MAF-II - cvičení 11.3.2021

1) Připomenutí pojmů z přednášky

- **Komplexní derivace**

$$\boxed{f'(z) \overset{def}{=} \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}}$$

$\in \mathbb{C} \to \mathbb{C}$

- $f(z)$ holomorfní v Ω otevřená $\subset \mathbb{C}$

$\overset{def}{\Longleftrightarrow} \quad \exists f'(z)$ pro $\forall z \in \Omega$

- Kalkulus:
  - $(f+g)'(z) = f'(z) + g'(z)$
  - $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$
  - $(f/g)'(z) = \dfrac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}$; $g(z)\neq 0$
  - $(f\circ g)'(z) = f'(g(z))\cdot g'(z)$

- Příklady holomorfních fcí
  - $P(z)$, $P\ldots$ polynom $\in H(\mathbb{C})$

    $(a_n z^n + a_{n-1} z^{n-1} \ldots)' = a_n n z^{n-1} + \ldots$
  - e^z $\quad (e^z)' = e^z$ $\quad \in H(\mathbb{C})$
  - $\dfrac{P(z)}{Q(z)}$ $\quad P, Q$ polynomy $\in H(\mathbb{C} \setminus \{z_0, Q(z_0)=0\})$
  - $\ln(z)$ ... libovolná větev (nebo jakákoli jiná větev) komplexního logaritmu

$\llcorner$

víme že: mít $f'(z)$ v $\mathbb{C}$ je daleko silnější vlastnost, než mít derivaci v $\mathbb{R}^2$

CHARAKTERIZACE – CAUCHY-RIEMANNOVY PODMÍNKY

Buď:
- $f(z): \mathbb{C} \to \mathbb{C}$ def na okolí $U(a)$
- $\tilde{f}(x,y): \mathbb{R}^2 \to \mathbb{C}$, $\tilde{f}(x,y) \overset{def}{=} f(x+iy)$
- $\hat{f}(x,y): \mathbb{R}^2 \to \mathbb{R}^2$, $\hat{f}(x,y) \overset{def}{=} (\underbrace{\operatorname{Re} f(x+iy)}_{\hat{f}_1(x,y)}, \underbrace{\operatorname{Im} f(x+iy)}_{\hat{f}_2(x,y)})$

potom: $\exists f'(a)$, $a = \alpha + i\beta$

$\iff$ $\hat{f}_1$ a $\hat{f}_2$ mají v (α,β) TOTÁLNÍ DIFERENCIÁL a platí

- $$\boxed{\begin{aligned} \frac{\partial \hat{f}_1}{\partial x}(\alpha,\beta) &= \frac{\partial \hat{f}_2}{\partial y}(\alpha,\beta) \\ \frac{\partial \hat{f}_1}{\partial y}(\alpha,\beta) &= -\frac{\partial \hat{f}_2}{\partial x}(\alpha,\beta) \end{aligned}}$$ reálný tvar C.-R.

$\iff$ $\tilde{f}$ má v (α,β) (kompl.) T.D. a platí

- $$\boxed{\frac{\partial \tilde{f}}{\partial x}(\alpha,\beta) = \frac{1}{i}\frac{\partial \tilde{f}}{\partial y}(\alpha,\beta)}$$ komplexní tvar C.-R.

- DALŠÍ HOLOMORFNÍ FCE

• $\sin z$, $\cos z$, $\sinh z$, $\cosh z$

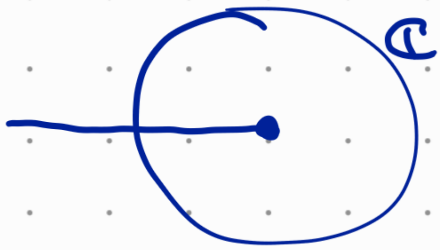
Lze je zavedené přes mocninné řady nebo pomocí exponenciály!

