7. LINEAR OPERATORS IN HILBERT SPACES

Let H be a vector space over $\mathbb{K}$. We say that H is the space with scalar product (or the space with inner product; or pre-Hilbert space) if there is a map $H \times H \to \mathbb{R}$ so that

(1) $(x,x)_H \geq 0 \quad \forall x \in H$ and equality holds if and only if $x=0$

(2) $(x+y,z) = (x,z) + (y,z)$ and $(\alpha x, z) = \alpha (x,z)$ $\quad \forall x,y,z \in H \quad \forall \alpha \in \mathbb{K}$

(3) $(x,y) = \overline{(y,x)}$

NOTE that
- $(x,y+z) = \overline{(y+z,x)} = \overline{(y,x)} + \overline{(z,x)} = (x,y) + (x,z)$
- $(x,\alpha z) = \overline{(\alpha z, x)} = \bar{\alpha}\overline{(z,x)} = \bar{\alpha}(x,z)$

Recall:
- $\|x\|_H := \sqrt{(x,x)_H}$ defines a norm on H
- if H is complete, then $\boxed{\text{H is called a Hilbert space}}$ $(H, \|\cdot\|_H)$
- two basic inequalities: $|(x,y)_H| \leq \|x\|_H \|y\|_H \quad \forall x,y \in H$ Cauchy-Schwarz-Buniakovski

  $\|x+y\|_H \leq \|x\|_H + \|y\|_H \quad \forall x,y \in H$ Minkowski or Δ-ineq.!
- nice identity called parallelogram identity (with respect to the rule in $\mathbb{R}^2$)

$$\|x+y\|_H^2 + \|x-y\|_H^2 = 2\|x\|_H^2 + 2\|y\|_H^2$$

Pf $(x+y,x+y) + (x-y,x-y) = 2\|x\|_H^2 + (x,y) + (y,x) + 2\|y\|_H^2 - (x,y) - (y,x)$ ∎

Hilbert spaces "differ" from Banach spaces by the presence of scalar product. This additional structural property allows one: to generalize the concept of orthogonality from $\mathbb{R}^d$ to H

to characterize/describe well all bdd linear functionals on H, i.e. to characterize H'.

7.1 Ortogonality

- Given $S \subseteq H$, define $\boxed{\operatorname{span}(S) \equiv \left\{ \sum_{i=1}^{N} \alpha_i x_i \ , \alpha_i \in \mathbb{K}, x_i \in S, N \geq 1 \right\}}$

Then $\operatorname{span}(S)$ is a subspace, but <u>not</u> necessarily closed.

define $V := \overline{\operatorname{span}(S)}$ ... a space generated by S

If $V = H$, then one says that S is total.
It means, if S is total, then $\forall x \in H \ \exists x_n \in \operatorname{span} S$ so that $\|x_n - x\|_H \to 0$.

- We say that $x, y \in H$ are orthogonal $\overset{df.}{\equiv} (x,y)_H = 0$.

- Given $S \subseteq H$, define $S^\perp \equiv \{ y \in H ; (y,x) = 0 \ \ \forall x \in S \}$.

<u>NOTE</u> $S^\perp$ is always <u>a closed</u> subspace of H. Verify!

Theorem 7.1 Let H be Hilbert and $V \subset H$ a <u>closed</u> subspace of H.

Then
(i) $\boxed{H = V \oplus V^\perp}$ i.e. $\forall x \in H \ \exists! \ y \in V$ and $z \in V^\perp$: $x = y + z$

(ii) y is the unique point in $\boxed{V}$ having minimal distance from x; $y \equiv P_V(x)$
z — " — in $\boxed{V^\perp}$ — " — x; $z = P_{V^\perp}(x)$

(iii) The perpendicular projections $x \mapsto P_V(x)$ and $x \mapsto P_{V^\perp}(x)$ are linear continuous with the norm ≤ 1.
In fact, if $V \neq \{0\}$ then $\|P_V\|_{\mathcal{L}(H,H)} = \|P_{V^\perp}\|_{\mathcal{L}(H,H)} = 1$.

Proof **Step 1** Let $x \in H$ be arbitrary, but fix.
Let $\alpha := d(x, V) = \inf_{y \in V} \|x - y\|_H$. Then $\exists \ y_n \in V$ so that $\lim_{n \to \infty} \|x - y_n\|_H = \alpha$. We shall show that $\{y_n\}$ is Cauchy.

If so and as V is closed, V is complete. Hence there is $y \in V$: $y_n \xrightarrow[\text{in } H]{} y$ and $\|x - y\|_H = \alpha$.

We need to show that y is a unique point satisfying $\|x - y\|_H = \alpha$.

Step 2 The facts that $\boxed{\{y_n\}_{n=1}^{\infty} \text{ is Cauchy}}$ and $\boxed{y \text{ is unique}}$ are proved using very similar arguments.

Recalling
$$\|u + v\|_H^2 + \|u - v\|_H^2 = 2\|u\|_H^2 + 2\|v\|_H^2 \qquad \forall u, v \in H$$

and taking $u = x - y_n$ and $v = x - y_m$, we get

$$\| y_n - y_m \|_H^2 = 2 \| x - y_m \|_H^2 + 2 \| x - y_n \|_H^2 - 4 \left\| x - \frac{y_n + y_m}{2} \right\|_H^2$$

Since $y_n, y_m \in V$, then $\frac{y_n + y_m}{2} \in V$ and $\left\| x - \frac{y_n + y_m}{2} \right\|_H^2 \geq \alpha^2$

Hence

$$\limsup_{m,n \to \infty} \| y_n - y_m \|_H^2 \leq 2 \limsup_{m \to \infty} \| x - y_m \|_H^2 + 2 \limsup_{n \to \infty} \| x - y_n \|_H^2 - 4 \liminf_{n,m \to \infty} \left\| x - \frac{y_n + y_m}{2} \right\|_H^2$$

$$\leq 2\alpha^2 + 2\alpha^2 - 4\alpha^2 = 0.$$

Similarly, if y and y' would be two minimizing points, then ($u := x - y$, $v = x - y'$)

$$\| y - y' \|_H^2 = 2 \| x - y \|_H^2 + 2 \| x - y' \|_H^2 - 4 \Big\| x - \underbrace{\frac{y + y'}{2}}_{\in V} \Big\|_H^2$$

$$\leq 4\alpha^2 - 4\alpha^2 = 0.$$

Hence $y = y'$. It means that the mapping (P_V) that gives to any $x \in H$ a point $y \in V$ (having the minimal distance) is well-defined.

