MAF II – cvičení 1.4.2021 Téma: REZIDUOVÁ VĚTA

OPAKOVÁNÍ TEORIE:

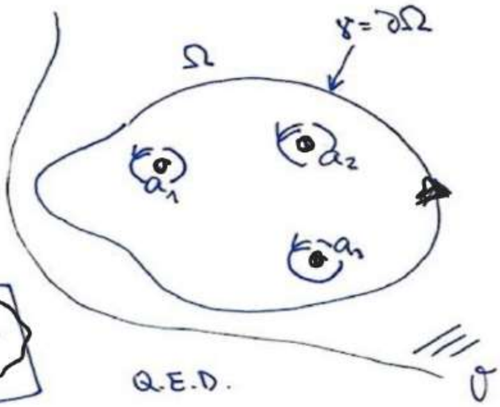
Věta 15.14 (Reziduová věta) Buď $\Omega \subset \overline{\Omega} \subset \mathcal{O} \subset \mathbb{C}$ a Ω je otevřená, omezená taková, že $\partial\Omega$ je popsána jako konečný součet regulárních vhodně orientovaných křivek (tak, aby platila pro Ω a $\partial\Omega$ Greenova věta). Nechť $A \subset \Omega$ konečná a $f \in H(\mathcal{O} \setminus A)$. Pak

$$\int_{\partial\Omega} f(z)\,dz = 2\pi i \sum_{a \in A} \operatorname{res}_a f$$

(Dk) je přímým důsledkem Cauchy-Goursatovy věty 15.5.

Všeobecně: Určitě existuje $\varepsilon > 0$ tak, že pro každé $a \in A$ $U_\varepsilon(a)$ neobsahuje žádný další bod z A. Funkce $f \in H(\mathcal{O} \setminus \bigcup_{a \in A} U_\varepsilon(a))$ a

tak $$\int_{\partial\Omega - \sum_{a \in A} \partial U_\varepsilon(a)} f(z)\,dz \overset{V15.5}{=} 0$$



Pak vidíme

$$\boxed{\int_{\partial\Omega} f(z)\,dz} = \sum_{a \in A} \int_{\partial U_\varepsilon(a)} f(z)\,dz \overset{\text{Motivační úvaha}}{=} \sum_{a \in A} 2\pi i \operatorname{res}_a f = \boxed{2\pi i \sum_{a \in A} \operatorname{res}_a f.}$$

Q.E.D.

$\boxed{\text{cíl}}$: počítat reálné integrály:
- převedeme na křivkový v $\mathbb{C}$
- využijeme rez. věta
- vypočítáme rezidua

Jak počítat rezidua?

15/35

$\boxed{\text{Tvrzení}}$ (jak určovat hodnoty residuí)

(1) Je-li $f \in H(U(a))$, pak $\operatorname{res}_a f = 0$

(2) Jsou-li $f, g \in H(U(a))$ a g má v a kořen násobnosti 1, pak $\operatorname{res}_a \frac{f}{g} = \frac{f(a)}{g'(a)}$

(3) Je-li $f \in H(U(a))$ a g má v a pól násobnosti 1 (tzn. v a je izolovaná singularita), pak $\operatorname{res}_a fg = f(a) \operatorname{res}_a g$

(4) Má-li f v a pól násobnosti ~~nejvýše~~ m,

⑤ pak $\text{res}_a f = \lim_{z\to a} \left[f(z)(z-a) \ldots \right] \frac{}{(n-1)!}$

Technické nástroje – 2 lemmata – Jordanovo a o obcházení pólu nás. 1

$\boxed{\text{Lemma A}}$ JORDANOVO $\lim_{R\to\infty}\int g(z) e^{i\gamma z}\,dz = 0$

- Buď $\langle\varphi\rangle : \langle a,b\rangle \to \mathbb{C}$ parametrizovaná zobrazením $\left[\varphi : \langle a,b\rangle \to \mathbb{C},\ \varphi(t) = Re^{it},\ t\in\langle a,b\rangle\right]$ $0 \le a < b \le 2\pi$
- Nechť f je spojitá v $\mathbb{C}\setminus B_K(0)$ pro K dostatečně velké.

Označme $M_R := \max_{z\in\langle\varphi\rangle} |f(z)|$.

