Fourierova transformace

$$\hat{f}(s) = \mathcal{F}[f](s) := \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot s}\, dx$$

$$\mathcal{F}^{-1}[f](s) = \int_{\mathbb{R}^d} f(x)\, e^{2\pi i x\cdot s}\, dx$$

Př. 3 Urči $\lambda>0$

$$\boxed{\mathcal{F}^{-1}\left[e^{-\frac{|x|^2}{\lambda}}\right](s) = (\lambda\pi)^{d/2} e^{-\pi^2\lambda|s|^2}}$$

neboli $\mu = \frac{1}{\lambda}$

$$\mathcal{F}^{-1}\left[e^{-\mu|x|^2}\right](s) = \left(\frac{\pi}{\mu}\right)^{d/2} e^{-\frac{\pi^2|s|^2}{\mu}} \quad (*)$$

Speciálně: $\mu=\pi$

$$\mathcal{F}^{-1}\left[e^{-\pi|x|^2}\right](s) = e^{-\pi|s|^2} \quad (**)$$

odvodíme na zač. příští přednášky.

DÚ: Odvoďte $(*)$ ze vzorce $(**)$ pomocí Věty 16.2.(v).

Chceme spočíst: $\mathcal{F}\left[e^{-\pi|x|^2}\right](s) = \int_{\mathbb{R}^d} e^{-\pi|x|^2} e^{-i2\pi x\cdot s}\, dx$

$x\cdot s = x_1s_1 + \dots + x_ds_d$

$|x|^2 = x_1^2 + x_2^2 + \dots + x_d^2$

$x = (x_1, x_2, \dots, x_d)$

$s = (s_1, s_2, \dots, s_d)$

$$= \int_{\mathbb{R}^d} e^{-\pi x_1^2} e^{-i2\pi x_1 s_1} \cdot \dots \cdot e^{-\pi x_d^2} e^{-i2\pi x_d s_d}\, dx$$

Fubini

$$= \underbrace{\left(\int_{\mathbb{R}} e^{-\pi x_1^2} e^{-i2\pi x_1 s_1}\, dx_1\right)}_{\mathcal{F}\left[e^{-\pi x_1^2}\right](s_1)} \cdot \dots \cdot \left(\int e^{-\pi x_d^2} e^{-i2\pi x_d s_d}\, dx_d\right)$$

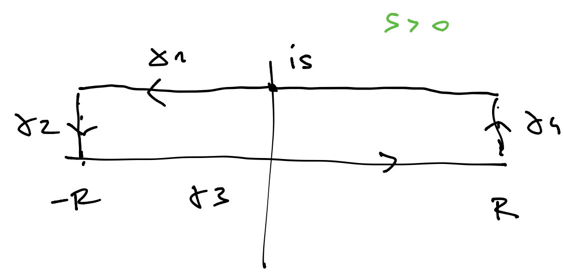
$$\mathcal{F}\left[e^{-\pi x_1^2}\right](s_1) \cdot \dots \cdot \mathcal{F}\left[e^{-\pi x_d^2}\right]$$

• Počítám v $\mathbb{R}$: $\mathcal{F}[e^{-\pi x^2}](s) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-i2\pi x s}\,dx$

$$= \int_{-\infty}^{\infty} e^{-\pi \underbrace{(x^2+2isx-s^2)}_{(x+is)^2}} e^{-\pi s^2}\,dx = e^{-\pi s^2} \underbrace{\int_{-\infty}^{\infty} e^{-\pi (x+is)^2}\,dx}_{I}$$