$\boxed{\text{Step 3}}$ We show that $P_V(x)$ can be characterised as the unique $y \in V$: $x - y \in V^\perp$.

$\boxed{\text{Existence}}$ For arbitrary $v \in V$ consider the mapping

$$\underset{\in \mathbb{R}}{t} \longmapsto \| x - (y + tv) \|_H^2 = \| x - y \|_H^2 + t^2 \| v \|_H^2 + 2 \operatorname{Re}(x - y, tv)$$

By Step 1 and 2, this mapping attains minimum at $t = 0$.

Hence $\operatorname{Re}(x - y, v) = 0 \quad \forall v \in V$

Replacing v by $-iv$, we conclude that

$$\operatorname{Im}(x - y, v) = \operatorname{Re}(x - y, -iv) = 0.$$ Hence $x - y \in V^\perp$

$\boxed{\text{Uniqueness}}$ If there are two such points $y, y' \in V$, then

$$\| y - y' \|_H^2 = (y - y', y - y') = (\underbrace{y - y'}_{\in V}, \underbrace{x - y'}_{\in V^\perp}) - (\underbrace{y - y'}_{\in V}, \underbrace{x - y}_{\in V^\perp}) = 0.$$

$\boxed{\text{Step 4}}$ Properties of P_V and $P_{V^\perp}$.

- If $y = P_V(x)$, $y' = P_V(x')$, then for $\alpha, \alpha' \in \mathbb{K}$, $\alpha y + \alpha' y' \in V$ and $\alpha x + \alpha' x' - \alpha y - \alpha' y' \in V^\perp$.

  Hence, by Step 3, $P_V(\alpha x + \alpha' x') = \alpha y + \alpha' y'$. Hence P_V is linear. As $P_{V^\perp} = I - P_V$, $P_{V^\perp}$ is linear as well.

- $$\| x \|_H^2 = \| \underbrace{x - P_V(x)}_{V^\perp} + \underbrace{P_V(x)}_{V} \|_H^2 = \| x - P_V(x) \|_H^2 + \| P_V(x) \|_H^2$$ (Pythagora's theorem)

  $$\Rightarrow \sup_{\|x\|_H = 1} \| P_V(x) \|_H \leq 1, \quad \sup_{\|x\|_H = 1} \| (I - P_V)(x) \|_H^2 \leq 1.$$

  If $V \neq \{0\}$, then $P_V(x) = x$ for $x \in V$.

$\boxed{7.2}$ Linear functionals on a Hilbert space. Riesz representation theorem.

$\boxed{\text{Theorem 7.2}}$ Let H be a Hilbert space. Then it holds:

(1) For every $x \in H$: $y \mapsto (y, x)_H \in H^*$, i.e. $\phi^x := \boxed{y \mapsto (y,x)_H}$ is linear continuous map of H into $\mathbb{K}$.

$\boxed{\text{Mapping: } \underset{\in H}{x} \mapsto \overset{\in H^*}{\phi^x} \text{ is isometry.}}$

(2) For every $\phi \in H^*$: $\exists!\, a \in H$ $\langle \phi, x \rangle_{H^*, H} = \phi(x) = (a, x)_H$ for all $x \in H$.

$\boxed{\text{Proof}}$ $\boxed{\text{Ad (1)}}$ Simple. Do it yourself! Or see below!

$\boxed{\text{Ad (2)}}$ Let $\phi \in H^*$ be given. ▶ If $\phi(y) = 0$ for all $y \in H$, the the conclusion holds with $a = 0$.

▶ If ϕ is nontrivial, then $\operatorname{Ker}\phi$ is closed subspace of H that is proper, i.e. $\operatorname{Ker}\phi \neq V$. Then $\exists\, b \in [\operatorname{Ker}\phi]^\perp$ that can be normalized; $\|b\|_H = 1$. Since <u>for any fixed $x \in H$</u>:

$$\phi\big(b\phi(x) - x\phi(b)\big) = \phi(b)\phi(x) - \phi(x)\phi(b) = 0,$$

the vector $b\phi(x) - x\phi(b) \in \operatorname{Ker}\phi$ and is then ortogonal to b, which implies that

$$0 = b \cdot b\phi(x) - b \cdot x\phi(b) \Rightarrow \phi(x) = \overline{\phi(b)}\, b \cdot x.$$

Hence, setting $a = \overline{\phi(b)}\, b$, we are done with the existence part of Theorem.

▶ Regarding uniqueness, assume that there are two points $a_1, a_2 \in H$ such that $\phi(x) = (a_i, x)$ $\forall x \in H$. Then $(a_1 - a_2, x) = 0$ $\forall x \in H$, which however implies that $a_1 = a_2$.

For the sake of completness, we add a proof of the part (1).

$\boxed{\text{Ad (1)}}$ On one hand, we have: $|\phi^x(y)| \le \|y\|_H \|x\|_H \Rightarrow \boxed{\|\phi^x\|_{H^*} \le \|x\|_H}$.

Note that linearity of ϕ^x is trivial. Hence $\phi^x \in \mathcal{L}(H; \mathbb{K}) = H^*$.

On the other hand, for $\frac{x}{\|x\|}$, which is at the unit sphere in H, we have

$$\phi^x\left(\frac{x}{\|x\|}\right) = \frac{x \cdot x}{\|x\|_H} = \|x\|_H,$$

which leads to $\boxed{\|x\|_H \le \|\phi^x\|_{H^*}}$.

Hence $\|x\|_H = \|\phi^x\|_{H^*}$ ///

$\boxed{\text{Theorem 7.3}}$ (Dual to H or $\boxed{H^* = H}$) Let H be a Hilbert space.

