Recall

(P)
$$\boxed{\begin{aligned} &\frac{\partial u}{\partial t} - \operatorname{div}\vec{q} = g \quad \text{in } Q \\ &\nabla u = \frac{q}{(1+|q|^a)^{1/a}} =: \mathcal{F}(q) \quad \text{in } Q \\ &u(0,\cdot) = u_0 \end{aligned}}$$

DATA $g, u_0, a>0$, T, L

Sols $u, \vec{q} : (0,T)\times\mathbb{R}^d \to \mathbb{R}\times\mathbb{R}^d$

Ω- periodic

$Q = (0,T)\times\Omega$

Def. (of weak sol.) Let $u_0 \in L^2(\Omega)$, $g \in L^2(Q)$ and $T, L, a \in (0,+\infty)$

We say that $(u, \vec{q})$ is W.S. to (P) if

- $u \in L^2(0,T; W^{1,2}(\Omega)) \cap \underbrace{W^{1,2}(0,T; L^2(\Omega))}_{\iff \frac{\partial u}{\partial t} \in L^2(Q)}$
- $q \in L^1(0,T; L^1(\Omega)^d)$
- $\int_\Omega \frac{\partial u}{\partial t}\varphi + q\cdot\nabla\varphi = \int_\Omega g\varphi \quad \forall \varphi \in W^{1,\infty}(\Omega)$ a.e. in $(0,T)$
- $\nabla u = \mathcal{F}(q)$ a.e. in Q
- $\|u(t,\cdot) - u_0\|_2^2 \to 0$ as $t \to 0+$.

Theorem ▸ Let $a>0$, $g \in L^2(Q)$ and $u_0 \in W^{1,\infty}_{per}$ satisfy
$$\|\nabla u_0\|_\infty =: U < 1.$$
Then $\exists!$ W. Sol. to (P). Moreover,
$$u \in L^2(0,T; W^{2,2}_{per})$$

▸ If in addition $\boxed{g \in W^{1,2}(0,T; L^2(\Omega))}$ and $\boxed{u_0 \in W^{2,2}_{per}}$

then $\frac{\partial u}{\partial t} \in L^\infty(0,T; L^2)$

and if
$$a \in \left(0, \frac{2}{d+1}\right)$$
then
$$q \in L^s(Q) \quad \text{with } s = \frac{(1-a)(d+1)}{d-1} > 1.$$

TODAY key steps of the proof of the first part.

The result in the second part indicated last time

Proof ▷ Uniqueness : Simple, do it for HW.

▷ Existence

$\boxed{\text{Step 1}}$ ε-Approximation & its solvability $\qquad \varepsilon>0$

$$(P_\varepsilon) \quad \begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \operatorname{div} q^\varepsilon = g \\ \nabla u^\varepsilon = \mathcal{F}(q^\varepsilon) + \varepsilon q^\varepsilon \\ u^\varepsilon(0,\cdot) = u_0 \end{cases}$$

$\Leftrightarrow \underline{q = L^\varepsilon(\nabla u)}$

[Lipschitz, strongly monotone]

Existence of $(u^\varepsilon, q^\varepsilon)$ solving (P_ε) is done by the Galerkin method :

$$u^N(t,x) = \sum_{r=1}^{N} c_r^N(t)\, w^r(x)$$

$$\frac{dc_s^N}{dt} = \left(\frac{du^N}{dt}, w^s\right) + \Big(\underset{L^\varepsilon(\nabla u^N)}{\overset{\|}{q^N}}, \nabla w^s\Big) = (g, w^s) \qquad s=1,\dots,N \quad (E_s)$$

$$u^N(0,x) = P^N u_0$$

It is convenient to consider $\{w^r\}_{r=1}^\infty$ as the set of eigenfunctions of the following eigenvalue problem:

to find $w^r \in W^{1,2}_{per}$.

$$(\nabla w^r, \nabla z) + \cancel{(w^r, z)} = \lambda_r (w^r, z) \qquad \forall z \in W^{1,2}_{per}$$

$$(\nabla w^s, \nabla z) = (\lambda_s w^s, z)$$

$\{w^r\}$ is orthogonal in L^2, also in $W^{1,2}$

▷ local $\exists$ (and uniqueness) follows from Picard-Lindelöf theorem.