$$I := \int_{-\infty}^{\infty} e^{-\pi (x+is)^2}\,dx$$

$$g(z) := e^{-\pi z^2}$$



γ_1: $z = x + is \quad x \in [R, -R]$

γ_2: $z = -R + it \quad t \in [1, 0]$

γ_3: $z = x \quad x \in [-R, R]$

γ_4: $z = R + it \quad t \in [0, 1]$

$\Gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$

$g(z) = e^{-\pi z^2} \in H(\mathbb{C})$

• $\int_\Gamma g(z)\,dz = 0$

$$\int_{\gamma_1} g(z)\,dz \xrightarrow{R\to\infty} -I$$

$$\int_{\gamma_3} g(z)\,dz = \int_{-R}^{R} e^{-\pi x^2}\,dx = \left| \sqrt{\pi}\,x = y \right| = \frac{1}{\sqrt{\pi}} \int_{-\sqrt{\pi}R}^{+\sqrt{\pi}R} e^{-y^2}\,dy \xrightarrow{R\to\infty} \frac{1}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-y^2}\,dy}_{\sqrt{\pi}}$$

$$\int_{\gamma_3} g(z)\,dz \longrightarrow 1$$

$$\int_{\gamma_2} g(z)\,dz = \int_1^0 e^{-\pi(-R+it)^2}\,dt = -\int_0^1 e^{-\pi(R^2+2iRt-t^2)}\,dt$$

$$= -\int_0^1 e^{-\pi(R^2-t^2)} e^{-i2\pi Rt} = -e^{-\pi R^2} \int_0^1 e^{\pi t^2} e^{-i2\pi Rt}\,dt$$

$$\left| \int_{\gamma_2} g(z)\,dz \right| \le e^{-\pi R^2} \int_0^1 \left| e^{\pi t^2} \right| dt \xrightarrow{R\to\infty} 0$$

$$\int_{\gamma_4} g(z)\,dz \;\ldots\; \xrightarrow{R\to\infty} 0$$

Celkově mám: $0 = -I + 1 \Rightarrow \underline{I = 1}$

$$\Rightarrow \quad \mathcal{F}[e^{-\pi x^2}] = e^{-\pi s^2} \qquad x \in \mathbb{R}$$

$$\Rightarrow \quad \mathcal{F}[e^{-\pi |x|^2}] = e^{-\pi |s|^2} \qquad x \in \mathbb{R}^d$$

+ škálování F.T.: Věta 16.2.(v) $\quad \widehat{f(\alpha x)}(s) = \frac{1}{|\alpha|^d} \hat{f}\left(\frac{s}{\alpha}\right)$

$$\boxed{\mathcal{F}\left[e^{-\frac{|x|^2}{\lambda}}\right]} = \mathcal{F}\left[e^{-\pi\left(\frac{|x|}{\sqrt{\pi\lambda}}\right)^2}\right] = (\pi\lambda)^{\frac{d}{2}} e^{-\pi \pi\lambda|s|^2} = \boxed{(\pi\lambda)^{d/2} e^{-\pi^2\lambda|s|^2}}$$

$$\uparrow \quad \alpha = \frac{1}{\sqrt{\pi\lambda}}$$

$$\alpha = \sqrt{\frac{m}{\pi}} \rightsquigarrow \mathcal{F}\left[e^{-m|x|^2}\right] = \left(\frac{\pi}{m}\right)^{\frac{d}{2}} e^{-\frac{\pi^2|s|^2}{m}}$$

(Př) $f(x) = x e^{-ax^2}, \quad a > 0$

- $f \in \mathcal{S}(\mathbb{R})$
- $$\mathcal{F}[f](s) = \int_{-\infty}^{\infty} \underbrace{x e^{-ax^2}}_{-\frac{1}{2a}\frac{d}{dx}\left(e^{-ax^2}\right)} e^{-i2\pi x s}\, dx = \int_{-\infty}^{\infty} \frac{d}{dx}\left(e^{-ax^2}\right) \frac{1}{(-2a)} e^{-i2\pi x s}\, dx$$

$$= -\frac{1}{2a} \mathcal{F}\left[\frac{d}{dx}\left(e^{-ax^2}\right)\right] \overset{\text{věta}}{=} -\frac{1}{2a}(i2\pi s)\, \underbrace{\mathcal{F}\left[e^{-ax^2}\right]}_{=\sqrt{\frac{\pi}{a}}\, e^{-\frac{\pi^2}{a}s^2}}$$

$$= \underline{-i\left(\frac{\pi}{a}\right)^{3/2} s\, e^{-\frac{\pi^2 s^2}{a}}}$$

Tvrzení: Je-li $f \in L^1(\mathbb{R}^d)$ sudá/lichá v proměnné x_j, pak je $\hat{f}$ sudá/lichá v s_j.

