1. INTRODUCTION

Functional analysis (FA) is a study of the mappings
$$L : X \to Y$$

where X, Y are vector spaces over the field of scalars $\mathbb{K}$ where $\mathbb{K}$ stands for real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$.

The vector spaces X, Y are (usually) equipped with additional structures that allows one to talk a continuity of the mappings etc.

X with scalar product $(\cdot|\cdot)_X$ ... pre-Hilbert spaces or spaces with inner product or scalar-product spaces

X with norm $\|\cdot\|_X$ .... normed spaces

X with distance $d(\cdot,\cdot)$ ..... metric spaces

X with topology $\mathcal{T}$ .... topological (vector) spaces.

Q: Why „functional" analysis? A: An important subclass of the mappings are those where $Y = \mathbb{K}$, i.e. $L : X \to \mathbb{K}$ maps points (vectors) of X into the numbers.
Such mappings are called functionals.

- LINEAR FA studies LINEAR mappings L (see definition below)
  - generalizes tools and the results of linear algebra; studies similarities/differences and connections between X's so that $\boxed{\dim X < +\infty}$ and X's with $\boxed{\dim X = \infty}$
  - provides an elegant way to solve linear ODE and/or PDE problems
- NONLINEAR FA analyzes NONLINEAR mappings L and includes
  - fixed-point theory
  - bifurcation theory
  - calculus of variation and optimization theory

A central topic in the both areas is the question of compactness (distinguishes X with $\dim X < +\infty$ to those X with $\dim X = +\infty$).

History of FA is connected with

(i) John von NEUMANN (1926) following the goal to develop mathematical foundation of quantum mechanics:

> there is a correspondance between the point in a Hilbert space of the norm 1 and a physical state of quantum system

(ii) Jean LERAY (1929-34) following the words of theoretical physicist Carl-Wilhelm Oseen on theoretical foundation of hydrodynamics, i.e. on Navier-Stokes equations.

> Leray looks at the solution of the nonlinear system of PDEs as a point in the vector space.

founder of modern analysis.

2. LINEAR OPERATORS BETWEEN BANACH SPACES

In the first year (2^{nd} semester), we use $\mathbb{R}^d$ to gradually build definitions and relations for Hilbert, Banach (normed), metric and topological vector spaces.

Let us do it differently now. Given a vector space X over the field of numbers $\mathbb{K}$ (where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$), we wish to introduce a distance $d(\cdot,\cdot): X\times X \to \mathbb{R}$ between the points of X. With distance we can specify topology (open, closed sets), convergence, continuity of mappings, convergence of series etc.

Since X has the algebraic structure of vector space, the distance d should be consistent with the structure. It is thus natural to require:

(P1) $d(x+z,y+z) = d(x,y) \quad \forall x,y,z \in X$ (invariance of distance w.r.t. the translation)

(P2) $d(\lambda x,\lambda y) = |\lambda|\, d(x,y) \quad \forall x,y\in X, \forall \lambda\in\mathbb{K}$ ("invariance" w.r.t. scaling)

(P3) $B_r(x_0) := \{y\in X;\ d(x_0,y)<r\}$ is convex for all $x_0\in X$, $\forall r>0$.

The invariance (P1) implies that $d(\cdot,\cdot)$ is determined once we specify the function $x \longmapsto \|x\|_X := d(x,0)$ called norm.
The story of FA can start from here.

Definition Let X be a vector space over $\mathbb{K}$ (field of numbers).
Then a map $x \longmapsto \|x\|_X : X \mapsto \mathbb{R}$ is called norm on X if it satisfies: for all $x\in X$: $\|x\| \geq 0$ and

(N1) $\|x\|_X = 0 \iff x = 0$;

(N2) $\|\lambda x\| = |\lambda| \, \|x\| \quad \forall x \in X \ \forall \lambda \in \mathbb{K}$;

(N3) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$.

$\boxed{(X, \|\cdot\|_X)}$ is then called a normed space.

Assertion 1 (Norm generates a distance) Let $(X, \|\cdot\|_X)$ be a normed space. Then
$$d(x,y) := \|x-y\|_X$$
defines the distance between the points of X, i.e.

(D1) $\boxed{d(x,y) \geq 0 \ \forall x,y \in X}$ and $\boxed{d(x,y) = 0 \iff x = y}$

(D2) $d(x,y) = d(y,x) \quad \forall x,y \in X$

(D3) $d(x,z) \leq d(x,y) + d(y,z). \quad \forall x,y,z \in X$

Moreover, this distance fulfills the properties (P1)–(P3) above.