Př. $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$

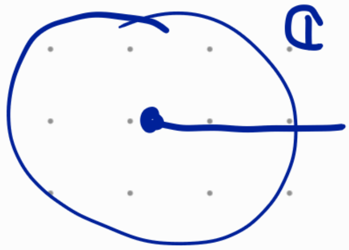
nebo $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ $\quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$

⋮

• <u>$\ln z$</u> <u>jednoznačná</u> větev logaritmu



nebo

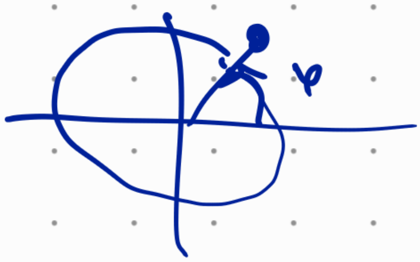


φ? ∞

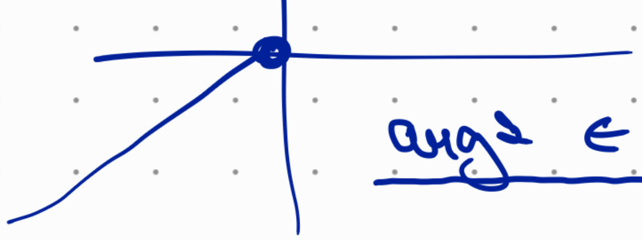
$z = \underbrace{|z|}_{e^{\ln|z|}} e^{i \arg z} = e^{\ln|z| + i\arg z}$

?

$\mathbb{C} \setminus \langle \varphi \rangle$

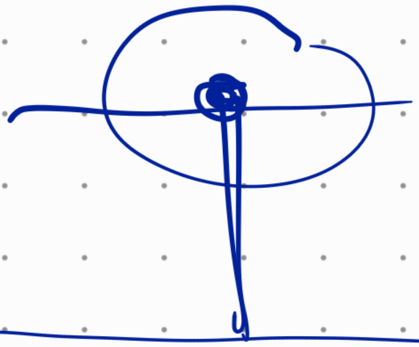


• $\underset{\in \mathbb{C}}{\ln z} = \underset{\in \mathbb{R}}{\ln|z|} + i \arg z$



$\arg z \in [0, 2\pi)$

. . . $\mathbb{C} \setminus \{0\}$



- $f(z) = \bar{z} = x - iy$ není holom. nikde v $\mathbb{C}$
  $z = x+iy$

- $f(z) = |z| = |x+iy| = \sqrt{x^2+y^2} \in \mathbb{R}$

? holomorfní

$f_1(x,y) = \sqrt{x^2+y^2}$
$f_2(x,y) = 0$

- $\dfrac{\partial f_1}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}$ $\quad (x,y) \neq (0,0)$

$$\frac{\partial f_1(0,0)}{\partial x} = \lim_{h_1 \to 0+} \frac{f_1(0+h_1, 0) - f_1(0,0)}{h_1} =$$

$$= \lim_{h_1 \to 0+} \frac{\sqrt{h_1^2} - 0}{h_1} = \lim_{h_1 \to 0+} \frac{|h_1|}{h_1} \quad \nexists$$

v $(0,0)$ $\nexists$ T.D. $\Rightarrow$ $\nexists$ $f'(0)$.

- $\dfrac{\partial f_2}{\partial y} \equiv 0$
- $\dfrac{\partial f_1}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}$ $\qquad \dfrac{\partial f_2}{\partial x} \equiv 0$

- v $(0,0)$ $\nexists$ T.D. navíc nejsou splněny C-R. podm.

$\Rightarrow$ $\boxed{f(z) = |z|}$ není hol. nikde v $\mathbb{C}$

- $z = x + iy$ C-R.
- $z = |z| e^{i\varphi}$ potřebujeme tvar C-R. podm.

$f(z)$ ... zavedu $\tilde{g} : \mathbb{R}^2 \to \mathbb{C}$

$(r, \varphi) \to \tilde{g} \in \mathbb{C}$

$$\boxed{\tilde{g}(r,\varphi)} = \tilde{f}(r\cos\varphi, r\sin\varphi) = f(r\cos\varphi + i r\sin\varphi)$$

$$= f(re^{i\varphi}) = f(z)$$

C-R. $\frac{\partial \tilde{g}}{\partial r} = ?\ \frac{\partial \tilde{g}}{\partial \varphi}$ | $\boxed{\frac{\partial \tilde{f}}{\partial x} = \frac{1}{i}\frac{\partial \tilde{f}}{\partial y}}$

f je holo. $\Rightarrow \tilde{f}$ splňuje C-R.