A mapping that maps $\phi^{\in H^*}$ to $a^{\in H}$ is one-to-one isometry of H^* onto H and denoting this mapping Ψ we have

$$\Psi(\phi_1 + \phi_2) = \Psi(\phi_1) + \Psi(\phi_2) \quad \text{and} \quad \Psi(\alpha\phi) = \bar{\alpha}\Psi(\phi)$$
$$\forall \phi_1, \phi_2 \in H^* \quad \forall \alpha \in \mathbb{K}.$$

If $\mathbb{K} = \mathbb{R}$, then Ψ is bijection isometry of H^* onto H.

Pf It follows from the previous theorems that $\Psi : \phi^{\in H^*} \mapsto a^{\in H}$ is bijection and isometry.

If $\phi_1, \phi_2 \in H^*$ and $a_1 = \Psi(\phi_1)$, $a_2 = \Psi(\phi_2)$ then

for all $x \in H$: $(\phi_1 + \phi_2)(x) = (a_1, x) + (a_2, x) = (a_1 + a_2, x)$

As Ψ is one-to-one

$$\Psi(\phi_1 + \phi_2) = a_1 + a_2 = \Psi(\phi_1) + \Psi(\phi_2).$$

▨

$\boxed{\text{Theorem 7.4}}$ (Reflexivity of Hilbert spaces)

Every Hilbert space is reflexive

Pf Pick any $\Phi \in H^{**}$. Our goal is to find $x \in H$ so that

$$J_x = \Phi \qquad \text{where } J : H \to H^{**} \text{ is canonical embedding.}$$
$$\Updownarrow$$
$$\varphi(x) = \boxed{\langle J_x, \varphi \rangle = \Phi(\varphi) \quad \forall \varphi \in H^*}$$

Consider, for all $h \in H$, mapping $\varphi_h : x \mapsto (x, h)$ $(x \in H)$

Then $\varphi_h \in H^*$. ~~Furthermore~~,

$$h \mapsto \overline{\Phi(\varphi_h)} \quad \text{maps } H \to \mathbb{K}$$

and it is bounded and linear functional.

By Riesz representation theorem : $\exists!\, x \in H$

$$\overline{\Phi(\varphi_h)} = (h, x)_H \quad \forall h \in H.$$

To conclude: for $\varphi \in H^*$ we find $h \in H$ so that $\varphi = \varphi_h$

then $\Phi(\varphi) = \Phi(\varphi_h) = \overline{(h, x)} = (x, h) = \varphi_h(x) = \varphi(x) = J_x(\varphi)$.

▨

Well-posedness for positive definite operators. Lax-Milgram Lemma.

A basic problem of LA: $\boxed{\text{to solve } Ax=b,\ A\in\mathbb{R}^{n\times n},\ b\in\mathbb{R}^n.}$

If $Ax\cdot x>0\ \forall x\neq 0$, then LA says: $\exists!\ x\in\mathbb{R}^n$ solving the problem.

Similar result holds in infinite-dimensional (Hilbert) spaces.

Def (H over $\mathbb{R}$) $A: H\to H$ is strictly positive definite:

$$\exists\beta>0:\quad (Au,u)\geq\beta\|u\|_H^2\quad\forall u\in H. \quad (*)$$

Theorem 7.5 H .. Hilbert over $\mathbb{R}$. Let $A\in\mathcal{L}(H;H)$ fulfills $(*)$.
Then $\forall f\in H\ \exists!\ u\in H$ ($u:=A^{-1}f$) so that $Au=f$.
The inverse $A^{-1}\in\mathcal{L}(H,H)$ satisfies $\|A^{-1}\|_{\mathcal{L}(H,H)}\leq\frac{1}{\beta}$.

Pf Goal: to show that $(*)$ implies A is onto and one-to-one.

Step 1 $(*)$ implies that

$$\beta\|u\|_H^2=(Au,u)\leq\|Au\|_H\|u\|_H\Rightarrow\boxed{\beta\|u\|_H\leq\|Au\|_H\quad\forall u\in H}\quad(**)$$

Hence if $Au=0$, then $u=0$ and A is one-to-one.

Step 2 $\boxed{\text{Range } A \text{ is closed}}$ Consider $v_n\in\text{Range } A$, $v_n\to v$ in H.
We look for $u\in H$: $Au=v$. However, by assumption $v_n=Au_n$ for some $u_n\in H$. Hence

$$\|u_n-u_m\|\leq\frac{1}{\beta}\|Au_n-Au_m\|=\frac{1}{\beta}\|v_m-v_n\|\longrightarrow 0$$

Hence $\{u_m\}_{m\in\mathbb{N}}$ is Cauchy and $\exists u\in H$: $u_n\to u$ in H
But, by Heine def. of continuity, $Au_n\to Au$ in H
Hence $Au=v$.

Step 3 $\boxed{\text{Range } A=H}$. If not, as Range A is closed, $\exists\ w\neq 0$ such that $w\in[\text{Range}]^\perp$ (see Theorem 7.1). But then

$$\beta\|w\|_H^2\leq(Aw,w)=0,$$

which gives ↯ with $w\neq 0$.

Step 4 As $A\in\mathcal{L}(H,H)$ is bijection, $Au=f$ has ! solution for any $f\in H$, denoted $A^{-1}f$. Then by $(**)$

$$\beta\|A^{-1}f\|_H=\beta\|u\|_H\leq\|Au\|_H=\|f\|_H,$$

which implies that $\|A^{-1}f\|_H\leq\frac{1}{\beta}$. Koniec

$\boxed{\text{Theorem 7.6}}$ Let H be a Hilbert space and $\mathbb{K} = \mathbb{R}$. Assume that $B : H \times H \to \mathbb{R}$ satisfies

LINEARITA
$$\begin{cases} B(\alpha u_1 + \beta u_2, v) = \alpha B(u_1, v) + \beta B(u_2, v) & \forall u_1, u_2, v \in H \quad \forall \alpha_1, \alpha_2 \in \mathbb{R} \\ B(u, \gamma v_1 + \delta v_2) = \gamma B(u, v_1) + \delta B(u, v_2) & \forall u, v_1, v_2 \in H \quad \forall \beta_1, \beta_2 \in \mathbb{R} \end{cases}$$

BOUNDEDNESS $\quad \exists C > 0 : \quad |B(u,v)| \leq C \|u\|_H \|v\|_H \quad \forall u, v \in H$

H-coercivity $\quad \exists \beta > 0 \quad B(u,u) \geq \beta \|u\|_H^2 \quad \forall u \in H.$

Then, for every $f \in H$, $\exists!\, u \in H$:
$$B(u, \varphi) = (f, \varphi) \quad \forall \varphi \in H$$

Moreover,
$$\|u\|_H \leq \frac{\|f\|}{\beta}$$

(Pf) For every $u \in H$: $\varphi \longmapsto B(u, \varphi) \in H^*$. Hence, by Riesz representation theorem 7.2, $\exists!$ vector in H, we call it Au so that
$$B(u, \varphi) = (Au, \varphi)_H \quad \forall \varphi \in H.$$

We will show that A is bounded, strictly positive definite linear operator.