(Pf) • (D1) follows directly from (N1).

• To prove (D2), we write
$$d(y,x) = \|y-x\|_X = \|(-1)(x-y)\|_X \overset{(N2)}{=} |(-1)| \, \|x-y\|_X = \|x-y\|_X = d(x,y).$$

• (D3) follows from (N3) as follows:
$$d(x,z) = \|x-z\| = \|x-y+y-z\| \overset{(N3)}{\leq} \|x-y\| + \|y-z\| = d(x,y) + d(y,z)$$

• (P1) is a consequence of the definition

• (P2) $d(\lambda x, \lambda y) = \|\lambda x - \lambda y\| = \|\lambda(x-y)\| \overset{(N2)}{=} |\lambda| \, \|x-y\| = |\lambda| \, d(x,y)$.

• (P3) It is enough, due to (P1), to prove (P3) for the balls centered at the origin. Let $x,y \in B_r(0)$, and $\theta \in \langle 0,1 \rangle$. Then
$$\underbrace{\|\theta x + (1-\theta)y\|} \leq \|\theta x\| + \|(1-\theta)y\| \leq \theta\|x\| + (1-\theta)\|y\|$$
$$< \theta r + (1-\theta) r = r$$

$\blacksquare$

Having $(X, \|\cdot\|_X)$ or $(X; d(\cdot,\cdot))$, we can define <u>open</u> and <u>closed</u> <u>balls</u>, and then open sets (each point has a ball contained in the set), closed sets (their complement is open) and thus <u>topology</u>.

Furthermore, for $\{x_n\}_{n=1}^{\infty} \subset X$, $x \in X$:

- $x_n \to x$ in X $\stackrel{df.}{\equiv}$ $\|x_n - x\|_X \to 0$ as $n \to \infty$
- $\{x_n\}$ is <u>cauchy</u> $\equiv$ $\forall \varepsilon > 0\ \exists n_0\ \forall n, m \geq n_0\ \|x_n - x_m\|_X < \varepsilon$
- $\sum_{n=1}^{\infty} x_n \in X$ $\equiv$ $\forall \varepsilon > 0\ \exists m_0\ \forall n \geq m_0\ \forall m \in \mathbb{N}\ \|\sum_{k=m}^{m+M} x_k\| < \varepsilon$

  $\sum_{n=1}^{\infty} x_n = x$ $\equiv$ $\|x - \sum_{n=1}^{N} x_n\|_X \to 0$ as $N \to \infty$.

Replacing norm $\|\cdot\|_X$ by the metric $d(\cdot,\cdot)$ we get the analogous definitions of concepts in any metric space.

$\boxed{\text{Def.}}$ (COMPLETNESS) $\left.\begin{matrix}(X, (\cdot,\cdot)_X) \\ (X, \|\cdot\|_X) \\ (X, d(\cdot,\cdot))\end{matrix}\right\}$ is $\left\{\begin{matrix}\text{Hilbert} \\ \text{Banach} \\ \text{Fréchet}\end{matrix}\right\}$ space $\stackrel{df}{\equiv}$ $\boxed{\text{Every cauchy sequence is converging in } X}$

<u>Examples</u>*) • $(\mathbb{R}^d, |\cdot|_p)$, $1 \leq p \leq +\infty$ are Banach spaces, for $p = 2$ Hilbert.

- $\left(C(\bar{\Omega}); \|f\|_\infty := \max_{x \in \bar{\Omega}} |f(x)|\right)$ is Banach

$\Omega \subset \mathbb{R}^d$ open

$\left(C(\Omega); \int_\Omega |f| dx\right)$ is not complete

$\left(C^1(\bar{\Omega}); \|f\|_{C^1(\bar{\Omega})} := \|f\|_\infty + \sum_{i=1}^{d} \|\frac{\partial f}{\partial x_i}\|_\infty\right)$ is Banach

$\left(C^1(\bar{\Omega}); \|f\|_{C^1(\bar{\Omega})} := \|f\|_\infty\right)$ is not

- $(\ell_p, \|\cdot\|_{\ell_p}) = \left(\{x = \{x_n\}_{n=1}^{\infty};\ \sum_{n=1}^{\infty} |x_n|^p < \infty\}; \|x\|_{\ell_p} := \begin{cases} \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} \\ \sup_{n \in \mathbb{N}} |x_n| & \boxed{p=\infty} \end{cases}\right)$

  are, for $1 \leq p \leq +\infty$, Banach

$\boxed{\ell_2}$ is a Hilbert space (By abstract Fourier series theory, we know that each separable Hilbert space is isometric to ℓ_2.)