$$\frac{\partial \tilde{g}}{\partial r} = \frac{\partial \tilde{f}}{\partial x}\cos\varphi + \underbrace{\frac{\partial \tilde{f}}{\partial y}}_{CR.}\sin\varphi$$

$$= \frac{\partial \tilde{f}}{\partial x}(\cos\varphi + i\sin\varphi) = \frac{\partial \tilde{f}}{\partial x} e^{i\varphi}$$

$$\frac{\partial \tilde{g}}{\partial \varphi} = \frac{\partial \tilde{f}}{\partial x}\cdot(-r\sin\varphi) + \frac{\partial \tilde{f}}{\partial y} r\cos\varphi =$$

$$= i\frac{\partial \tilde{f}}{\partial x} \quad \text{C.R.}$$

$$= \frac{\partial \tilde{f}}{\partial x}\, ir(\cos\varphi + i\sin\varphi)$$

$$= \frac{\partial \tilde{f}}{\partial x} e^{i\varphi}\, ir$$

$$\Rightarrow \boxed{\frac{\partial \tilde{g}}{\partial \varphi} = ir\frac{\partial \tilde{g}}{\partial r}}$$

polární tvar C-R. podmínek.

$\tilde{g}(r, \varphi)$ je holomorfní $\Rightarrow \tilde{g}$ splňuje

ločímo/ideas
iφ C-R podmínek.

Př: $f(z) = \ln z = \ln|z|e^{i\varphi} =$

$= \ln \underbrace{|z|}_{r} + i\varphi$

$\tilde{g}(r,\varphi) = \underline{\ln r} + i\underline{\varphi}$

- $\frac{\partial \tilde{g}}{\partial r} = \frac{1}{r}$
- $\frac{\partial \tilde{g}}{\partial \varphi} = i$

$\Rightarrow ir\frac{\partial \tilde{g}}{\partial r} = i$ ✓

$\shortparallel$

$\frac{\partial \tilde{g}}{\partial \varphi}$

- mějme nechť $f(z)$ $\mathbb{C} \to \mathbb{C}$ má v $U(a)$ $(\underline{a = \alpha + i\beta})$ derivaci

a nechť $u(x,y) \overset{\text{def}}{=} \operatorname{Re} f(x+iy)$

$\shortparallel$

$f_1(x,y)$

$v(x,y) \overset{\text{def}}{=} \operatorname{Im} f(x+iy)$

$\shortparallel$

(a navíc $u, v \in C^2(\mathcal{U}(\alpha, 3)))$ $f_2'(x,y)$

pak $\Delta u = 0$ & $\Delta v = 0$

v $\mathcal{U}((\alpha, 3))$ (jsou harmonické)

$\llcorner$

Platí i následující tvrzení (opačného typu)

Buď $\underline{u(x,y)}$ v $\mathcal{U}((\alpha, 3))$ $\underline{\text{harmonická}}$

a $\underline{C^2(\mathcal{U}(\alpha, 3))}$ a C^2

pak $\exists$ $v(x,y)$ $\underline{\text{harmonická}}$ v $\mathcal{U}((\alpha, 3))$

splňující spolu s u $\underline{\text{Cauchy-Riemannovy p.}}$

a tedy

$$f(z) \stackrel{def}{=} \underline{u}(\underset{x}{\mathrm{Re}\, z}, \underset{y}{\mathrm{Im}\, z}) + i\, \underline{v}(\mathrm{Re}\, z, \mathrm{Im}\, z)$$

má v. $\mathcal{U}((\alpha + i3))$ derivaci.

Tedy jsme v úloze

HARMONICKY SDRUŽENÁ k u

Př. máme $u(x,y) = x^2 - y^2 \in C^\infty(\mathbb{R}^2)$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

harmonická všude v $\mathbb{R}^2$

? v harm. sdružená ?

- $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x}$ (1. CR)
- $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ (2. CR)

(1. CR): $\dfrac{\partial v}{\partial y} = 2x$

$$v = 2xy + c(x)$$

(2. CR) $\dfrac{\partial v}{\partial x} = 2y + c'(x) =$

$-(-2y)$

$$\Rightarrow \quad c'(x) = 0 \quad \Rightarrow \quad c(x) = C$$

(const.)