(D) jen v $\mathbb{R}^1$:
$$\mathcal{F}[f](-s) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi(-s)x}\, dx = \left| \begin{matrix} x = -y \\ dx = -dy \end{matrix} \right|$$
$$= -\int_{\infty}^{-\infty} f(-y)\, e^{-i2\pi s y}\, dy$$
$$= \int_{-\infty}^{\infty} f(-y)\, e^{-i2\pi s y}\, dy$$

- f sudá: $= \mathcal{F}[f](s)$ ... sudá
- f lichá: $= -\mathcal{F}[f](s)$ ... lichá

Př: $f(x) = \frac{x}{a^2+x^2}$, $a > 0$ Na přednášce: $\frac{1}{1+x^2} \to \hat{f} = \pi e^{-2\pi|s|}$

- $f \notin L^1(\mathbb{R})$ $(\sim \frac{1}{x})$, ale $f \in L^2(\mathbb{R})$

  $\mathcal{F}[f]$ pro $f \in L^2(\mathbb{R}^d)$ <u>není</u> dána integrálem $\int_{\mathbb{R}^d} f(x) e^{-i2\pi x\cdot s}\, dx$

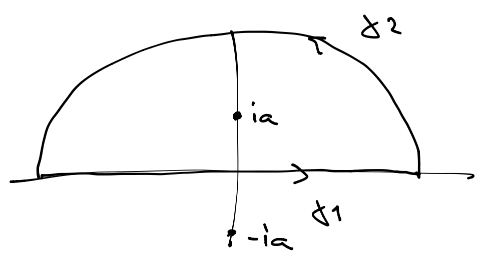
  Jinak: $f_n \to f$

- Počítejme: $\mathcal{F}[f](s) = \lim_{R\to\infty} \int_{-R}^{R} \underbrace{\frac{x}{a^2+x^2}\, e^{-i2\pi x s}}_{g(x)}\, dx$

- $f(x)$ je lichá $\Rightarrow$ $\hat{f}(s)$ je lichá

  $\Rightarrow$ počítejme pro $\underline{s<0}$, v $\mathbb{C}$, $g(x) \to g(z)$, res. věta.

- $g(z) := \frac{z}{a^2+z^2}\, e^{-i\underbrace{2\pi s}_{<0} z}$ $s < 0$



$\Gamma = \gamma_1 + \gamma_2$

- $\int_\Gamma g(z)\, dz = 2\pi i \operatorname{Res}_{ia} g(z)$

γ_1: $z = x$ $x \in [-R, R]$

- $I_1 = \int_{\gamma_1} g(z)\, dz = \int_{-R}^{R} \frac{x}{a^2+x^2}\, e^{-i2\pi x s}\, dx \xrightarrow{R\to\infty} \hat{f}(s)$

- $I_2 \xrightarrow{R\to\infty} 0$: Jordan s $\gamma = -2\pi s > 0$, $M_R \sim \frac{1}{R} \xrightarrow{R\to\infty} \underline{0}$

Tedy: $\hat{f}(s) = 2\pi i \operatorname{Res}_{ia} \frac{z}{(z+ia)(z-ia)} e^{-i2\pi s z} = \cancel{2}\pi i \frac{i\cancel{a}}{\cancel{2}i\cancel{a}} e^{2\pi s a}$

$= i\pi e^{2\pi a s} = \underline{i\pi e^{-2\pi a|s|}} \quad s<0$

$\longrightarrow$ liché rozšíření: $\hat{f}(s) = -i\pi \operatorname{sgn}(s)\, e^{-2\pi a|s|} \quad s \neq 0$

- Pro $s=0$: $\hat{f}(0) = \int_{-\infty}^{\infty} \frac{x}{a^2+x^2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \underbrace{\frac{x}{a^2+x^2}}_{\text{liché}}\,dx = 0$