$\boxed{\text{Linearity}}$
$$(A(u+v), \varphi) = B(u+v, \varphi) \underset{\text{lin. of } B}{=} B(u, \varphi) + B(v, \varphi)$$

$\boxed{\text{Boundedness}}$
$$\|Au\|_H = \sup_{\|\varphi\|_H = 1} |(Au, \varphi)| \leq \sup_{\|\varphi\|_H = 1} |B(u, \varphi)| \underset{\text{boundedness of } B}{\leq} C \|u\|_H$$

which implies
$$\|A\|_{\mathcal{L}(H,H)} \leq C$$

$\boxed{\text{Strict positive definiteness}}$ follows from
$$(Au, u) = B(u,u) \geq \beta \|u\|_H^2$$

Hence, by Theorem 7.5, we conclude that $Au = f$ has unique solution $u = A^{-1} f$ satisfying $\|u\|_H \leq \frac{\|f\|_H}{\beta}$.

Application of Lax-Milgram lemma: Dirichlet and Neumann problems for linear elliptic equations

Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with C^1 boundary $\partial\Omega$ consisting of two disjoint parts Γ_N and Γ_D so that $\partial\Omega = \Gamma_N \cup \Gamma_D$.
Let $\mathbb{A} : \Omega \to \mathbb{R}^{d\times d}$ satisfy:

(1) $\exists \beta > 0: \quad \mathbb{A}(x)\xi \cdot \xi \geq \beta |\xi|^2 \quad$ for $\forall \xi \in \mathbb{R}^d$ and a.a. $x \in \Omega$.

For any $f : \Omega \to \mathbb{R}$, we look for $u : \Omega \to \mathbb{R}$ solving

(P)
$$\boxed{\begin{aligned} -\operatorname{div}\left(\mathbb{A}(x)\nabla u\right) &= f && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma_D \\ \mathbb{A}(x)\nabla u \cdot n &= 0 && \text{on } \Gamma_N \end{aligned}}$$

← This is due to (1) an elliptic equation
← homogeneous Dirichlet bc's
← homogeneous Neumann bc's

Our goal is to show that for any $f \in L^2(\Omega)$ (or even more general) there exists unique weak solution u to (P).

We proceed as follows:

1. via apriori estimate we „identify" proper function space
2. define weak solution to (P) and show its compatibility with the concept of classical solution
3. formulate well-posedness result within the context of weak solutions
4. Prove this result as an application of Lax-Milgram lemma and Poincaré's inequality.

Ad 1 Multiply $(P)_1$ by u, integrate over Ω and use Gauss theorem: (together with $(P)_{2,3}$)

$$\int_\Omega \mathbb{A}(x)\nabla u \cdot \nabla u \, dx = \int_\Omega f u \, dx$$

This and (1) lead to

$$\beta \|\nabla u\|_2^2 \leq \|f\|_2 \|u\|_2$$

Assume that there is $c_P > 0$:

(2)
$$\boxed{\|u\|_2 \leq c_P \|\nabla u\|_2 \quad \text{for all "admissible" } u}$$

then one concludes

$$\beta\|\nabla u\|_2^2 \le c_P\|f\|_2\|\nabla u\|_2 \overset{\text{Young}}{\le} \frac{\beta}{2}\|\nabla u\|_2^2 + \frac{c_P^2\|f\|_2^2}{2\beta},$$

and thus

$$\beta\|\nabla u\|_2^2 \le \frac{c_P^2}{\beta}\|f\|_2^2,$$

which together with (2) implies

$$\boxed{(3)\qquad \|u\|_{1,2}^2 \le c_P^2\|\nabla u\|_2^2 + \frac{c_P^2}{\beta^2}\|f\|_2^2 \le \frac{c_P^2}{\beta^2}(1+c_P^2)\|f\|_2^2,}$$

where

$$(4)\qquad \|u\|_{1,2}^2 := \|u\|_2^2 + \|\nabla u\|_2^2.$$

Conclusion: we get apriori bound (3): $\boxed{\|u\|_{1,2} \le \frac{c_P}{\beta}\sqrt{1+c_P^2}\,\|f\|_2}$ provided that the inequality (2) holds.

The inequality (2) is called Poincaré's inequality. It does not hold in general: consider constant functions. It reveals that if constant functions are eliminated (by bc's or by zero mean-value condition), then (2) holds.

The apriori estimate suggest the "correct" function space, where the solution should be looked for.

Definition Sobolev spaces

$$W^{1,2}(\Omega) := \{u \in L^2(\Omega);\ \underbrace{\frac{\partial u}{\partial x_i} \in L^2(\Omega) \text{ for } i=1,2,\dots,d}_{\text{i.e. } \nabla u \in L^2(\Omega)^d := L^2(\Omega)\times\dots\times L^2(\Omega)}\}$$

$$W^{1,2}_{\Gamma_D} := \{u \in W^{1,2}(\Omega);\ u|_{\Gamma_D} = 0\}$$

$$W := \{u \in W^{1,2}(\Omega);\ \int_\Omega u(x)\,dx = 0\}.$$

$$W^{1,2}_0(\Omega) := \{u \in W^{1,2}(\Omega);\ u|_{\partial\Omega} = 0\} \quad (= W^{1,2}_{\partial\Omega})$$