▷ global $\exists$: then follows from the first energy estimates.

$$\boxed{\text{Estimates for } u^N, q^N}$$

1) "Testing by u" ~ Multiply (E_s) by $c_s^N(t)$, $\sum_{s=1}^{N}$ $\Rightarrow$

$$\boxed{\frac{1}{2}\frac{d}{dt}\|u^N\|_2^2 + \underbrace{(q^N, \nabla u^N)}_{\displaystyle\int_\Omega \frac{|q^N|^2}{(1+|q^N|^2)^{1/2}} + \varepsilon\underbrace{\int_\Omega |q^N|^2}} = (g, u^N)}$$

$$\nabla u^N = \underbrace{f(q^N)}_{L^\infty} + \underbrace{\varepsilon q^N}_{\in L^2}$$

$\Downarrow$

$$\underline{\nabla u^N \text{ bdd in } L^2(Q)}$$

2) "Testing by $\frac{\partial u}{\partial t}$" ~ Multiply (E_s) by $\frac{dc_s^N(t)}{dt}$, $\sum_{s=1}^{N}$

$\Rightarrow$ $\left\{\frac{\partial u^N}{\partial t}\right|$ bdd in $L^2(Q)$

3) "Testing by $-\Delta u$" to get info on spatial derivatives

Multiply (E_s) by $\lambda_s c_s^N(t)$, $\sum_{s=1}^{N}$

in E_s: i $\underline{(g, w_s)}$ ii $(g, \lambda_s c_s^N(t) w_s) = (\nabla g, c_s^N(t)\nabla w_s)$

eigenvalue problem

iii $\left(\nabla g, \sum_{s=1}^{N} c_s^N(t)\nabla w_s\right)$

$$= \left(\nabla g, \nabla \sum c_s^N(t) w_s\right)$$

$$= (\nabla g, \nabla u^N)$$

$$\Rightarrow \underline{\left(\frac{\partial \nabla u^N}{\partial t}, \nabla u^N\right) + (\nabla q^N, \nabla^2 u^N) = (\nabla g, \nabla u^N)}$$

$$\Rightarrow \frac{d}{dt}\|\nabla w^N\|_2^2 + 2\varepsilon\|\nabla q^N\|_2^2 + 2\int_\Omega |\nabla q^N|_{A(q^N)}$$

$$\le \|\nabla g\|_2^2 + \|\nabla w^N\|_2^2$$

$\boxed{\text{Limit } N \to \infty}$ is standard (omitted here)

In addition, by lower-weak semicontinuity of the norms, the derived estimates are kept for limit functions $(u^\varepsilon, q^\varepsilon)$.

$\exists!$ of $(u^\varepsilon, q^\varepsilon)$ satisfying the estimates ($*$) is established for $\varepsilon > 0$ arbitrary, but fixed

Estimates ($*$)

$\{u^\varepsilon\}$ bdd in $L^\infty(L^2)$

$\{q^\varepsilon\}$ bdd in $L^1(L^1)$

$\left\{\frac{\partial u^\varepsilon}{\partial t}\right\}$ bdd in $L^2(L^2)$

Last lecture

$$\sup_\varepsilon \int_0^T\int_\Omega |\nabla q^\varepsilon|^2_{A(q^\varepsilon)}\, dt\, dt \underbrace{}_{+\,\varepsilon\int_0^T\int_\Omega |\nabla q^\varepsilon|^2} < +\infty \Rightarrow \sup_\varepsilon \int_0^T\int_\Omega |\nabla^2 u^\varepsilon|^2 < +\infty$$

$\boxed{\text{Step 2}}$ $\boxed{\text{Limit } \varepsilon \to 0+}$ $\exists\, u, q$

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(W^{1,2}) \cap W^{1,2}(0,T;L^2) \cap L^2(W^{2,2})$$

$$\left. W^{2,2} \hookrightarrow\hookrightarrow W^{1,2} \hookrightarrow L^2 \right\} \Rightarrow \boxed{u^\varepsilon \to u \quad \text{in } L^2(0,T; W^{1,2})}$$

u $\qquad \frac{\partial u}{\partial t}$

Next we show that

(1) $q^\varepsilon \to q$ a.e.

$$\overset{}{\underset{\text{Fatou}}{\Rightarrow}} \int_Q |q| \, dx\, dt < \infty \quad \text{i.e.} \quad \underline{\underline{q \in L^1(Q)}}$$

