Cvičení Matematika pro fyziky 1
11.12.2020 10:45–12:15

December 11 is the 345th day of the year (346th in leap years) in the Gregorian calendar. 20 days remain until the end of the year.
in year 220 – Emperor Xian of Han is forced to abdicate the throne by Cao Cao 's son Cao Pi, ending the Han dynasty.
(and so on...)

Dobrý den! Dnes začneme v 10:50.

Křivkový a plošný integrál

1. Spočtěte $\overbrace{\int_S (y-z)\,dy\,dz + (z-x)\,dz\,dx + (x-y)\,dx\,dy}^{\text{Plošný int. 2. druhu}}$, kde S je "vnější strana" kužele $x^2+y^2=z^2$, $0 \le z \le h$.

4.5. Plošný integrál

← Kopáček Příklady III

Definice. Nechť φ je parametrizace jednoduché regulární plochy na Ω a $\vec{T}$ je vektorové pole, definované na $\langle\varphi\rangle = \varphi(\Omega)$. Pak definujeme *plošný integrál 2. druhu* z vektorového pole $\vec{T}$ přes plochu $\langle\varphi\rangle$ s orientací danou parametrizací φ (tj. kladně vzhledem k normálovému vektoru $\nu = ((\partial\varphi/\partial u)\times(\partial\varphi/\partial v))/||(\partial\varphi/\partial u)\times(\partial\varphi/\partial v)||$) předpisem (pokud integrál napravo existuje jako Lebesgueův integrál)

$$\int_\varphi \vec{T}\,d\vec{S} = \int_\varphi T_1\,dydz + T_2\,dzdx + T_3\,dxdy =$$
$$= \int_\Omega (\vec{T}\circ\varphi, (\partial\varphi/\partial u)\times(\partial\varphi/\partial v))\,dudv,$$

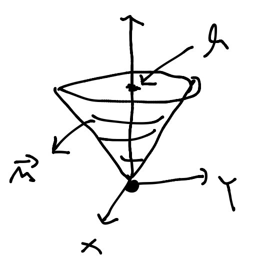
$$I = \int \vec{f}\cdot d\vec{S}, \quad \text{kde} \quad \vec{f} = (y-z, z-x, x-y)$$

Teď

$\boxed{\text{1. druhu}}$ $\displaystyle\int_{\langle\varphi(\Omega)\rangle} f\,dS \overset{df}{=} \int_\Omega f(\vec{\varphi}(u,v)) \left\| \frac{\partial\vec{\varphi}}{\partial u}\times\frac{\partial\vec{\varphi}}{\partial v}\right\|_2 du\,dv$

$\boxed{\text{2. druhu}}$ $\displaystyle\int_{\langle\varphi(\Omega)\rangle} \vec{f}\cdot d\vec{S} = \int_{\langle\varphi(\Omega)\rangle} \vec{f}\cdot\vec{n}\,dS = \int_\Omega \vec{f}(\vec{\varphi}(u,v))\cdot\left(\frac{\partial\vec{\varphi}}{\partial u}\times\frac{\partial\vec{\varphi}}{\partial v}\right) du\,dv$

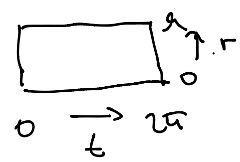
Výpočet: • parametrizace plochy
• výpočet integrálu



Plocha: $x^2+y^2=z^2$, $0 \le z \le h$

$\vec{\varphi}$: $x = r\cos t$ $\quad r\in(0,h)$
$y = r\sin t$ $\quad t\in(0,2\pi)$
$z = r$

Plocha v $\mathbb{R}^3$ ⟶



Tečné vektory: $\dfrac{\partial\vec{\varphi}}{\partial r} = (\cos t, \sin t, 1)$

$\dfrac{\partial\vec{\varphi}}{\partial t} = (-r\sin t, r\cos t, 0)$

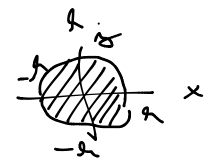
Normálový vektor:
$\dfrac{\partial\vec{\varphi}}{\partial r}\times\dfrac{\partial\vec{\varphi}}{\partial t} = (-r\cos t, -r\sin t, r) \xrightarrow{\text{orientace}} \vec{n} = (r\cos t, r\sin t, -r)$

$$I = \int_S \vec{f}\cdot d\vec{S} = \int_{\langle\vec{\varphi}(r,t)\rangle} \vec{f}\cdot d\vec{S} = \int_\Omega \vec{f}(\vec{\varphi}(r,t))\cdot \underbrace{\left(\frac{\partial\vec{\varphi}}{\partial r}\times\frac{\partial\vec{\varphi}}{\partial t}\right)}_{\vec{n}}\,dr\,dt \qquad \vec{f} = (y-z, z-x, x-y)$$

$$= \int_{r=0}^{R}\int_{t=0}^{2\pi} (r\sin t - r, r - r\cos t, r\cos t - r\sin t)\cdot(r\cos t, r\sin t, -r)\,dr\,dt$$

$$= \int_0^R\int_0^{2\pi} (\cancel{r\sin t}-r)\,r\cos t + (r-\cancel{r\cos t})\,r\sin t + (r\cos t - r\sin t)(-r)\,dr\,dt$$

$$= \int_0^R\int_0^{2\pi} -2r^2\cos t + 2r^2\sin t\,dr\,dt = 2\int_0^R r^2\,dr \underbrace{\int_0^{2\pi}\sin t - \cos t\,dt}_{0} = \underline{0}$$