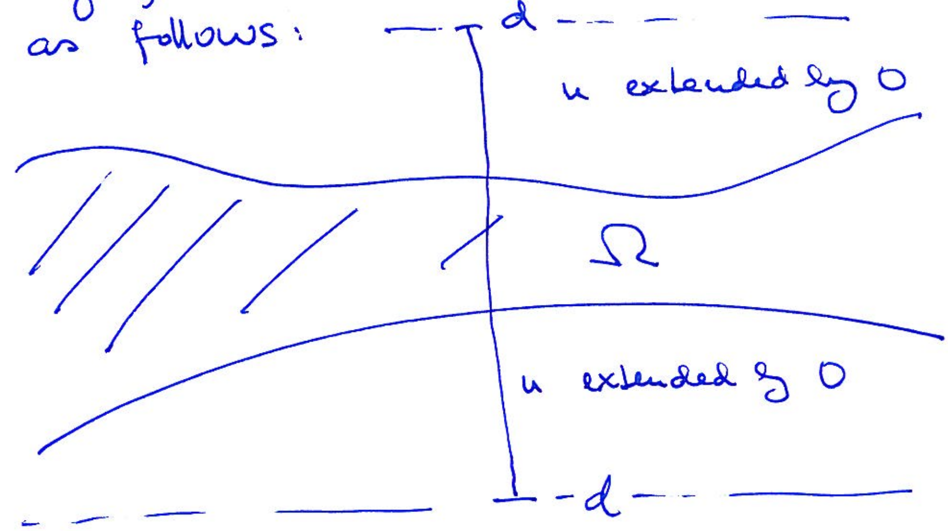
Note

- $W^{1,2}_0(\Omega) \subset W^{1,2}_{\Gamma_D} \subset W^{1,2}(\Omega)$
- W and $W^{1,2}_{\Gamma_D}$ and $W^{1,2}_0(\Omega)$ are closed subspaces of $W^{1,2}(\Omega)$

Lemma (Poincaré inequality) The inequality (2) holds for all $u \in W^{1,2}_{\Gamma_D}$ if $\Gamma_D \neq \emptyset$. It also holds for $u \in W$.

(Pf) Under even other situations, see PDE I course.

If Ω can be placed into a channel, say $[-d,d] \times \Omega'$, see Fig 1, and $u=0$ on $\partial\Omega$ then we can proceed as follows:



For $x \in \Omega$:

$$u(x) - \underbrace{u(-d,x')}_{=0} = \int_{-d}^{x_1} \frac{\partial u}{\partial x_1}(s,x_2,\dots,x_d)\,ds$$

Hence

$$|u(x)|^2 \le \left(\int_{-d}^{d} 1 \cdot \left|\frac{\partial u}{\partial x_1}(s,x_2,\dots,x_d)\right| ds\right)^2 \le 2d \int_{-d}^{d} |\nabla u(s,x_2,\dots,x_d)|^2 ds$$

By integration over Ω

$$\|u\|_2^2 \le 4d^2 \|\nabla u\|_2^2 \text{, which is (2) with } c_p = 2d.$$

Consequently:

$\boxed{\text{on } W^{1,2}_{\Gamma_D}}$ the norms $\|\nabla u\|_2$ and $\left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^{1/2}$ are equivalent

$\boxed{\text{on } W}$ — " —

Also: if $\partial\Omega$ is Lipschitz or C^1, then

$$W^{1,2}_{\Gamma_D} = \text{closure of } \underbrace{\{u \in C^1(\Omega) \cap C(\bar\Omega) \,;\, u|_{\Gamma_D} = 0\}}_{= V_{\Gamma_D}} \text{ w.r.t. } \|\cdot\|_{1,2}$$

$$W^{1,2}_0(\Omega) = \text{closure of } \mathcal{D}(\Omega) \text{ w.r.t. } \|\cdot\|_{1,2}.$$

$\boxed{\text{Ad } ②}$ Assume that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a <u>classical solution</u> to (P). Multiplying $(P)_1$ by $\varphi \in V_\Gamma$, integrating over Ω, using Gauss theorem and bc's $(P)_{2-3}$, we obtain

$$\boxed{\int_\Omega \mathbb{A}(x)\nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \qquad \forall \varphi \in V_\Gamma} \quad (4)$$

If we relax the requirements on u and, following the apriori estimates in ①, require that $\boxed{u \in W^{1,2}_{\Gamma_D}}$ that, by density, we conclude) that

(from (4))

$$\boxed{\int_\Omega \mathbb{A}(x)\nabla u \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle \qquad \forall \varphi \in W^{1,2}_{\Gamma_D}} \quad (5)$$

holds provided that

$$\mathbb{A} \in [L^\infty(\Omega)]^{d\times d} \quad \text{and} \quad f \in \left(W^{1,2}_{\Gamma_D}\right)^* \qquad (6)$$

assume (6) holds.

$\boxed{\text{Def.}}$ We say that $u : \Omega \to \mathbb{R}$ is a <u>weak solution to (P)</u> if $\boxed{u \in W^{1,2}_\Gamma(\Omega)}$ $\boxed{\text{satisfies (5)}}$

$\boxed{\text{Assertion}}$ (On compatibility of the concepts of classical and weak solutions).

Let $\mathbb{A}$, f be regular (smooth) enough. Let u be sufficiently regular. Then the concepts of classical and weak solutions are equivalent)

Pf $\boxed{\Rightarrow}$ above

$\boxed{\Leftarrow}$ Since u is ~~smooth~~ weak solution and $u \in W^{1,2}_{\Gamma_D}$, we observe that $u|_{\Gamma_D} = 0$, i.e. u fulfills $(P)_2$.

As f regular enough, $\langle f, \varphi \rangle = \int_\Omega f \varphi \, dx$ for all φ smooth.