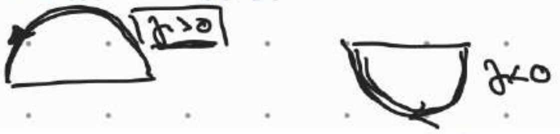
$\langle a,b\rangle \in [0,\pi]$

$\langle a,b\rangle \in [-\pi,0]$

Pak platí:

$\boxed{\gamma=0}$ (a) Je-li $\boxed{RM_R} \xrightarrow{R\to\infty} 0$, pak $\int_{\langle\varphi\rangle} f(z)\,dz \to 0$ pro $R\to\infty$

$\gamma\neq 0$ (b) Je-li $M_R \xrightarrow{R\to\infty} 0$ a $\left\{\begin{matrix}\langle a,b\rangle \subset \langle 0,\pi\rangle \text{ pro } \gamma>0 \\ \langle a,b\rangle \subset \langle -\pi,0\rangle \text{ pro } \gamma<0\end{matrix}\right\}$ pak $\int_{\langle\varphi\rangle} e^{i\gamma z} f(z)\,dz \to 0$ pro $R\to\infty$



$\boxed{\text{Lemma B}}$ O obcházení jednonásobného pólu 15/38

Má-li f v a pól násobnosti 1, a je-li γ část kružnice parametrizovaná $\varphi(t) = a + \varepsilon e^{it}$, $t \in \langle\alpha,\beta\rangle$. Pak

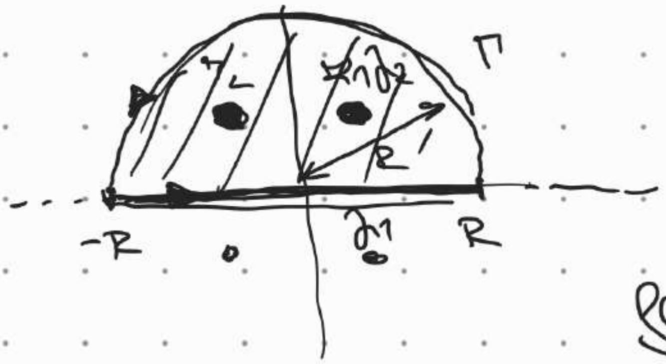
$$\int_\gamma f(z)\,dz \xrightarrow[\varepsilon\to 0]{} i(\beta-\alpha)\,\text{res}_a f$$

(P_1) $\int_0^\infty \frac{x^2+1}{x^4+1}\,dx$	(P_2) $\int_{-\infty}^\infty \frac{x\cos x}{x^2-2x+10}\,dx$	(P_3) v.p. $\int_{-\infty}^\infty \frac{dx}{(x^2+a^2)(x-\alpha)}$, $\alpha\in\mathbb{R}$
$\int_{-\infty}^\infty \frac{P(x)}{Q(x)}\,dx$	$\int_{-\infty}^\infty \frac{P(x)}{Q(x)} \cdot \sin x / \cos x \,dx$	—‖—
st. $P <$ st. $Q - 1$	st. $P <$ st. Q, póly mimo $\mathbb{R}$	1-násobné póly na $\mathbb{R}$

$$\left(x_{1,2} = \frac{2\pm\sqrt{4-40}}{2} = 1 \pm 3i \notin \mathbb{R} \right) \qquad x=\alpha$$

Př

$$I = \int_0^\infty \underbrace{\frac{x^2+1}{x^4+1}}_{\text{sudá}}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2+1}{x^4+1}\,dx$$



$$= \lim_{R\to\infty}\frac{1}{2}\int_{-R}^{R}\frac{x^2+1}{x^4+1}\,dx$$

$$f(x) = \frac{x^2+1}{x^4+1} \to \boxed{f(z) = \frac{z^2+1}{z^4+1}}$$

$\gamma_1: z=t,\ t\in[-R,R]$

$\gamma_2: z = Re^{it},\ t\in[0,\pi]$

$$\int_\Gamma f(z)\,dz = 2\pi i \underbrace{\left(\sum_{z_i\in \mathrm{Int}\,\Gamma} \mathrm{Res}\, f(z)\right)}_{\text{dobře vyčíslím a P}}$$

$$\| $$

$$\underbrace{\int_{\gamma_1} f(z)\,dz}_{I_1^R} + \underbrace{\int_{\gamma_2} f(z)\,dz}_{I_2^R}$$

- $I_1^R = \int_{-R}^{R} f(t)\,dt = \int_{-R}^{R}\frac{t^2+1}{t^4+1}\,dt \xrightarrow{R\to\infty} \underline{2I}$ do duci