- Výpočet pomocí jiné parametrizace

Explicitní: $z = z(x,y) = \sqrt{x^2+y^2}$

$x^2+y^2 = z^2 \longrightarrow$ $(x,y)\in\Omega : \{(x,y) : x^2+y^2 < R^2\}$

$$\vec{\varphi} : (u,v) \longrightarrow (u,v,\psi(u,v)) \qquad \psi(u,v) = \sqrt{u^2+v^2}$$

$$\left.\begin{aligned} \frac{\partial\vec{\varphi}}{\partial u} &= \left(1,0,\frac{\partial\psi}{\partial u}\right)\\ \frac{\partial\vec{\varphi}}{\partial v} &= \left(0,1,\frac{\partial\psi}{\partial v}\right)\end{aligned}\right\} \frac{\partial\vec{\varphi}}{\partial u}\times\frac{\partial\vec{\varphi}}{\partial v} = \left(-\frac{\partial\psi}{\partial u}, -\frac{\partial\psi}{\partial v}, 1\right) \ldots \text{normálový vektor}$$

$\pm \ldots$ orientace

$$\Rightarrow \int_S \vec{f}\cdot d\vec{S} = \int_{\langle\vec{\varphi}(u,v)\rangle} \vec{f}(\vec{\varphi}(u,v))\cdot\left(\frac{\partial\vec{\varphi}}{\partial u}\times\frac{\partial\vec{\varphi}}{\partial v}\right)$$

$$= \int_\Omega f_x(\vec{\varphi}(u,v))\left(-\frac{\partial\psi}{\partial u}\right) + f_y(\vec{\varphi}(u,v))\left(-\frac{\partial\psi}{\partial v}\right) + f_z(\vec{\varphi}(u,v))\cdot 1\,du\,dv$$

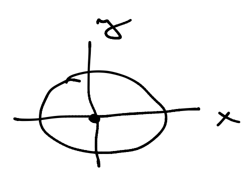
$$z = \sqrt{x^2+y^2} \qquad \frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$\hookrightarrow (x,y) \longrightarrow (x,y,z(x,y))$

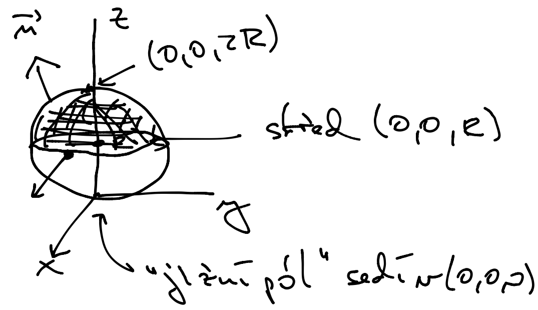
$$\int_S \vec{f}\cdot d\vec{S} = \int_\Omega f_x\left(-\frac{\partial z}{\partial x}\right) + f_y\left(-\frac{\partial z}{\partial y}\right) + f_z\cdot 1\,dx\,dy$$

$$= \int_\Omega (\cancel{y} - \underset{\sqrt{x^2+y^2}}{z})\frac{-x}{\sqrt{x^2+y^2}} + (\sqrt{x^2+y^2} - \cancel{x})\cdot\frac{-y}{\sqrt{x^2+y^2}} + (x-y)\cdot 1\,dx\,dy$$

$$= \int_\Omega (2x-2y)\,dx\,dy = 0$$



3. Spočtěte $\int_S (z-R)^2\,dx\,dy$, kde S je část kulové plochy $x^2+y^2+z^2 = 2Rz$, $R \le z \le 2R$, orientovaná normálou ven.



$$I = \int \vec f \cdot \vec{dS}\ ,\quad \text{kde}\quad \vec f = \left(0,0,(z-R)^2\right)$$

Plocha: $\quad x^2+y^2+(z-R)^2 = R^2$

(a) Výpočet „podle vzorečku"

Parametrizace: $\vec\varphi$:

$$x = R\sin\vartheta\cos t \qquad \vartheta \in \left(0,\tfrac{\pi}{2}\right)$$
$$y = R\sin\vartheta\sin t \qquad t \in (0,2\pi)$$
$$z = R\cos\vartheta + R$$

$$\frac{\partial\vec\varphi}{\partial\vartheta} = \left(R\cos\vartheta\cos t,\ R\cos\vartheta\sin t,\ -R\sin\vartheta\right)$$