Hence, by integration by parts at the left-hand side of (5), we obtain

$$(7) \quad \boxed{\int_\Omega \left(-\operatorname{div}(\mathbb{A}(x)\nabla u) - f\right)\varphi \, dx + \int_{\Gamma_N} \mathbb{A}(x)\nabla u \cdot n \, \varphi \, dS = 0 \qquad \forall \varphi \in V_\Gamma.}$$

Since $\mathcal{D}(\Omega) \subset V_\Gamma$, we conclude from (7) that

$$\int_\Omega (-\operatorname{div}(A(x)\nabla u) - f)\varphi \, dx = 0 \quad \forall \varphi \in \mathcal{D}(\Omega)$$

which implies (it is sufficient to know that $-\operatorname{div}(A(x)\nabla u) - f \in L^1_{loc}(\Omega)$) that

$$-\operatorname{div}(A(x)\nabla u) = f \quad \text{in } \Omega$$

which is $(P)_1$. Inserting this into (7), we get that

$$\int_{\Gamma_N} A(x)\nabla u \cdot \vec{n}\, \varphi \, dS = 0 \qquad \forall \varphi \in V_\Gamma$$

which similarly gives $(P)_2$. ▨

$\boxed{\text{HW}}$ Check the proof in the case $\Gamma_D = \emptyset$; i.e. $W^{1,2}_{\Gamma_D} = W^{1,2}$.
Note that in this case the right-hand side has to satisfy

(8) $$\int_\Omega f \, dx = 0 \qquad \text{or} \qquad \langle f, 1\rangle = 0.$$

Pf (on the level of weak solution) Since the space of test functions include the whole $W^{1,2}(\Omega)$, we can, in particular, take $\varphi \equiv 1$. Then $\nabla\varphi \equiv 0$ and $\langle f, 1\rangle = 0$.

Q.E.D.

$\boxed{\text{Theorem 7.7}}$ • Let (6) holds and $\Gamma_D \neq \emptyset$. Then there exists unique $u \in W^{1,2}_{\Gamma_D}$ weak solution to (P).

• Let (6) holds, $\Gamma_D = \emptyset$ and f satisfy (8). Then there exists unique $u \in W$ weak solution to (P).

$\boxed{\text{Proof.}}$ Set $\quad B(u,\varphi) = \int_\Omega A(x)\nabla u \cdot \nabla\varphi \, dx.$

Clearly B is bilinear, bounded and due to Poincaré inequality and (1): $B(u,u) \geq \beta_* \|u\|^2_{1,2}$ for $u \in \begin{cases} W^{1,2}_{\Gamma_D} \\ W \end{cases}$.

Then by Lax-Milgram theorem and its consequences $\exists!\, u \in \begin{cases} W^{1,2}_{\Gamma_D} \\ W \end{cases}$

$$B(u,\varphi) = \langle f, \varphi\rangle \quad \forall \varphi \in W^{1,2}_{\Gamma_D} \quad (\text{or } W^{1,2}_{\Omega})$$

and

$$\|u\|_{1,2} \leq \frac{\|f\|_{(W^{1,2}_{\Gamma_D})^*}}{\beta_*}.$$

▣

Weak convergence in the Hilbert spaces takes, due to Riesz representation theorem, slightly simpler form. Recall

$x_n \to x$ weakly in $H \iff \phi(x_n) \to \phi(x) \quad \forall \phi \in H^*$

By Riesz representation theorem we can associate with any ϕ unique $a_\phi \in H$ so that $\phi(x) = \langle \phi, x\rangle = (a_\phi, x)_H$ for all $x \in X$

We can then say

$$\boxed{x_n \to x \text{ weakly in } H \iff (y, x_n)_H \to (y, x) \quad \text{for all } y \in H}$$

Uniqueness of the limit can be re-checked again. If $x_n \to x$ weakly in H and $x_n \to \tilde{x}$ weakly in H, then

$$0 = \lim_{n\to\infty} (x-\tilde{x}, x_n)_H - \lim (x-\tilde{x}, x_n)_H = (x-\tilde{x}, x) - (x-\tilde{x}, \tilde{x})$$
$$= (x-\tilde{x}, x-\tilde{x}) = \|x-\tilde{x}\|_H^2$$

$\square$

Theorem 7.8 Let $x_n \to x$ weakly in H and $L \in \mathcal{L}(H,H)$ be compact. Then $\lim_{n\to\infty} \|Lx_n - Lx\|_H = 0$ (strong convergence).

Pf Since $x_n \to x$ weakly in H, $\{x_n\}_{n\geq 1}^\infty$ is bdd. As L is compact $\{Lx_n\}_{n\geq 1}^\infty$ contains subsequence that is converging, i.e. there is $y \in H$: $\|Lx_{n_\ell} - y\|_H \to 0$ as $\ell \to \infty$.

It remains to show that $Lx = y$. However, using the adjoint operator, we have

$$(v, Lx_n - Lx) = (L^* v, x_n - x) \xrightarrow[n\to\infty]{} 0 \quad \forall v \in H$$

Hence $Lx_n \to Lx$ weakly in H, but also $Lx_{n_\ell} \to y$ weakly in H. From the uniqueness of the weak limit $Lx = y$.

To check that the whole sequence Lx_n converges to Lx, assume, on the contrary, that there is a subsequence $\{x_{n_\ell}\} \subset \{x_n\}$ so that $L(x_{n_\ell} - x)$ does not converge to zero, i.e. $\exists\, \varepsilon_0 > 0$ so that $\|L(x_{n_\ell} - x)\|_H \geq \varepsilon_0 > 0$. But as $x_{n_\ell} - x$ is bdd, there is a subsequence (denoted for simplicity again x_{n_ℓ}) so that as L is compact $Lx_{n_\ell} \to z$ as $\ell \to \infty$. However, from the first part of the proof $z = Lx$, which contradicts to

$$\|Lx_{n_\ell} - Lx\|_H \geq \varepsilon_0 > 0 \quad \text{for all } \ell \in \mathbb{N}.$$

EXAMPLE The goal is to see ① how wild (= rapidly oscillating) sequences can converge weakly, and ② how compact operators transfer weakly converging sequences to strongly converging sequences.

Ad ① ▷ Consider $H = L^2((0,\pi))$ and $f_n(x) = \sin^2 nx$. The sequence $\{f_n\}_{n=1}^{\infty}$ is bounded in $L^{\infty}(0,\pi)$ by 1, and hence $\{f_n\}_{n=1}^{\infty}$ is bdd in H. $f_n \geq 0$ are more and more oscilating, f_n do not converge pointwise in $(0,\pi)$. Also f_n does not converge strongly in H. (Otherwise, there is a subsequence converging pointwise (a.e.).)