- $I_2^R \xrightarrow{R\to\infty} 0$ (Jordan) b)

a) ... $f(z)$ ... klasifikace singularit?

$$f(z) = \frac{P(z)}{Q(z)} \qquad P, Q \text{ polynomy}$$

$$f(z) \in H\setminus\{z_i;\ Q(z_i)=0\}$$

? kořeny jmenovatele $z^4+1=0$ $\quad z^4 = -1 = e^{i\pi + 2k\pi i}$ $k\in\mathbb{Z}$

$z = e^{i\pi/4 + \frac{k\pi i}{2}}$ $k\in\mathbb{Z}$

$\Longleftrightarrow \quad z = e^{\dots}$

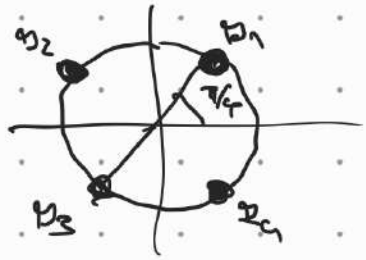
singularity $\quad z_1 = e^{i\pi/4}$

$z_2 = e^{3i\pi/4}$

$z_3 = e^{5i\pi/4}$

$z_4 = e^{7i\pi/4}$

jednonásobné póly



$$RHS = 2\pi i \left(\operatorname{Res}_{z_1} f(z) + \operatorname{Res}_{z_2} f(z) \right)$$

b) Jordanovo lemma

nejprve bez něj

$$I_2^R = \int_{\gamma_2} f(z)\,dz = \int_0^{\pi} \frac{(Re^{i\alpha})^2 + 1}{(Re^{i\alpha})^4 + 1}\, iRe^{i\alpha}\, d\alpha$$

$\gamma_2 : z = Re^{i\alpha}, \ \alpha \in [0, \pi]$

$f(z) = \dfrac{z^2+1}{z^4+1}$

$$\bullet \ |I_2^R| \le \int_0^{\pi} \left| \text{——} \right| |iRe^{i\alpha}|\, d\alpha \le$$

$$\le \int_0^{\pi} \frac{R^2+1}{|R^4-1|} R\, d\alpha = \pi \frac{R^3+R}{R^4-1} \sim \frac{1}{R} \xrightarrow{R\to\infty} 0$$

$\xrightarrow{R\to\infty} 0$

$\left(\le \frac{K}{R} \right)$

porovnal

$1 + \deg P < \deg Q$

$\boxed{\text{Jordan}}$: $\boxed{\lambda = 0}$, $f(z) = \dfrac{z^2+1}{z^4+1}$

$$\left[\quad M_R = \max |f(z)| = \max_{\gamma_2} \left| \frac{z^2+1}{z^4+1} \right| \le \frac{R^2+1}{R^4-1} \sim \frac{1}{R^2} \right.$$

$\gamma_2 \qquad z = Re^{i\alpha}$

$$R \cdot M_R = \frac{R^2+1}{R^4-1} R \xrightarrow{R\to\infty} 0$$
$$\sim \frac{1}{R}$$

$$\Rightarrow \lim_{R\to\infty} \int_{\gamma_2^R} f(z)\,dz = 0$$ (dle Jordanova lemmatu)

zbývá spočítat rezidua

$\lim_{R\to\infty}$:
$$2I = 2\pi i \left(\operatorname{Res}_{z_1}\left(\frac{z^2+1}{z^4+1}\right) + \operatorname{Res}_{z_2}\left(\frac{z^2+1}{z^4+1}\right)\right)$$

z_1 i z_2 jsou 1-násob. póly „h/g'" (z_1 [illegible])

dle věty 2
$$\operatorname{Res}_{z_1}\left(\frac{z^2+1}{z^4+1}\right) = \frac{z^2+1}{4z^3}\Bigg|_{z\to z_1} =$$

$$= \frac{z_1^2+1}{4z_1^3} = \frac{e^{i\pi/2}+1}{4\underbrace{e^{3i\pi/4}}} = \frac{(1+i)}{4} e^{-\frac{3i\pi}{4}}$$

$z_1 = e^{i\pi/4}$ $\qquad -\frac{1+i}{\sqrt{2}}$

$$= \frac{(1+i)}{4}\left(\frac{-1-i}{\sqrt{2}}\right) = -\frac{(1+i)^2}{4\sqrt{2}} = -\frac{(\cancel{1}+2i-\cancel{1})}{4\sqrt{2}}$$

$$= \frac{-i}{2\sqrt{2}}$$

$$\operatorname{Res}_{z_2} f(z) = \text{„stejně"} = \frac{z^2+1}{4z^3}\Bigg|_{z\to z_2} =$$

