Power-law fluids. One special case

1) Motivation
2) Formulate $\boxed{\text{our special problem}}$ — why is difficult
3) Preliminary consideration
4) Definition of solution to

Classical power-law fluids

$\boxed{2\mu_* = 1}$

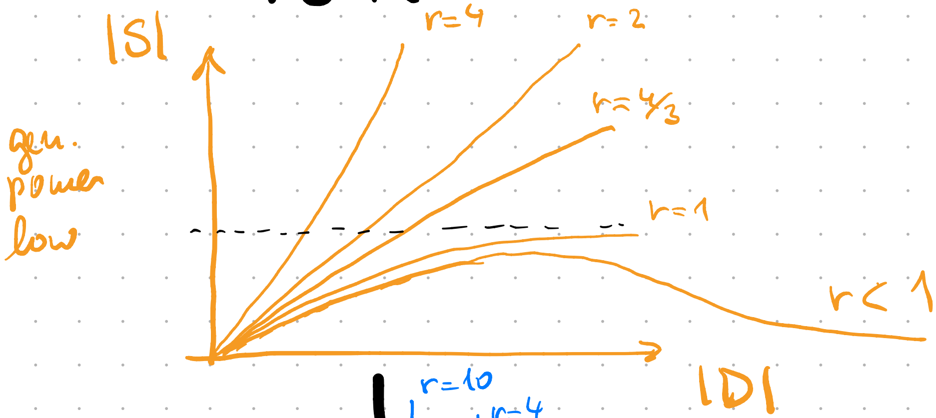
$$\boxed{\mathbb{S} = 2\mu_* |\mathbb{D}v|^{r-2}\,\mathbb{D}v} \iff \boxed{\mathbb{D}v = |\mathbb{S}|^{r'-2}\,\mathbb{S}}$$

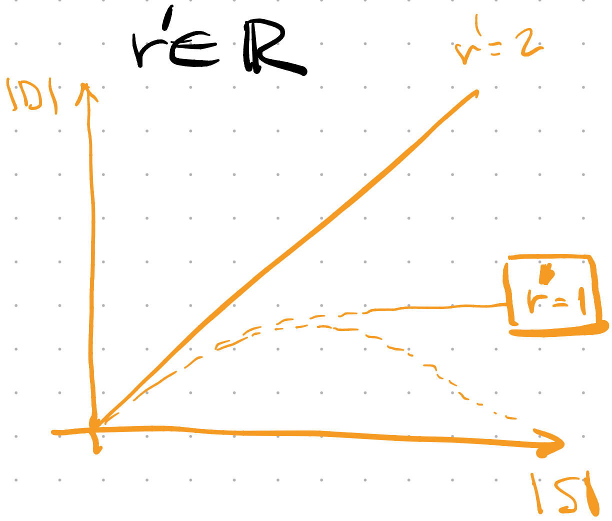
$\boxed{r=1}$ 1-laplacian $\qquad \boxed{r \in (1,+\infty)} \qquad$ $r'=1$ $\Uparrow$ $\boxed{r=+\infty}$ ∞-laplacian

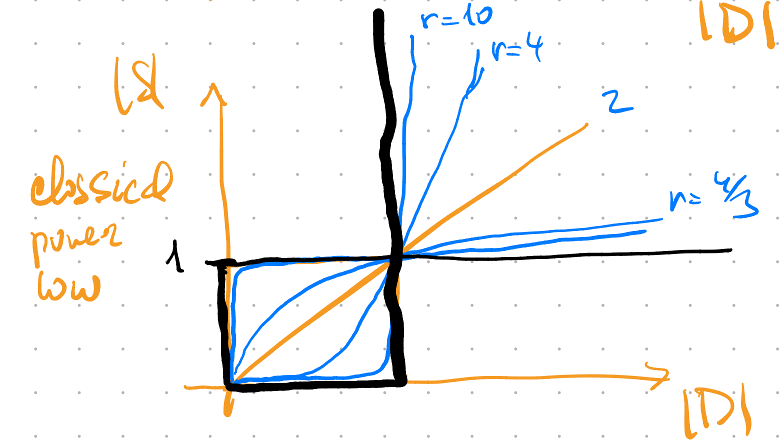
Generalized power-law fluids

$$\boxed{\mathbb{S} = (1+|\mathbb{D}v|^2)^{\frac{r-2}{2}}\,\mathbb{D}v} \qquad \boxed{\mathbb{D}v = (1+|\mathbb{S}|^2)^{\frac{r'-2}{2}}\,\mathbb{S}}$$