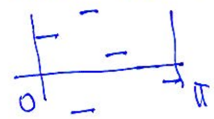
However,

(a)
$$f_n \to \frac{1}{2} \text{ weakly in } H.$$

Indeed: our goal is to check that
$$\int_0^{\pi} \sin^2 nx \, \varphi(x)\, dx = \frac{1}{2}\int_0^{\pi} \varphi(x)\, dx \qquad \forall \varphi \in L^2((0,\pi))$$

- Let first $\varphi = \chi_{(0,b)}$, $b \in (0,\pi)$. Then
$$\int_0^b \sin^2 nx \, dx = \int_0^b \sin nx \sin nx \, dx = \left[-\frac{\sin nx \cos nx}{n}\right]_0^b + \int_0^b (1-\sin^2 nx)\, dx$$
$$\Rightarrow \int_0^b \sin^2 nx \, dx = \frac{b}{2} - \frac{\sin 2nb}{4n} \to \frac{b}{2} = \frac{1}{2}\int_0^b \varphi(x)\, dx.$$

- By linearity, the same holds for every piece-wise constant φ



- For a general $\varphi \in L^2(0,\pi)$ and for any $\varepsilon > 0$ there is piecewise constant $\tilde{\varphi}$: $\|\varphi - \tilde{\varphi}\|_{L^2} < \varepsilon$

Then
$$\int_0^{\pi} \left(\sin^2 nx - \frac{1}{2}\right)\varphi(x)\, dx = \int_0^{\pi} \left(\sin^2 nx - \frac{1}{2}\right)(\varphi(x) - \tilde{\varphi}(x))\, dx$$
$$+ \int_0^{\pi} \left(\sin^2 nx - \frac{1}{2}\right)\tilde{\varphi}(x)\, dx = I_1^n + I_2^n$$

$|I_n^2| < \frac{\varepsilon}{2}$ for $n \gg 1$ due to previous step.

$$|I_n^1| \overset{\text{Hölder}}{\leq} \left\|\sin^2 nx - \frac{1}{2}\right\|_{L^2} \|\varphi - \tilde{\varphi}\|_{L^2} \leq \frac{3}{2}\pi \|\varphi - \tilde{\varphi}\|_{L^2} < \frac{3\varepsilon\pi}{2} \quad ✓$$

Thus (a) holds.

NOTE that
$$\|f_n - \underset{\frac{1}{2}}{\underset{\parallel}{f}}\|_2^2 = \int_0^{\pi} \left(\sin^2 nx - \frac{1}{2}\right)^2 dx = \frac{\pi}{8} \neq 0$$
no strong convergence in $L^2(0,\pi)$. ✓

Ad ② ▷ Consider the operator $L(f)(x) := \int_0^x f(y)\,dy$.

By Kolmogorov criterion is this linear bounded*) operator compact. Indeed: • for any $f \in L^2(0,\pi)$: $\|f\|_{L^2} \le 1$ we get

$\|Lf\|_2 \le \pi^{3/2}\|f\|_2 \le \pi^{3/2}$ (see footnote), i.e. $L(B_1(0))$ is bdd.

• $$\|Lf(\cdot+h) - Lf(\cdot)\|_2^2 = \int_0^\pi \left(\left| \int_x^{x+h} 1 f(y)\,dy \right| \right)^2 dx \overset{\text{Hölder}}{\le} \int_0^\pi h \|f\|^2_{L^2(0,\pi)}\,dx$$

$$= \pi h \|f\|^2_{L^2(0,\pi)} \to 0 \text{ as } |h| \to 0 \quad \forall f : \|f\|_{L^2(0,\pi)} \le 1.$$

(f are extended zero outside $(0,\pi)$)

Hence $L \in \mathcal{L}(L^2(0,\pi))$ is compact.

It is of interest to see what happens with $\{f_n\}_{n=1}^\infty$ $= \sin^2 nx$ satisfying (a) above.

$$Lf_n = \int_0^x \sin^2 ny\,dy = \frac{x}{2} - \frac{\sin 2nx}{4n} \underset{n\to\infty}{\rightrightarrows} \frac{x}{2} = \frac{1}{2}\int_0^x dy = Lf$$

As $Lf_n \rightrightarrows Lf$ either $\|Lf_n - Lf\|_{L^\infty} \underset{n\to\infty}{\to} 0 \Rightarrow \|Lf_n - Lf\|_{L^2} \to 0$.

*) **Boundedness of L**

$$\|Lf\|^2_{L^2(0,\pi)} = \int_0^\pi \left| \int_0^x f(y)\,dy \right|^2 dx \le \int_0^\pi \left(\int_0^x |f(y)|\,dy \right)^2 dx$$

$$\le \int_0^\pi \left(\int_0^\pi |f(y)| \cdot 1\,dy \right)^2 dx \overset{\text{Hölder}}{\le} \pi^3 \|f\|^2_{L^2(0,\pi)}$$

LAX-MILGRAM THEOREM OVER $\mathbb{K}$

$J(\alpha x+y)=\bar{\alpha}J(x)+J(y)$

Theorem 7.6* Let H be over $\mathbb{K}$, and $B: H\times H\to\mathbb{K}$ be sesquilinear (i.e. linear at the first component and conjugate linear in the second component)

Assume that there are $C>0$ and $\beta>0$ so that

(1) $|B(v,u)| \le C\|u\|_H\|v\|_H \quad \forall u,v\in H$ — Boundedness or continuity

(2) $\operatorname{Re} B(u,u) \ge \beta\|u\|_H^2 \quad \forall u\in H$ — Coercivity

Then there exists a unique map $A: H\to H$ so that

$$B(v,u) = (v, Au) \quad \forall u,v\in H$$

In addition, $A\in\mathcal{L}(H)$ is invertible and $\|A\|_{\mathcal{L}(H)}\le C$ and $\|A^{-1}\|_{\mathcal{L}(H)}\le\frac{1}{\beta}$.

Pf [1] $\forall u\in H$: $B(\cdot,u)\in H'$ (linear continuous functional on H due to (1))

By Riesz representation theorem, there is a unique element (vector) $Au\in H$ so that $B(v,u)=(v,Au)_H$ for all $u\in H$

Moreover $\|Au\|_H = \|B(\cdot,u)\|_{H^*}\le C\|u\|_H$

Since B and the scalar product H are conjugate linear in the second argument, it follows that A is linear.