$$\boxed{v(x,y) = 2xy + C}$$

$$\in C^{\infty}(\mathbb{R}^2) \quad \checkmark$$

$$\Delta v = 0 \quad \text{v } \mathbb{R}^2$$

$$f(z) = ?$$

$$u(x,y) + i v(x,y) =$$

$$= \underline{x^2 - y^2 + i2xy} + \tilde{C}$$

$$= (x+iy)^2 + \tilde{C}$$

$$\boxed{f(z) = z^2 + c} \in H(\mathbb{C})$$

nidependance (Feynmann)

u, v ($\Delta u = 0$, $\Delta v = 0$)

u, v $(\Delta u = 0, \Delta v = 0)$

$$\nabla u \cdot \nabla v = (2x, -2y) \cdot (2y, 2x) = 0$$

$$\nabla u \cdot \nabla v = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \left(\underbrace{\frac{\partial v}{\partial x}}_{= -\frac{\partial u}{\partial y}}, \underbrace{\frac{\partial v}{\partial y}}_{= \frac{\partial u}{\partial x}}\right)$$

$$= 0$$

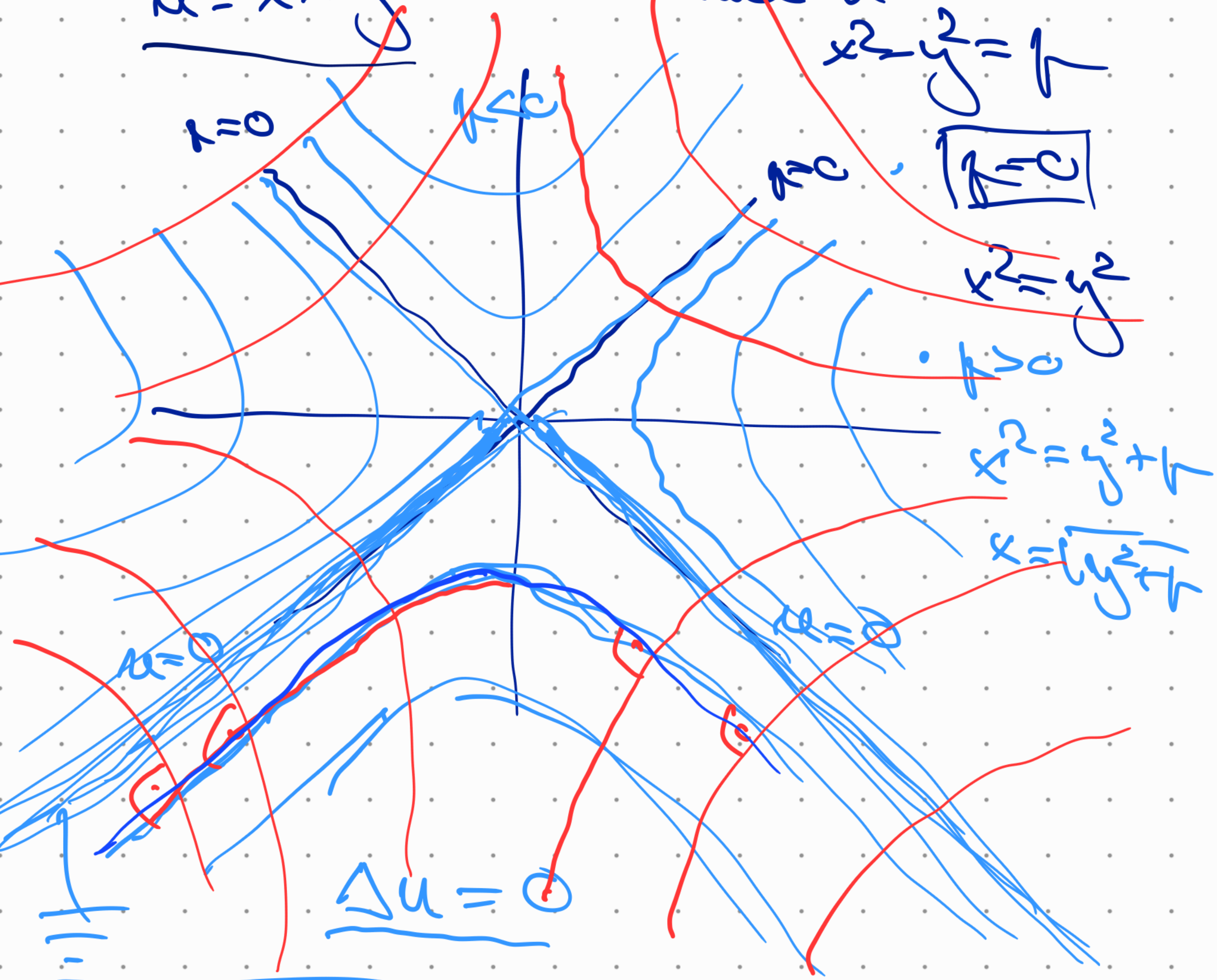
u, v splňujú C-R $\Rightarrow$ $\underline{\nabla u \cdot \nabla v = 0}$

$\Delta u = 0$

$\Delta v = 0$

$\Rightarrow$ u, v harm. združené jsou 2 řešení Lapl. rce jejichž siločáry ($\parallel$ s ∇u, ∇v) jsou na sebe <u>kolmé</u> a tedy i ekvipotenciály jsou na sebe kolmé.