$r \in \mathbb{R}$ $\qquad\qquad$ $r' \in \mathbb{R}$







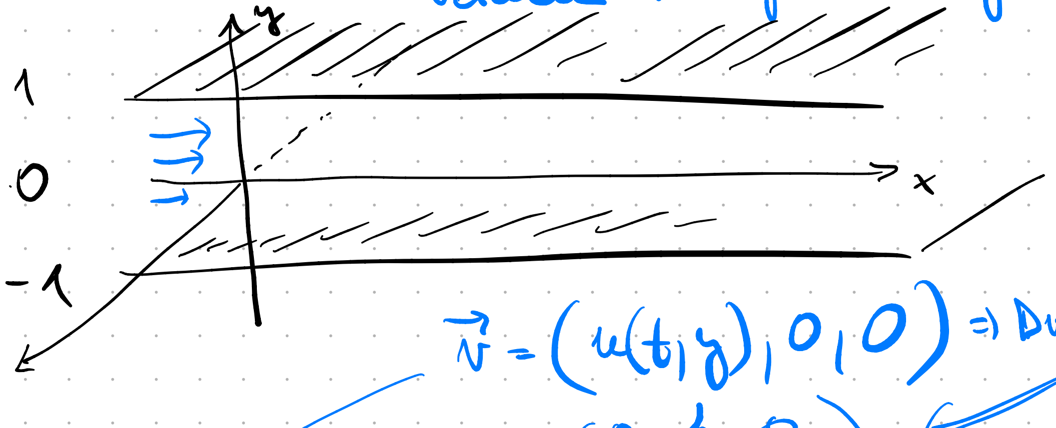
AIM: to provide nice PDE analysis (in the spirit of Leray) for the following problem:

$$\left[\begin{aligned} &\operatorname{div} \vec{v} = 0 \\ &\frac{\partial \vec{v}}{\partial t} + v_2 \frac{\partial v}{\partial x_2} - \operatorname{div} \mathbb{S} = -\nabla p \\ &\mathbb{D}v = \frac{\mathbb{S}}{(1+|\mathbb{S}|^2)^{1/2}} \end{aligned}\right.$$

$\boxed{\text{IC's + BC's}}$ +

Problem is difficult? ???

Simplifications: Consider unsteady simple shear flows between two parallel plates.



$$\vec{v} = (u(t,y),0,0) \Rightarrow Dv = \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial u}{\partial y} & 0 \\ \frac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$D_{12} = D_{xy}$

$$D = \frac{S}{(1+|S|^2)^{1/2}}$$

$$S = \begin{pmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\sigma = \sigma(t,y)$

$$\boxed{\begin{array}{l} \operatorname{div} \vec{v} = 0 \\ v_k \frac{\partial v}{\partial x_k} = \vec{0} \end{array}}$$

$$\boxed{\underbrace{\frac{\partial u}{\partial t} - \frac{\partial \sigma}{\partial y}}_{t,y} = \underbrace{-\frac{\partial p}{\partial x}}_{t,x}}, \quad \underbrace{-\frac{\partial p}{\partial y} = 0 = -\frac{\partial p}{\partial z}}$$

p depends only on x.

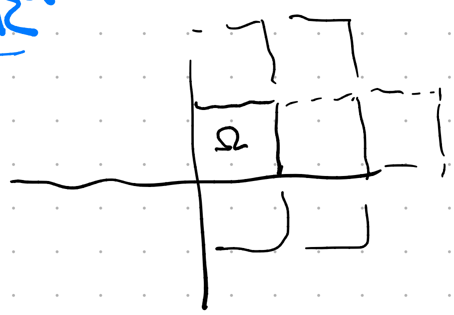
$$\rightarrow \boxed{\frac{\partial u}{\partial t} - \frac{\partial \sigma}{\partial y} = g(t)} \quad \text{and} \quad -\frac{\partial p}{\partial x} = g(t)$$

$$\rightarrow \frac{\partial u}{\partial y} = \frac{\sigma}{(1+|\sigma|^2)^{1/2}}$$

Problem

$(u, \vec{q}) : \underline{(0,T) \times \mathbb{R}^d \longrightarrow \mathbb{R} \times \mathbb{R}^d}$

(P)
$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} \vec{q} = g \\ \nabla u = \frac{\vec{q}}{(1+|q|^a)^{1/a}} \\ u(0,\cdot) = u_0 \\ u, \vec{q} \text{ are } \Omega\text{-periodic, where } \Omega = (0,L)^d \end{cases}$$



Data: $\boxed{a > 0}$

$g : (0,T) \times \mathbb{R}^d \longrightarrow \mathbb{R}$ Ω-periodic

$u_0 : \mathbb{R}^d \longrightarrow \mathbb{R}$ Ω-periodic

$$\Phi[u] = \int_\Omega \sqrt{1+|\nabla u|^2}\,dx$$

$\Downarrow$ (E-L)

$$\left[\; -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \qquad u = u_D \right.$$

$$\boxed{-\operatorname{div}\vec{q} = 0 \qquad \vec{q} = \frac{\nabla u}{(1+|\nabla u|^2)^{1/2}}}$$

- Why (P) is difficult?
- Preliminary considerations!

