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Na cvičení jsme si ukázali, že řešení počáteční úlohy pro vlnovou rovnici

$$
\begin{aligned}
\Box_c u(t,x) &= f(t,x) && \text{v } \mathcal{D}'(\mathbb{R}\times\mathbb{R}^d)\\
u(0,x) &= u_0(x) && \text{v } \mathcal{D}'(\mathbb{R}^d)\\
u_{,t}(0,x) &= u_1(x) && \text{v } \mathcal{D}'(\mathbb{R}^d),
\end{aligned}
$$

kde $\Box_c u := \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2}$ a $f \equiv 0$ pro $t<0$, lze nalézt ve tvaru

$$
u(t,x) = \frac{\partial}{\partial t}(u_F \overset{(x)}{\star} u_0) + (u_F \overset{(x)}{\star} u_1) + (H\, u_F \overset{(t,x)}{\star} f), \tag{1}
$$

kde $\overset{(x)}{\star}$ značí konvoluci v prostorové proměnné $x$ a $\overset{(t,x)}{\star}$ značí konvoluci v čase i prostoru, $H = H(t)$ je Heavisideova skoková funkce. Příště si ukážeme, ze pro $d=1$ platí

$$
\boxed{u_F(t,x) = \frac{1}{2c}\chi_{[-c|t|,c|t|]}(x)\operatorname{sgn}(t)} \equiv \frac{1}{2c}\left(\chi_{K_+} - \chi_{K_-}\right)(t,x) \tag{2}
$$

kde $\chi_{[a,b]}$ značí charakteristickou funkci intervalu $[a,b]$, viz. obrázek pro vizualizaci nosiče a hodnot funkce $u_F$.

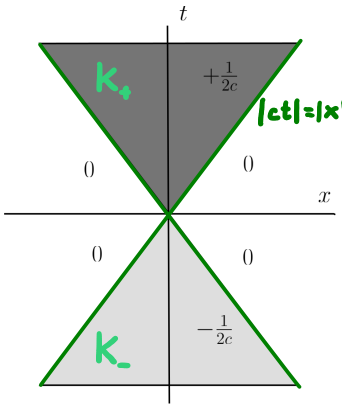
Vhodná parametrizácia $K_\pm$

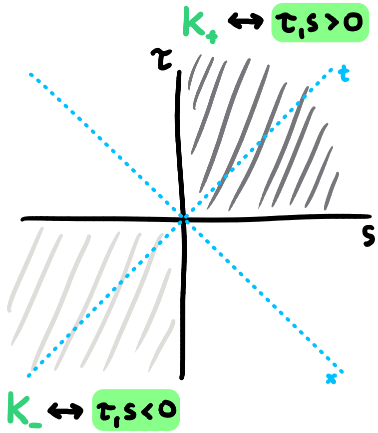
$\tau \equiv ct - x \qquad s \equiv ct + x$

$$
\Psi: \begin{cases} t = \dfrac{\tau+s}{2c} \\[4pt] x = \dfrac{s-\tau}{2} \end{cases}
$$

$$
\Rightarrow D\Psi \equiv \begin{pmatrix} \frac{\partial t}{\partial \tau} & \frac{\partial t}{\partial s} \\ \frac{\partial x}{\partial \tau} & \frac{\partial x}{\partial s} \end{pmatrix} = \begin{pmatrix} \frac{1}{2c} & \frac{1}{2c} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}
$$

$$
\Rightarrow |\det D\Psi| = \left|\frac{1}{4c} + \frac{1}{4c}\right| = \frac{1}{2c}
$$




1. Odvoďte z formule (2) výpočtem dle (1) d'Alembertovu formuli (její obecnou variantu pro nehomogenní rovnici).
2. Ověřte výpočtem, že platí

   a)
   $$\Box_c u_F = 0 \qquad \text{v } \mathcal{D}'(\mathbb{R}\times\mathbb{R})$$

   b)
   $$\Box_c(H(t)u_F) = \delta(t)\otimes\delta(x) \qquad \text{v } \mathcal{D}'(\mathbb{R}\times\mathbb{R})$$

   Nápověda: Postupujte dle definic, tedy uvažujte libovolnou testovací fci $\varphi \in \mathcal{D}(\mathbb{R}\times\mathbb{R})$ a počítejte $\langle \Box_c u_F, \varphi\rangle = \langle u_F, \Box_c\varphi\rangle$ a dále upravujte s využitím toho, že $u_F$ je evidentně regulární distribuce a můžete tedy dualitu nahradit integrací.