Note that looking at ℓ_p as a vector space, one can get the meaning only to finite summation.

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*) that we know and we studied them earlier.

Hence; while $\mathrm{span}\{e_1, \dots, e_N\} = \mathbb{R}^d$ for $\{e_i\}_{i=1}^N$ being canonical bases,

in l_p: $\underbrace{\mathrm{span}\{e_1, \dots, e_N, \dots\}}_{\text{Here, we get only those sequences that have only finite non-zero components.}} \subsetneqq l_p$

"generate"

- $\left(\mathcal{L}^p(\Omega), \|\cdot\|_{\mathcal{L}^p(\Omega)}\right)$ are Banach, for $1 \le p \le \infty$, even Hilbert for $p = 2$.

classes of functions that may differ on the sets of zero measure

Examples (we did not investigate yet) For $1 \le p \le +\infty$, $k \in \mathbb{N}$

SOBOLEV SPACES.

- $$W^{k,p}(\Omega) := \left\{ f \in L^p(\Omega);\ D^\alpha f \in L^p(\Omega)\ \ \forall\, \alpha = (\alpha_1, \dots, \alpha_d),\ |\alpha| \le k \right\}$$

  are Banach spaces with

  $$\|f\|_{W^{k,p}(\Omega)} := \left(\sum_{\alpha=0}^{k} \|D^\alpha f\|_p^p \right)^{1/p}$$

- For $p = 2$:

  $H^k(\Omega) := W^{k,2}(\Omega)$ are Hilbert spaces

  For example

  $$\left(H^1, (\cdot,\cdot)_{H^1}\right);\ (f,g)_{H^1(\Omega)} := \int_\Omega f(x) g(x) + \sum_{i=1}^{d} \int_\Omega \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i}\, dx$$

  if $f, g : \Omega \to \mathbb{R}$.

Topology (and hence metric, norm or scalar product) enables us to talk about continuity of mappings.

▷ Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two normed spaces over the same field $\mathbb{K}$.

$\boxed{\text{Def.}}$ We say that $f: X \to Y$ is continuous in X

$\stackrel{\text{df.}}{=}$ $\forall x_0 \in X \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \ \|x - x_0\|_X < \delta \Rightarrow \|f(x) - f(x_0)\|_Y < \varepsilon$

$\iff$ $\quad x_n \xrightarrow[n\to\infty]{} x_0 \Rightarrow f(x_n) \to f(x_0)$

▷ For X, Y as above, we say that f is bounded if it maps bounded sets in X into bounded sets in Y.

▷ For X, Y as above.

$\boxed{\text{Def.}}$ • A mapping $\boxed{L : \text{Dom}(L) \subset X \to Y \text{ is linear}}$

if • $L(x+y) = Lx + Ly \quad \forall x, y \in \text{Dom}\, L$

• $L(\alpha x) = \alpha L x \quad \forall x \in \text{Dom}\, L, \forall \alpha \in \mathbb{K}$

Here, $\text{Dom}(L)$ is called domain; it is a subspace of X where L is defined.

The sets

• $\text{Range}\, L = \text{Im}\, L = \{ y \in Y ;\ y = Lx \text{ for certain } x \in \text{Dom}\, L \}$

• $\text{Ker}\, L = \text{Null}\, L = \{ x \in \text{Dom}\, L ;\ Lx = 0 \}$

are called image resp. kernel (or null space)

• L is nonlinear $\stackrel{\text{df.}}{=}$ L is not linear.

▷ Def. (boundedness of linear operator) Let $L : X \to Y$ be linear $(\text{Dom}\, L = X)$. We say that L is bounded if $\quad \|L\| := \sup\limits_{\|x\|_X \le 1} \|Lx\|_Y < +\infty$

Note that if L is linear & bounded and $x \neq 0$ then

$$\underbrace{\|L(x)\|}_{Y} = \Big\|\|x\| L\Big(\frac{x}{\|x\|}\Big)\Big\|_Y \stackrel{(N2)}{=} \|x\|_X \Big\|L\Big(\underbrace{\frac{x}{\|x\|}}_{\in B_1(0)}\Big)\Big\|_Y \le \underbrace{\|L\|\ \|x\|_X}$$

which implies that bounded sets in X are mapped into bounded sets in Y.

The above inequality holds trivially for $x = 0$. Hence for L linear bounded:

$$\|Lx\|_Y \le \|L\|\ \|x\|_X \quad \forall x \in X.$$