$\frac{3\pi}{}$

$e^{\frac{\ }{4}}$

$= \dots \frac{-i}{2\sqrt{2}}$ (sami!

$$2I = \cancel{2}\pi i \left(-\frac{i}{\cancel{2}\sqrt{2}} - \frac{i}{\cancel{2}\sqrt{2}} \right) = \frac{2\pi}{\sqrt{2}}$$

$$\Rightarrow \boxed{I = \frac{\pi}{\sqrt{2}}}$$

Q: co kdyby

$$2I + \underset{\substack{\downarrow \\ 0}}{I_2} = -2\pi i \left(\mathrm{Res}_{z_3} f(z) + \mathrm{Res}_{z_4} f(z) \right)$$

SAMi

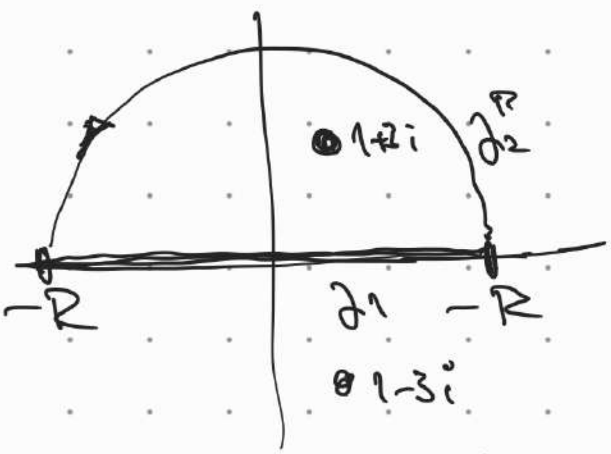
(P) $I = \int_{-\infty}^{\infty} \frac{x}{x^2-2x+10} \underbrace{\cos x}_{\mathrm{Re}(e^{ix})} \, dx = \mathrm{Re} \int_{-\infty}^{\infty} \frac{x}{x^2-2x+10} e^{ix} \, dx$

?

$$f(z) = \frac{z}{z^2-2z+10} e^{iz}$$

$$f(z) = \underline{g(z)\, e^{1 \cdot iz}}$$

Jordan pro $\boxed{\lambda = 1} > 0$



γ_1^R: $z = t$, $t \in [-R, R]$

γ_2^R: $z = Re^{it}$, $t \in [0, \pi]$

$f(z)$ ... ? singularity -- kořeny jmenovatele

$$z^2 - 2z + 10 = 0$$

$$z_{1,2} = \frac{2 \pm \sqrt{4-40}}{2} = \underline{1 \pm 3i}$$

$1+3i \in \mathrm{Ind}\,\Gamma$ 1-rész. pól.

- $I_1^R = \int_{-R}^{R} \frac{t}{t^2-2t+10} e^{it}\,dt \xrightarrow{R\to\infty} =: J \quad (\mathrm{Re}\,J = I)$

- $I_2^R \xrightarrow{R\to\infty} 0$?

Jordan: $f(z) = e^{1 \cdot iz} \underbrace{\frac{z}{z^2-2z+10}}_{g(z)}$

$$\underline{M_R} = \max_{\substack{\gamma_2 \\ z=Re^{it}}} |g(z)| \leq \frac{R}{R^2-2R-10} \sim \underline{\frac{1}{R}}$$

$\lambda = 1$

$\boxed{\lambda > 0}$: $\underline{M_R} \xrightarrow{R\to\infty} 0$ ✓ $\overset{\text{Jordan}}{\Longrightarrow}$ $\lim_{R\to\infty} I_2^R = 0$ ✓

rez. vétel $\int_{\Gamma_R} f(z)\,dz = \boxed{2\pi i}\, \underline{\mathrm{Res}_{1+3i} f(z)}$

$$I_1^R + I_2^R$$

$$I_1^R \xrightarrow{R\to\infty} J, \qquad I_2^R \xrightarrow[(\text{Jordan})]{R\to\infty} 0$$

$$\mathrm{Res}_{1+3i}\left(\frac{z e^{iz}}{z^2-2z+10}\right) \overset{h/g'}{=} \frac{z e^{iz}}{2z-2}\Bigg|_{z\to 1+3i}$$

$$= \frac{(1+3i)\, e^{i(1+3i)}}{2\cdot 3i} = \frac{(1+3i)}{6i} e^{-3+i} =$$

$$= \frac{(1+3i)}{6i} e^{-3} (\cos 1 + i \sin 1)$$

$$J = 2\pi i \left(\quad \right) = \frac{\pi}{3}(1+3i) e^{-3} (\cos 1 + i \sin 1)$$

