \quad u - Ku = v}$ knowing that there are $u_n \in H$: $u_n - Ku_n = v_n$.

The point/difficulty is that we do not know that u_n converges to some u.

[If this would hold, then $u_n \to u$ implies: $\begin{matrix} u_n - Ku_n = v_n \\ \downarrow \quad \downarrow \\ u - Ku = v. \end{matrix}$]

To overcome this difficulty, we project u_n on $\text{Ker}(I-K)$ and its orthogonal complement:

$$u_n = \underbrace{\tilde{u}_n}_{\in \text{Ker}(I-K)} + \underbrace{z_n}_{\in [\text{Ker}(I-K)]^\perp} \qquad \text{where } z_n := u_n - \tilde{u}_n.$$

Note $v_n = u_n - Ku_n = z_n - Kz_n$

By Step 1, see (8.1),

$$\|v_n - v_m\|_H \geq \beta \|z_n - z_m\|_H$$

As $\{v_n\}$ converges, there is $u \in H$: $z_n \to u$ in H.

Then $u - Ku = \lim_{n\to\infty} z_n - Kz_n = \lim_{n\to\infty} v_n = v$, which completes the proof.

$\boxed{\text{Ad } 3}$ Since $\text{Im}(I-K)$ and $[\text{Ker}(I-K^*)]^\perp$ are closed subspaces, the assertion 3 is equivalent to

(8.2) $$[\text{Im}(I-K)]^\perp = \text{Ker}(I-K^*)$$

However,

$$x \in \text{Ker}(I-K^*) \Leftrightarrow (I-K^*)x = 0 \Leftrightarrow (y, (I-K^*)x) = 0 \;\; \forall y \in H$$
$$\Leftrightarrow ((I-K)y, x) = 0 \;\; \forall y \in H$$
$$\Leftrightarrow x \in [\text{Im}(I-K)]^\perp$$

and (8.2) is verified.

Ad 4 $\boxed{\Rightarrow}$ Assume: $\boxed{\text{Ker}(I-K) = \{0\}}$, i.e. $I-K$ is injective, $\boxed{\text{and } \text{Im}(I-K) \neq H}$. The goal is to get an contradiction.

Set $H_1 := \text{Im}(I-K) \subsetneq H$. By 2 H_1 is closed subspace of H.

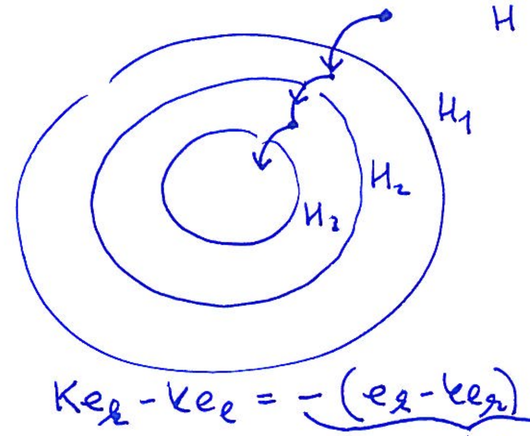
As $I-K$ is injective, but not onto $H_2 := (I-K)H_1 \subsetneq H_2$,

etc.: $H \supset H_1 \supset H_2 \supset \ldots \supset H_n$ where each H_n is a closed subspace of H.

For each $n \in \mathbb{N}$: pick $e_n \in H_n \cap H_{n+1}^\perp$ with $\|e_n\|_H = 1$,

see figure 1.

Figure 1.



If $k < l$: $\quad Ke_k - Ke_l = \underbrace{-(e_k - Ke_k) + (e_l - Ke_l)}_{} + e_k - \underline{e_l}$

$$= e_k + z_k \in H_{k+1} \quad \text{(by construction and order of indices)}$$

Since $e_k \in H_{k+1}^{\perp}$, by Pythagora's theorem

$$\|Ke_k - Ke_l\|_H \geq \|e_k\|_H = 1$$

and we get the contradiction w.r.t. compactness of K.

$\boxed{\Leftarrow}$ Assume that $\boxed{\mathrm{Im}\,(I-K) = H}$. By Theorem 6.1.