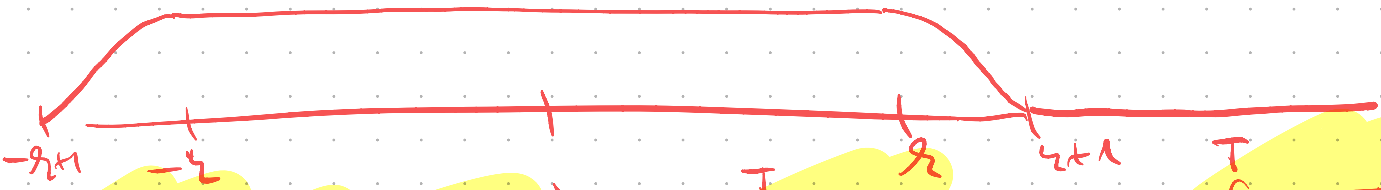
$$\Rightarrow \quad \underline{\nabla u = \mathcal{F}(q)} \quad \text{a.e.}$$

$\boxed{\text{(2) } (u,q) \text{ fulfill weak formulation by renormalization}}$

$$\left(\frac{\partial u^\varepsilon}{\partial t}, \varphi\right) + (q^\varepsilon, \nabla\varphi) = (g, \varphi)$$

$$\varphi = T_k(|q^\varepsilon|)\,\psi \qquad \psi \in W^{1,\infty}_{per}$$

$$T_k(z) \nearrow 1$$



$$\int_0^T \left(\frac{\partial u^\varepsilon}{\partial t}, T_k(|q^\varepsilon|)\psi\right) + \int_0^T \left(q^\varepsilon, T_k(|q^\varepsilon|)\nabla\psi\right) = \int_0^T \left(g, T_k(|q^\varepsilon|)\psi\right)$$

$$- \int_0^T \left(q^\varepsilon, \psi \nabla T_k(q^\varepsilon)\right)$$

$$\int_0^T \left(\frac{\partial u}{\partial t}, \psi\right) \qquad \int_0^T (q, \nabla\psi) \qquad \int_0^T (g, \psi)$$

$$\xrightarrow[\varepsilon \to 0+]{k \to +\infty} 0 \; ?$$

$$-\int \psi \frac{q^\varepsilon}{(1+|q^\varepsilon|^q)^{1/q}} \cdot \nabla T_k(|q^\varepsilon|)(1+|q^\varepsilon|^q)^{1/q} =$$

$$\mathcal{F}(q^\varepsilon)$$

$$\underbrace{T_k'(|q^\varepsilon|)(1+|q^\varepsilon|^q)^{1/q}\nabla|q^\varepsilon|}_{\nabla G_k(|q^\varepsilon|)}$$

$$-\int \psi\, \mathcal{F}(q^\varepsilon)\cdot \nabla G_k(|q^\varepsilon|)$$

$$= \int \nabla\psi\, \underbrace{\mathcal{F}(q^\varepsilon)}_{\infty}\, G_k(|q^\varepsilon|) + \int \psi\, A(q^\varepsilon)\nabla q^\varepsilon \cdot G_k(|q^\varepsilon|)$$

$$\Rightarrow |G_k(|q^\varepsilon|)| \le C(1+|q^\varepsilon|)$$

$$\Rightarrow \{\text{supp } G_k\} \subset \{|q^\varepsilon| \ge k\}$$

$k \to \infty$

$$\|q^\varepsilon\|_1 < +\infty \Rightarrow \int_{\{|q^\varepsilon|\ge k\}} |q^\varepsilon|\,dx \xrightarrow{k\to\infty} 0$$

Cauchy-Schwarz

$$\underbrace{\left(\int |\nabla q^\varepsilon|^2_{A(q^\varepsilon)}\right)^{1/2}}_{\le K < +\infty} \left(\int |G_k(|q^\varepsilon|)|^2_{A(q^\varepsilon)}\right)^{1/2}$$

$$\int \frac{|q^\varepsilon|^2}{(1+|q^\varepsilon|^q)^{1/q+1}} \le \int_{\{|q^\varepsilon|\ge k\}} |q^\varepsilon| \xrightarrow{k\to\infty} 0.$$