Hence $A\in\mathcal{L}(H)$ with $\|A\|_{\mathcal{L}(H)}\le C$.

[2] • Moreover

$$\beta\|u\|_H^2 \le \operatorname{Re}B(u,u) = \operatorname{Re}(u,Au) \underset{C-S}{\le} \|u\|_H\|Au\|_H,$$

which implies

$$\boxed{\beta\|u\|_H \le \|Au\|_H \quad \forall u\in H \quad (*)}$$

Hence $Au=0$ gives $u=0$ and A is injective.

• Furthermore, $\operatorname{Im} A$ is closed:

$Au_k\to y$ as $k\to\infty$

$\Rightarrow \quad \|u_k-u_l\|_H \underset{\substack{\uparrow\\ \text{linearity of } T}}{\overset{(*)}{\le}} \frac{1}{\beta}\|Tu_k-Tu_l\| \xrightarrow{k,l\to\infty} 0$

$\Rightarrow \quad u_k\to u$ as $k\to\infty$, $u\in H$

$\Rightarrow \quad Au_k\to Au$ as $k\to\infty$ (T is continuous)

$\Rightarrow \quad y=Au \Rightarrow y\in\operatorname{Im}A$.

• $\boxed{\operatorname{Im}A=H}$ If not, by "projection" theorem 7.1 and since $\operatorname{Im}A$ is closed, there is $u_0\in H\setminus\operatorname{Im}A$, $u_0\neq 0$ so that $(v,u_0)=0 \ \forall v\in\operatorname{Im}A$.

Then $\|\beta u_0\|_H^2 \le \operatorname{Re}(u_0,Tu_0)=\operatorname{Re}(Tu_0,u_0)=0$, which gives $\lightning$.

[3] The results in Step [2] imply that $A : H \to H$ is bijective. Then (*) implies $\|A^{-1}f\|_H \le \frac{1}{\beta}\|f\|_H \quad \forall f \in H$,

which gives $\|A^{-1}\|_{\mathcal{L}(H)} \le \frac{1}{\beta}$. ∎

$\boxed{\text{Consequences}}$ By Theorem 7.6* / Theorem 7.6: $\left[\text{for all } f \in H^*: \exists!\, u \in H \text{ so that } B(u,v) = \langle f, v\rangle \ \forall v \in H\right]$

(Pf) • For $f \in H^*$ $\exists!\, \tau f \in H$: $\langle f, v\rangle = (v, \tau f)_H$ by Riesz.

Also: $B(u,v) = (v, Au)_H$ by Theorem 7.6*

Consequently, (to find solution u: $B(u,v) = \langle f, v\rangle \ \forall v \in H$)

is equivalent to (to find u: $(v, Au)_H = (v, \tau f)_H \ \forall v \in H$)

Hence $\boxed{Au = \tau f}$

Then from (*), see Pf of Th. 7.6*,

$$\beta\|u\|_H \le \|\tau f\|_H \underset{\text{Riesz}}{=} \|f\|_{H^*} \Rightarrow \boxed{\|u\|_H \le \frac{\|f\|_{H^*}}{\beta}} \quad (**)$$

<u>Conclusion</u> ▪ The solution operator satisfies (**). It implies <u>the stability of the solution</u> w.r.t. the right-hand side: if f_1 close to f_2, then u_1 close to u_2:

$$\begin{aligned} B(u_1,v) &= \langle f_1, v\rangle \ \forall v \\ B(u_2,v) &= \langle f_2, v\rangle \ \forall v \end{aligned} \Rightarrow B(u_1-u_2, v) = \langle f_1 - f_2, v\rangle \ \forall v \in V$$

and from (**): $\|u_1 - u_2\|_H \le \frac{\|f_1 - f_2\|_{H^*}}{\beta}$.

By HADAMARD, a problem is <u>WELL-POSED</u>, if there is unique solution that is stable w.r.t. perturbation in data

$\Updownarrow$

<u>Existence</u>, <u>Uniqueness</u>, <u>Stability</u> = continuous dependence on data.

▪ Theorem 7.6* can be reformulated in the terms of operators as follows:

Let H Hilbert over $\mathbb{K}$, Let $A \in \mathcal{L}(H)$ be <u>coercive</u>, i.e.

$\exists \beta > 0$: $\operatorname{Re}(u, Au)_H \ge \beta \|u\|_H^2$ $\Leftrightarrow$ <u>strictly positive definite</u>

Then A is invertible, with $C = \|A\|_{\mathcal{L}(H)}$

(Pf) Setting $B(v,u) := (v, Au)_H$

we observe that B is sesquilinear form satisfying the assumption of Theorem 7.6* with $C = \|A\|_{\mathcal{L}(H)}$ ∎

■ If $B(\cdot,\cdot)$ is a scalar product, then in addition u is determined as an absolute (unique) minimizer of the functional

$$E(v) := \frac{1}{2} B(v,v) - \operatorname{Re} \langle f, v \rangle$$

Pf. Let $u, v \in H$. Then

$$E(v) - E(u) = \frac{1}{2}\big(B(v,v) - B(u,u)\big) - \operatorname{Re}\langle f, v-u \rangle$$

$$= \frac{1}{2}\big(B(v,v) - B(u,u)\big) - \operatorname{Re} B(v-u, u)$$

$$= \frac{1}{2}\big(B(v,v) - B(v,u) - B(u,v) + B(u,u)\big)$$

$$= \frac{1}{2} B(v-u, v-u) \geq \frac{\beta}{2} \|u-v\|_H^2$$

$$\boxed{\begin{aligned} &B(v,u) = \operatorname{Re} B(v,u) + i \operatorname{Im} B(v,u) \\ &B(u,v) = \overline{B(v,u)} = \operatorname{Re} B(v,u) - i \operatorname{Im}(B(v,u)) \end{aligned} \Rightarrow \operatorname{Re} B(v,u) = \frac{1}{2} B(v,u) + \frac{1}{2} B(u,v)}$$

which implies the assertion. ■