$\mathrm{Ker}\,(I-K^*) = [\mathrm{Im}\,(I-K)]^{\perp} = H^{\perp} = \{0\}$. Since K^* is compact, by previous implication $\boxed{\Rightarrow}$: $\mathrm{Im}\,(I-K^*) = H$.
Using Theorem 6.1 again, we get

$$\mathrm{Ker}\,(I-K) = [\mathrm{Im}\,(I-K^*)]^{\perp} = H^{\perp} = \{0\}, \text{ q.e.d.}$$

Ad ⑤ Step (i) We first show, by contradiction, that *)

$$\dim \mathrm{Ker}\,(I-K) \geq \dim\,[\mathrm{Im}\,(I-K)]^{\perp} = \dim\,[\mathrm{Ker}\,(I-K^*)]$$

So, let us assume that

(Ass) $$\dim \mathrm{Ker}\,(I-K) < \dim\,[\mathrm{Im}\,(I-K)]^{\perp}$$

Then $\exists\, A : \mathrm{Ker}\,(I-K) \longrightarrow [\mathrm{Im}\,(I-K)]^{\perp}$ which is one-to-one, but not onto.

We extend $A : H \to [\mathrm{Im}\,(I-K)]^{\perp}$ by requiring $Au = 0$ on $[\mathrm{Ker}\,(I-K)]^{\perp}$.
Since, by (8.2) and ① (K^* is compact), $[\mathrm{Im}\,(I-K)]^{\perp}$ is finite-dim., $\mathrm{Im}\,A$ is finite-dimensional, and hence A is compact and also $K+A$ is compact.

We will show below that $\boxed{\mathrm{Ker}\,(I-(K+A)) = \{0\}}$ (⊚)

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*) Note that we already know that $[\mathrm{Im}\,(I-K)]^{\perp} = \mathrm{Ker}\,(I-K^*)$ and as K is compact and hence K^* is compact, by ①
$\dim\,(I-K) < +\infty$ and $\dim\,[\mathrm{Im}\,(I-K)]^{\perp} = \dim \mathrm{Ker}\,(I-K^*) < \infty$.

Indeed, take any $u \in H$ and decompose it as

$u = u_1 + u_2$ where $u_1 \in \operatorname{Ker}(I-K)$ and $u_2 \in [\operatorname{Ker}(I-K)]^\perp$

Then, due to definition of A,

$$(\bullet\bullet) \quad \big(I-(K+A)\big)(u_1+u_2) = (I-K)(u_2) - Au_1 \in \operatorname{Im}(I-K) \oplus \big[\operatorname{Im}(I-K)\big]^\perp$$

Since $\boxed{(I-K)u_2 \perp Au_1}$, it follows from $(\bullet\bullet)$ that $(I-K-A)u=0$ (which is equivalent to $(I-K)u_2 - Au_1 = 0$) if and only if

$(I-K)u_2 = 0$ and $Au_1 = 0$

As $(I-K)$ is injective on $[\operatorname{Ker}(I-K)]^\perp$ and A is injective on $[\operatorname{Ker}(I-K)]^\perp$ then $u_1 = u_2 = 0$, i.e. $u=0$ and $(\bullet)$ is proved.

Next, as $(\bullet)$ holds, by (4) (already proved): $\operatorname{Im}(I-(K+A)) = H$.

But this leads to contradiction with the assumption (Ass).

Indeed, it follows from (Ass) and the fact that

$A: \operatorname{Ker}(I-K) \to [\operatorname{Im}(I-K)]^\perp$ is not onto that

there is $v \in [\operatorname{Im}(I-K)]^\perp$ and $v \notin \operatorname{Im} A$

But then by $(\bullet\bullet)$, the equation $u - Ku - Au = v$ does not have solution, which contradicts to .

Consequently, $\dim \operatorname{Ker}(I-K) \geq \dim [\operatorname{Im}(I-K)]^\perp = \dim [\operatorname{Ker}(I-K^*)]$

To prove that the opposite ineq. holds, we apply dual arguments: as $[\operatorname{Im}(I-K^*)]^\perp = \operatorname{Ker}(I-K)$, from the proved ineq. applied to K^* we have

$$\dim \operatorname{Ker}(I-K^*) \geq \dim [\operatorname{Ker}(I-K)],$$

The proof of Theorem 8.1 is complete. ▨

Question $\boxed{\text{Why } I-K \text{ operator, with } K \text{ being compact, appears naturally in solving linear elliptic problems?}}$ 2

Consider a more general linear elliptic operator

$$Lu := \underbrace{-\operatorname{div}(A(x)\nabla u)}_{\text{diffusion}} + \underbrace{\vec{b}(x)\cdot\nabla u}_{\text{transport}} + \underbrace{c(x)u}_{\text{source/gain}}$$

and, for simplicity, solve homogeneous Dirichlet problem:

$$\boxed{\begin{aligned} Lu &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}} \qquad \Omega \subset \mathbb{R}^d$$

- Our goal is to develop a theory assuming
$$A \in [L^\infty(\Omega)]^{d\times d},\ \vec{b} \in [L^\infty(\Omega)]^d \text{ and } c \in L^\infty(\Omega)$$
but without any additional smallness or structural assumption on these data (such as $\operatorname{div}\vec{b} = 0$).