$u = x^2 - y^2$ ... izočáry u $\quad k \in \mathbb{R}$

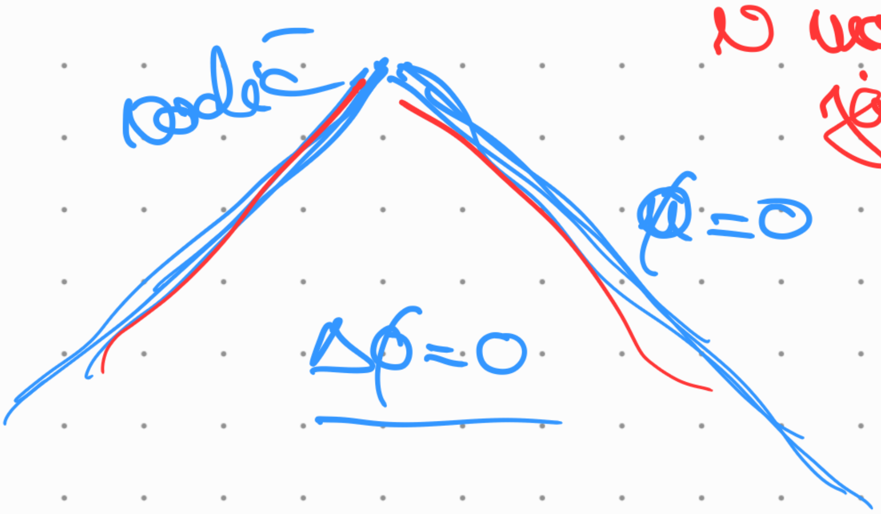


$\Delta u = 0$

$\boxed{f(z) = z^2}$

$v = 2xy + \cancel{c}$

v volíme interpretovat jako siločáry u



Př

- $u(x,y) = x^2 - y^2 + e^x(x\cos y - y\sin y)$

Hledejme v harmonicky sdr.
pokud to nalezneš.

- $\in C^\infty(\mathbb{R}^2)$ ✓
- $\Delta u = 0$ (sami)
- C–R k.

1CR.

- $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = 2x + e^x(x\cos y - y\sin y) + e^x\cos y$

$$v = 2xy + e^x x\sin y - e^x \int y\sin y + e^x \sin y + C(x)$$

$$\int y\sin y = \sin y - y\cos y \quad ✓$$

$$v = 2xy + e^x(x\sin y + y\cos y) + C(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = (*)$$

$$\parallel$$

$$\cancel{2y} + e^{x}(\cancel{x\sin y} + \cancel{y\cos y}) + e^{x}\cancel{\sin y} + C'(x)$$

$$(*) = -\left(+\cancel{2y} + e^{x}(\cancel{x\sin y} + \cancel{\sin y} + \cancel{y\cos y})\right)$$

$$\Rightarrow \quad C'(x) = 0 \qquad \underline{C(x) = C}$$

$$v(x,y) = 2xy + e^{x}(y\cos y + x\sin y) + C$$

$$\in C^{\infty}(\mathbb{R}^2)$$

∘ $\Delta v = 0$ ✓ (Dani)

$$f(z) = ?$$

$$u(x,y) + i\,v(x,y) =$$

$$= \underline{x^2 - y^2} + e^x(x\cos y - y\sin y)$$

$$+ i\left(\underline{2xy} + e^x(y\cos y + \underline{x\sin y})\right)$$

$$= \dots (x+iy)^2 +$$

$$= \quad + e^x\left(\underline{\cos y}(x+iy) + \underline{+ i\sin y}(x+iy)\right)$$

$$= (x+iy)^2 + \underbrace{e^x\left((x+iy)e^{iy}\right)}$$

$$= (x+iy)^2 + e^{x+iy}\cdot(x+iy)$$

- $$f(z) = \underline{z^2 + z\cdot e^z} \in H(\mathbb{C})$$