First observation

$$\Downarrow \quad \nabla u = \frac{q}{(1+|q|^a)^{1/a}}$$

$$|\nabla u| \le 1 \Rightarrow \nabla u \in L^\infty(\Omega)^d$$

Second obs. Eq. multiply $\cdot\, u$;

$$\frac{1}{2}\partial_t |u|^2 + \vec{q}\cdot\nabla u - \operatorname{div}(\vec{q}u) = gu$$

$$\int_\Omega \quad \frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \int_\Omega q\cdot\nabla u = \int_\Omega gu \le \|g\|_2\|u\|_2$$

$$\int_\Omega \frac{|\vec{q}|^2}{(1+|\vec{q}|^a)^{1/a}}\,dx \qquad \frac{1}{2}\|g\|_2^2 + \frac{1}{2}\|u\|_2^2$$

Gronwall

$$\Rightarrow \sup_t \|u(t)\|_2^2 + 2\int_0^T\int_\Omega \frac{|\vec{q}|^2}{(1+|\vec{q}|^a)^{1/a}} \le C\left(\int_0^T \|g\|_{2}^2 + \|u_0\|_2^2\right)$$

$$u \in L^\infty(0,T;W^{1,\infty}(\Omega))$$

!! gives control of $\vec{q}$ in $L^1(0,T;L^1(\Omega))$

Indeed:

$$\int_{Q_T} |\vec{q}| = \int_{Q_T} \frac{|\vec{q}|}{\underbrace{(1+|q|^q)^{1/2q}}} \underbrace{(1+|\vec{q}|^q)^{1/2q}}$$

$$\leq c\left(\int_{Q_T} (1+|\vec{q}|^q)^{\frac{2}{q}}\right)^{1/2}$$

$$\leq c_0 + c\left(\int_{Q_T} |q|\right)^{1/2}$$

$$\Rightarrow \boxed{\int_{Q_T} |\vec{q}|\,dx\,dt < +\infty}$$

$\boxed{3^{rd} \text{ observation}}$ Our problem could be seen as limiting case of $\nabla u = (1+|\vec{q}|^2)^{\frac{r'-2}{2}} \vec{q} =: \mathcal{F}(\vec{q})$ if $\boxed{r' \to 1}$, for particular value $\boxed{a=2}$

Observe that for $r' > 1$:

$$\left(\mathcal{F}(q_1) - \mathcal{F}(q_2)\right) \cdot (q_1 - q_2) \overset{?}{\geq} 0$$

$$= \int_0^1 \frac{d}{ds}\left((1+|\underbrace{q_2 + s(q_1-q_2)}_{q_{12}}|^2)^{\frac{r'-2}{2}} \underbrace{(q_2 + s(q_1-q_2))}_{q_{12}} \right) ds \cdot (q_1 - q_2)$$

$$= \int_0^1 (1+|q_{12}|^2)^{\frac{r'-2}{2}} |q_1-q_2|^2 + \frac{r'-2}{\cancel{2}} (1+|q_{12}|^2)^{\frac{r'-4}{2}} \cancel{2}\, q_{12}\cdot(q_1-q_2)^2\, ds$$

$$= \int_0^1 (1+|q_{12}|^2)^{\frac{r'-4}{2}} \left((1+|q_{12}|^2)|q_1-q_2|^2 + \underbrace{(r'-2)}_{\geq 0} \underbrace{(q_{12}\cdot(q_1-q_2))^2}_{\geq 0} \right) ds$$

$r' \geq 2$ $\int_0^1 (1+|q_{12}|^2)^{\frac{r'-2}{2}} ds\, |q_1-q_2|^2 \geq 0$

$\geq$ $r' \leq 2$

$r' \le 2$: $\displaystyle\int_0^1 (1+|q_{12}|^2)^{\frac{r'-4}{2}} \left\{ \underbrace{(1+|q_{12}|^2)}\,|q_1-q_2|^2 - (2-r')\,\underbrace{|q_{12}|^2 |q_1-q_2|^2} \right\}$

$1-2+r' = r'-1$

$r' \in (1,2)$

$$= \int_0^1 (1+|q_{12}|^2)^{\frac{r'-4}{2}} \left\{ \underbrace{(1+(r'-1)}\,|q_{12}|^2)\;|q_1-q_2|^2 \right\}$$

$$\ge (r'-1)(1+|q_{12}|^2)\,|q_1-q_2|^2$$

$$\ge (r'-1)\int_0^1 (1+|q_{12}|^2)^{\frac{r'-2}{2}} |q_1-q_2|^2\, ds$$

$r' \to 1$

$r'-2 \to -1$

$r'=1$

$$\ge \int_0^1 (1+|q_{12}|^2)^{\frac{r'-4}{2}} |q_1-q_2|^2\, ds$$

$r'-4 \overset{r'=1}{=} -3$

$= -\frac{3}{2}$

Observe that for $r'=1$, but a arbitrary $\in (0,+\infty)$

for $\Psi(\vec q) = \dfrac{q}{(1+|q|^a)^{1/a}}$

$$(\Psi(\vec q_1) - \Psi(q_2))\cdot(\vec q_1 - \vec q_2) \ge \int_0^1 (1+|q_{12}|^a)^{-\frac{1}{a}-1} |q_1-q_2|^2$$

check yourselves

$-1-a \in (-1,-\infty)$

no

$a \in (0,+\infty)$

$\boxed{a=2} \Rightarrow -\frac{3}{2}$