- The apriori estimates (multiply by u, $\int_\Omega dx$, Gauss + bc's) leads to

$$\alpha \|\nabla u\|_2^2 \le \int_\Omega A(x)\nabla u\cdot\nabla u \le \int_\Omega |c(x)|\,|u|^2 dx + \int_\Omega |b(x)|\,|\nabla u|\,|u|\,dx$$
$$\le \frac{\alpha}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\vec{b}\|_\infty \|u\|_2^2 + \|c\|_\infty \|u\|_2^2$$

$$\Rightarrow \boxed{\alpha\|\nabla u\|_2^2 \le \left(\|\vec{b}\|_\infty + 2\|c\|_\infty\right)\|u\|_2^2}$$

This would give a ~~apriori~~ bound, with help of Poincaré inequality $\|u\|_2^2 \le c_p^2 \|\nabla u\|_2^2$ only if

$$c_p^2\left(\|b\|_\infty + 2\|c\|_\infty\right) < \alpha \qquad \text{(smallness condition)}$$

We need to proceed differently. From above calculations follow the existence of $\gamma \gg 1$ so that the apriori estimate is available for

$$\boxed{\begin{aligned} L_\gamma u := Lu + \gamma u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}}$$

This, apparently different problem, has unique solution operator that gives $f \in L^2$ the solution $u \in W_0^{1,2}(\Omega)$.

As $W_0^{1,2}$ is compactly imbedded into $L^2(\Omega)$, the operator

$$f \longmapsto u = L_\gamma^{-1} f \quad \text{as a mapping from } L^2(\Omega) \text{ into } L^2(\Omega)$$

is compact. Note:

$\boxed{u \in W_0^{1,2}(\Omega) : Lu = f} \iff \boxed{u \in W_0^{1,2} : L_\gamma u := Lu + \gamma u = f + \gamma u}$

$\iff \boxed{u \in W_0^{1,2} \text{ satisfying } u = L_\gamma^{-1}(f + \gamma u)}$

$\iff \boxed{u \in W_0^{1,2} \text{ satisfies } u - Ku = h \text{ where } K := \gamma L_\gamma^{-1} \text{ and } h := L_\gamma^{-1} f}$

Hence, we can apply Fredholm alternative theory. ∎

For details, see PDE I.

Consider $u \in W_0^{1,2}(\Omega)$ solving (in a weak sense) the problem $\boxed{\begin{matrix} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{matrix}}$, where $Lu = -\operatorname{div}(\mathbb{A}(x)\nabla u) + \vec{b}(x) \cdot \nabla u + c(x) u$

where $\mathbb{A} : \Omega \to \mathbb{R}^{d \times d}$ (not necessarily symmetric)

$\vec{b} : \Omega \to \mathbb{R}^d$

$c : \Omega \to \mathbb{R}$

We know that $L : W_0^{1,2}(\Omega) \to [W_0^{1,2}(\Omega)]^*$

but also $L : \operatorname{Dom}(L) \subseteq W^{2,2}(\Omega) \hookrightarrow L^2(\Omega) \longmapsto L^2(\Omega)$

but also $\langle Lu, \varphi \rangle = B(u, \varphi) \quad \forall u, \varphi \in W_0^{1,2}$.

Goal: $\left[\text{Find } L^* : \operatorname{Dom}(L^*) \subseteq W^{2,2} \subset L^2(\Omega) \to L^2(\Omega) \text{ so that } \langle Lu, \varphi \rangle = \langle u, L^* \varphi \rangle \right]$

$$\langle Lu, \varphi \rangle = \int_\Omega \left[-\operatorname{div}(A(x)\nabla u) + b(x) \cdot \nabla u + c(x) u \right] \varphi(x)\, dx$$

$$\overset{\text{Gauss}}{=} \int_\Omega A(x)\nabla u \cdot \nabla \varphi + b(x) \cdot \nabla u \varphi + c(x) u \varphi - \int_{\partial\Omega} \underbrace{A(x)\nabla u \cdot n \varphi}_{0 \text{ if } \varphi \equiv 0 \text{ on } \partial\Omega, \text{ for example}}$$

$$\overset{\text{Gauss again}}{=} \int_\Omega \Big[\underbrace{-\operatorname{div}\left(\mathbb{A}^T(x)\nabla\varphi\right) - \operatorname{div}\left(\vec{b}(x)\varphi\right) + c(x)\varphi}_{L^*\varphi} \Big] u(x) + \int_{\partial\Omega} \underbrace{\mathbb{A}^T(x)\nabla\varphi \cdot n u}_{0} - \int_{\partial\Omega} \underbrace{\vec{b}(x) \cdot n \varphi u \, dS}_{0}$$

$$= \langle u, L^* \varphi \rangle.$$

An Application of Riesz representation theorem & Fredholm alternative in finite-dimensions