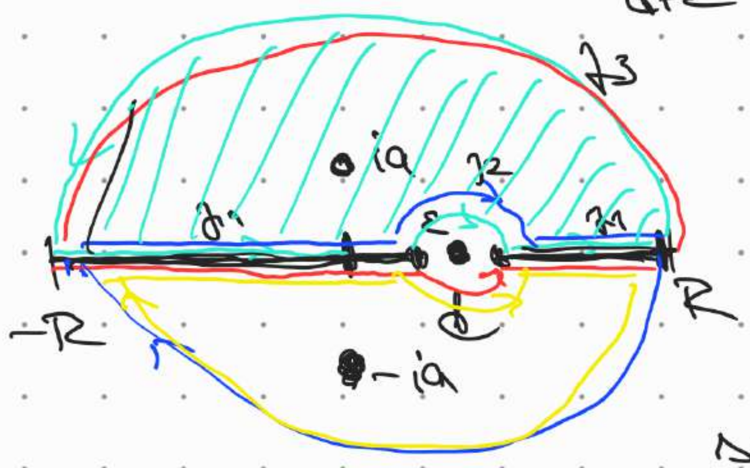
$$= \frac{\pi}{3} e^{-3} \left(\cos 1 - 3 \sin 1 + i \left(\quad \right) \right)$$

$$I = \operatorname{Re} J = \frac{\pi}{3} e^{-3} (\cos 1 - 3 \sin 1)$$

Pr $I = \text{v.p.} \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x-d)}$ $\quad d \in \mathbb{R}$

$$\stackrel{\text{def}}{=} \lim_{\varepsilon \to 0+} \left\{ \int_{-\infty}^{d-\varepsilon} \frac{1}{(x^2+a^2)(x-d)} dx + \int_{d+\varepsilon}^{\infty} \frac{1}{(x^2+a^2)(x-d)} dx \right\}$$

$$= \lim_{\substack{R \to \infty \\ \varepsilon \to 0+}} \int_{-R}^{d-\varepsilon} + \int_{d+\varepsilon}^{R}$$



$f(z) = \frac{1}{(z^2+a^2)(z-d)}$

sing $\pm ia, d$

π-nás.

$f(z)$ singularity: $f(z) = \dfrac{1}{(z+ia)(z-ia)(z-d)}$ $\quad a>0$

obr

$\gamma_1^{R\varepsilon}: \; z = t, \; t \in [-R, d-\varepsilon] \cup [d+\varepsilon, R]$

$\gamma_2^{\varepsilon}: \; z = d + \varepsilon e^{it}, \; t \in [\pi, 0]$

$\gamma_3^{R}: \; z = Re^{it}, \; t \in [0, \pi]$

REZ. VĚTA: $\displaystyle\int_{\Gamma^{R\varepsilon}} f(z)\,dz = 2\pi i \sum_{z_i \in \operatorname{Int}\Gamma} \operatorname{Res}_{z_i} f(z)$

$I_1 + I_2 + I_3 \overset{\text{obr 1.}}{=} 2\pi i \operatorname{Res}_{ia} f(z)$

- $I_1^{R\varepsilon} \xrightarrow[\varepsilon \to 0+]{R\to\infty}$ V.P. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x-d)}\,dx = I$ (hledaný)

- $I_3^{R} \xrightarrow{R\to\infty} 0$ ✓ (Jordan) $(\lambda = 0)$

$$M_R \sim \frac{1}{R^2} \qquad R \cdot M_R \sim \frac{1}{R^2} \xrightarrow{R\to\infty} 0 \; ✓$$

- $I_2^{\varepsilon} \xrightarrow{\varepsilon\to 0+} \, ?$

lemma o obcházení pól nás. 1:

$$I_2^{\varepsilon} \xrightarrow{\varepsilon \to 0} -i\pi \cdot \operatorname{Res}_d f(z)$$

$$I = i\pi \operatorname{Res}_{d} f(\omega) + 2\pi i \operatorname{Res}_{ia} f(\omega)$$

$$= \dots$$

$$= -\frac{\pi d}{a(a^2+d^2)}$$