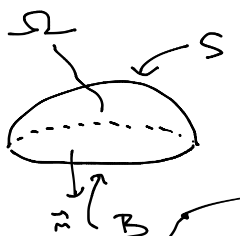
$$\frac{\partial\vec\varphi}{\partial t} = \left(-R\sin\vartheta\sin t,\ R\sin\vartheta\cos t,\ 0\right)$$

$$\frac{\partial\vec\varphi}{\partial\vartheta}\times\frac{\partial\vec\varphi}{\partial t} = \left(R^2\sin^2\vartheta\cos t,\ R^2\sin^2\vartheta\sin t,\ R^2\sin\vartheta\cos\vartheta\right) \ \dots\ \text{orientace } \checkmark$$

$$\Rightarrow I = \int_S \vec f\cdot\vec{dS} = \int_S f_z\cdot n_z\,dS = \int_{t=0}^{2\pi}\int_{\vartheta=0}^{\frac{\pi}{2}} \underbrace{R^2\cos^2\vartheta}_{f_z}\cdot\underbrace{R^2\sin\vartheta\cos\vartheta}_{n_z}\,d\vartheta\,dt$$

$$= R^4\cdot\underbrace{\int_0^{2\pi}1\,dt}_{2\pi}\cdot\int_0^{\frac{\pi}{2}}\cos^3\vartheta\sin\vartheta\,d\vartheta = 2\pi R^4\underbrace{\left[-\frac{\cos^4\vartheta}{4}\right]_0^{\frac{\pi}{2}}}_{-0+\frac{1}{4}} = \frac{\pi}{2}R^4$$

(b) Výpočet pomocí Gaussovy věty



$$\Omega \subset \mathbb{R}^3$$
$$\partial\Omega = S \cup B$$

$\boxed{\text{Gaussova věta}}$ se týká $\Omega \subset \mathbb{R}^3$ a její hranice $\partial\Omega$, která představuje plochu v $\mathbb{R}^3$. Typický příklad je $\Omega = B_1(0) \subset \mathbb{R}^3$ a $\partial\Omega = \partial B_1(0)$.

Platí:
$$\int_{\partial\Omega}\vec F\cdot\vec{dS} = \int_\Omega \operatorname{div}\vec F\,dx\,dy\,dz$$

neboli
$$\boxed{\int_\Omega \operatorname{div}\vec F\,dx\,dy\,dz = \int_{\partial\Omega}\vec F\cdot\vec n\,dS}$$

$$\int_{\partial\Omega}\vec f\cdot\vec{dS} = \underbrace{\int_S \vec f\cdot\vec{dS}}_{\text{to chceme}} + \int_B \vec f\cdot\vec{dS} = \int_\Omega \operatorname{div}\vec f\,dx\,dy\,dz$$

• $\int_B \vec{f}\cdot d\vec{S} = \int_B \vec{f}\cdot \vec{n}\, dS = \int_B (z-R)^2\cdot(-1)\, dS = 0$

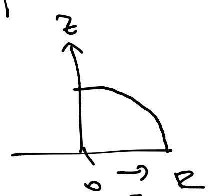
$(0,0,(z-R)^2)$ $\quad$ $(0,0,-1)$ $\quad$ "poklice" polosféry leží v rovině $z=R$

• $\int_\Omega \operatorname{div}\vec{f}\, dx\,dy\,dz = \int_\Omega 2(z-r)\,dx\,dy\,dz \left| \begin{matrix} x'=x \\ y'=y \\ z'=z-R \end{matrix} \right| = 2\int_{\Omega'} z'\,dz'$

jacobián $=1$

$\overset{\text{VOS}}{=} \left| \begin{matrix} x'=r\cos t \\ y'=r\sin t \\ z'=z' \end{matrix} \right| \overset{\text{FUBINI}}{=} 2\int_{t=0}^{2\pi}\left(\int_{r=0}^{R}\left(\int_{z=0}^{\sqrt{R^2-r^2}} z\,dz\right) r\,dr\right) dt$

Jac $= r$



$= \cancel{2}\cdot 2\pi\cdot\int_0^R \left[\frac{z^2}{\cancel{2}}\right]_0^{\sqrt{R^2-r^2}} r\,dr = 2\pi\int_0^R (R^2-r^2)\,r\,dr = 2\pi\int_0^R (R^2 r - r^3)\,dr$

$= 2\pi\left[R^2\frac{r^2}{2}-\frac{r^4}{4}\right]_0^R = 2\pi\left[R^2\frac{R^2}{2}-\frac{R^4}{4}\right] = 2\pi R^4 \underbrace{\left(\frac{1}{2}-\frac{1}{4}\right)}_{\frac{1}{4}} = \frac{\pi}{2}R^4$