$\boxed{(\mathbb{R}^d, (\cdot,\cdot)_d)}$ $(\mathbb{R}^n, (\cdot,\cdot)_n)$, $(\mathbb{R}^m, (\cdot,\cdot)_m)$

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ & & \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

$A \in M^{m \times n}$ n columns, m rows

$(A^T) \in M^{n \times m}$ m columns, n rows

$[A^T]_{ij} = A_{ji}$ is the transpose matrix, that can be defined as unique $n \times m$ - matrix such that

$$(Ax, y)_m = (x, A^T y) \quad \forall x \in \mathbb{R}^n \ \forall y \in \mathbb{R}^m$$

$\boxed{\mathbb{C}^d, (\cdot,\cdot)_d \text{ Hermitian scalar product}}$ $(\mathbb{C}^n, (\cdot,\cdot)_n)$, $(\mathbb{C}^m, (\cdot,\cdot)_m)$

For $A \in \mathbb{C}^{m \times n}$, define adjoint matrix $A^* \in \mathbb{C}^{n \times m}$:

$$(Ax, y)_m = (x, A^* y) \quad \forall x \in \mathbb{C}^n, y \in \mathbb{C}^m.$$

By Riesz representation theorem one can prove

Theorem 8.2 Let $(X, (\cdot,\cdot)_X)$ and $(Y, (\cdot,\cdot)_Y)$ be two <u>complex</u> Hilbert spaces.

Let $A \in \mathcal{L}(X,Y)$ be given

Then (a) $\exists!$ $A^* \in \mathcal{L}(Y,X)$ called <u>adjoint</u> of A:

$$(Ax, y)_Y = (x, A^* y)_X \quad \forall x \in X \ \forall y \in Y$$

The mapping $A \in \mathcal{L}(X,Y) \longmapsto A^* \in \mathcal{L}(X,Y)$ is semilinear: $\tau(A+B) = \tau(A) + \tau(B)$, $\tau(\alpha A) = \bar{\alpha}\tau(A)$

and $\|A^*\|_{\mathcal{L}(Y,X)} = \|A\|_{\mathcal{L}(X,Y)}$

(b) $(\operatorname{Im} A)^\perp = \ker A^*$, $(\operatorname{Im} A^*)^\perp = \ker A$

$Y = \ker A^* \oplus \overline{\operatorname{Im} A}$, $X = \ker A \oplus \overline{\operatorname{Im} A^*}$ (for any $A \in \mathcal{L}(X,Y)$)

If $(X, (\cdot,\cdot)_X)$ and $(Y, (\cdot,\cdot)_Y)$ are <u>real</u> Hilbert spaces, then similarly $\exists!$ $A^T \in \mathcal{L}(Y,X)$: $(Ax, y)_Y = (x, A^T y)$ $\forall x \in X$ $\forall y \in Y$, the mapping $A \mapsto A^*$ is linear and (b) holds just by replacing A^* by A^T.

Pf See* the proof of <u>Theorem 6.1</u> and the observation $Y = \overline{\operatorname{Im} A} \oplus (\overline{\operatorname{Im} A})^\perp$
$= \overline{\operatorname{Im} A} \oplus (\operatorname{Im} A)^\perp = \overline{\operatorname{Im} A} \oplus \ker A^*$

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$\boxed{\text{Theorem}}$ 8.3 Fredholm alternative in finite-dim. spaces

Let $A \in \mathbb{K}^{m\times n}$, $b \in \mathbb{K}^m$. Then

$\boxed{\text{EITHER}}$ $Ax = b$ has at least one solution $x \in \mathbb{K}^m$

$\boxed{\text{OR}}$ $Ax = b$ has <u>no</u> solution and there is at least one $y \in \mathbb{K}^m$: $\begin{cases} A^T y = 0 \text{ and } y^T b \neq 0 & \text{if } \boxed{\mathbb{K}=\mathbb{R}} \\ A^* y = 0 \text{ and } y^* b \neq 0 & \text{if } \boxed{\mathbb{K}=\mathbb{C}} \end{cases}$

Pf $\boxed{\text{Let } \mathbb{K}=\mathbb{C}.}$ The case $\mathbb{K}=\mathbb{R}$ is done analogously.