$\boxed{1}$

$$
\left(u_F(t,\cdot) \overset{(x)}{*} u\right)(x) \equiv \int_{\mathbb{R}} u(s)\,u_F(t,x-s)\,ds \equiv \frac{1}{2c}\int_{\mathbb{R}} u(s)\,\underbrace{\chi_{[-c|t|,c|t|]}(x-s)}\,\operatorname{sgn}(t)\,ds =
$$

$$
\begin{cases} 1 & -c|t| \le x-s \le c|t| \\ & x-c|t| \le s \le x+c|t| \\ 0 & \text{ináč} \end{cases}
$$

sgn(t) nám zruší $|\cdot|$

$$
= \frac{1}{2c}\operatorname{sgn}(t)\int_{x-c|t|}^{x+c|t|} u(s)\,ds = \frac{1}{2c}\int_{x-ct}^{x+ct} u(s)\,ds
$$

$$
\Rightarrow \frac{\partial}{\partial t}\left(u_F(t,\cdot) \overset{(x)}{*} u\right)(x) = \frac{1}{2}\left(u(x+ct) + u(x-ct)\right)
$$

[1 mínus kvôli derivácii spodnej hranice, 1 z $\frac{\partial}{\partial t}(x-ct)$]

$$
\left(H(t)u_F \overset{(t,x)}{*} f\right)(t,x) \equiv \int_{\mathbb{R}}\int_{\mathbb{R}} f(\tau,s)\,\underbrace{H(t-\tau)}_{\to 0 \text{ pre } \tau<0}\,u_F(t-\tau,x-s)\,ds\,d\tau = \left[\begin{matrix} 0 \le \tau \le t \quad \text{ak } t \le 0 \ \int \dots = 0 \\ x - c|t-\tau| \le s \le x + c|t-\tau| \end{matrix}\right] =
$$

v tomto prípade triviálne $=1$

$$= \frac{1}{2c}\,\mathrm{sgn}(t) \int_0^t \left( \int_{x-c|t-\tau|}^{x+c|t-\tau|} f(\tau,s)\,ds \right) d\tau = \frac{1}{2c} \int_0^t \left( \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\tau,s)\,ds \right) d\tau$$

Celkovo teda

$$u(t,x) = \frac{1}{2}\Big( u_0(x+ct) + u_0(x-ct) \Big) + \frac{1}{2c} \int_{x-ct}^{x+ct} u_1(s)\,ds + \frac{1}{2c} \int_0^t \left( \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\tau,s)\,ds \right) d\tau$$

$\boxed{2}$ Využijeme parametrizáciu zavedenú vyššie a

$$\frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial}{\partial x^{\mu}} \Rightarrow \begin{array}{l} \frac{\partial}{\partial \tau} = \frac{1}{2c}\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial}{\partial x} \\ \frac{\partial}{\partial s} = \frac{1}{2c}\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x} \end{array} \Rightarrow \frac{\partial}{\partial s}\frac{\partial}{\partial \tau} = \frac{1}{(2c)^2}\frac{\partial^2}{\partial t^2} - \frac{1}{4}\frac{\partial^2}{\partial x^2}$$

Takže $\Box_c = (2c)^2 \frac{\partial}{\partial s}\frac{\partial}{\partial \tau}$

a) $$\langle \Box_c u_F, \varphi \rangle \equiv \langle u_F, \Box_c \varphi \rangle \equiv \frac{1}{2c}\left[ \int_{K_+} \Box_c \varphi \, dt dx - \int_{K_-} \Box_c \varphi \, dt dx \right] =$$

$$= \cancel{\frac{1}{2c}} \left[ \int_0^\infty \int_0^\infty \cancel{(2c)^2} \frac{\partial}{\partial s}\frac{\partial}{\partial \tau}(\varphi \circ \psi) \cancel{|\det D\psi|}\, d\tau ds - \int_{-\infty}^0 \int_{-\infty}^0 \text{— || —} \right] =$$

$$= \left[ \text{2x Newtonov vzorček} \right] = \varphi(0,0) - \varphi(0,0) = 0 \equiv \langle 0, \varphi \rangle$$

b) rovnako, len bez druhého členu $[\, H(t) = 0$ na $K_- \,]$

$$\Longrightarrow \langle \Box_c (H(t) u_F), \varphi \rangle = \dots = \varphi(0,0) \equiv \langle \delta(t) \otimes \delta(x), \varphi \rangle$$