As $\mathbb{C}^m$ is finite-dimensional, $\operatorname{Im} A$ is closed.

By previous theorem, part (b),

$$\mathbb{C}^m = \operatorname{Ker} A^* \oplus \operatorname{Im} A$$

Therefore, $\boxed{\text{either}}$ $b \in \operatorname{Im} A$ and then $Ax = b$ has at least one solution

$\boxed{\text{or}}$ $b \notin \operatorname{Im} A$ and then $Ax = b$ has no solution and the projection ~~Py~~ of b on $\operatorname{Ker} A^*$, which cannot be zero as b is not ~~...~~ zero since $b \notin \operatorname{Im} A$,

then $y = Pb$ satisfies $A^* y = 0$ and $y^* b = y^* y \neq 0$. □

$\boxed{\text{Ad proof of Theorem 8.2}}$

$\forall y \in Y$: $\underset{\in X}{x} \mapsto (Ax, y)_Y \in \mathbb{K}$ is a continuous linear functional on X

$\phi_y \in X^*$

Riesz: $\phi_y(x) = (x, A^* x)_X \quad \forall x \in X$

$\exists!$ element, denoted $A^* y$, ~~∈X~~ so that $A^* y \in X$ and

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$\boxed{\text{Application}}$ of Riesz representation theorem for proving H–B theorem on Hilbert space

$\boxed{\text{Theorem}}$ Let $(H, (\cdot,\cdot)_H)$ be Hilbert over $\mathbb{K}$, and $Y \subset H$ be subspace of H, and $l : Y \to \mathbb{K}$ be continuous linear form on Y. Then $\exists\, \tilde{l} : H \to \mathbb{K}$

$$\tilde{l}(y) = l(y) \;\; \forall y \in Y \quad \text{and} \quad \|\tilde{l}\|_{H^*} = \|l\|_{Y^*}.$$

Such an extension is unique.

$\boxed{\text{Proof}}$ ~~...~~ Since $\mathbb{K}$ is complete, there is a unique continuous extension of l to $\hat{l} : \overline{Y} \to \mathbb{K}$ so that

$$\hat{l}(y) = l(y) \;\; \forall y \in Y \quad \text{and} \quad \|\hat{l}\|_{(\overline{Y})^*} = \|l\|_{Y^*}$$

Then $x \in H$: $\quad x = \underbrace{Px}_{\in \overline{Y}} + \underbrace{P^\perp x}_{\in (\overline{Y})^\perp}$

Set $\quad \tilde{l}(x) := \hat{l}(Px)$

Then $\tilde{l}$ is an extension of l since

$$\tilde{l}(y) = \hat{l}(Py) = \hat{l}(y) = l(y) \qquad \forall y \in Y$$

and
$$\|l\|_{Y^*} = \sup_{\substack{\|y\|=1 \\ y \in Y}} \frac{|l(y)|}{\|y\|} \le \sup_{\substack{\|x\|=1 \\ x \in H}} \frac{|\tilde{l}(x)|}{\|x\|} = \|\tilde{l}\|_{H^*}$$

$$= \sup_{\|x\|=1} \frac{|\hat{l}(Px)|}{\|x\|_H} \le \|\hat{l}\|_{\overline{Y}^*} = \|l\|_{Y^*}$$

since $\|Px\| \le \|x\|$ for all $x \in X$; hence $\|\tilde{l}\|_{H^*} = \|l\|_{Y^*}$

$\boxed{\text{Uniqueness}}$ We may assume (w.l.o.g.) that Y is closed (as the extension from Y to $\overline{Y}$ is unique). Let $l^\# \in H^*$ be an extension of $l \in H^*$ satisfying $\|l^\#\|_{H^*} = \|l\|_{Y^*}$. By Riesz:

$\exists!\; z \in H$: $\quad l^\#(x) = (x, z)_H \;\; \forall x \in X$ and $\|l^\#\|_{H^*} = \|z\|_H$

Since $l^\#(y) = l(y) = (y, z) = (y, Pz) \;\; \forall y \in Y$

it follows that $\|l\|_{Y^*} = \|Pz\|_H$. Hence $\|l^\#\|_{H^*} = \|l\|_{Y^*}$ implies

$\|Pz\|_H = \|z\|_H$, which gives $Pz = z$. Consequently:

$$l^\#(x) = (x, Pz) \overset{(Px + P^\perp x,\, Pz) = (Px, Pz)}{=} (Px, z) = l^\#(Px) = l(Px) \qquad \forall x \in H$$

$\blacksquare$