9. SPECTRUM. AN INTRODUCTION TO SPECTRAL THEORY

Def. An operator $T \in \mathcal{L}(X)$, X being Banach, is invertible $\overset{\text{df.}}{\equiv}$ there exists $S \in \mathcal{L}(X)$ so that $\boxed{TS = ST = I}$

Here, I is the identity operator on X: $Tx = x$. for all $x \in X$

▷ If such operator S exists, it is denoted $\boxed{T^{-1}}$

▷ If such operator S exists, it is uniquely determined:

Pf Let $TS_1 = I = S_2T$, then

$$S_2 = S_2TS_1 = IS_1 = S_1.$$

▷ $T \in \mathcal{L}(X)$ is invertible $\iff$ T is one-to-one & onto
(injective) (surjective)

Pf $\boxed{\Rightarrow}$ by definition, there is $S \in \mathcal{L}(X)$:

$$T(Sx) = S(Tx) = x \quad \forall x \in X.$$

If $Tx = Ty$, then $S(Tx) = x$ & $S(Ty) = y$ imply $x = y$ & T is injective.

If $x \in X$ is arbitrary, then set $z := Sx$ and $T(Sx) = x$ implies $Tz = x$, it means T is onto.

$\boxed{\Leftarrow}$ If $T \in \mathcal{L}(X)$ is bijective, then by a consequence of open mapping theorem $T^{-1} \in \mathcal{L}(X)$ and $TT^{-1} = T^{-1}T = I$. Hence T is invertible.

Theorem 9.1 (Sufficient condition for existence of invertible operators of the form $I - T$)

Let X be Banach, $T \in \mathcal{L}(X)$ and $\|T\|_{\mathcal{L}(X)} < 1$.

Then
- $I - T$ is invertible
- $(I - T)^{-1} = \sum T^m$

Pf Set $T^0 := I$ and $S_m := I + T + \dots + T^m$ for all $m \in \mathbb{N}$

Since $\|T^n\|_{\mathcal{L}(X)} \leq \|T\|^n_{\mathcal{L}(X)}$ (verify) and $\|T\|_{\mathcal{L}(X)} < 1$, we observe that

$$\|S_n - S_{n-1}\|_{\mathcal{L}(X)} \leq \|T\|^n_{\mathcal{L}(X)} \xrightarrow{n\to\infty} 0$$

and consequently

$\{S_n\}_{n=1}^{\infty}$ is a cauchy sequence in $\mathcal{L}(X)$.

Hence, as $\mathcal{L}(X)$ is a Banach space, there is $S \in \mathcal{L}(X)$

$$(\bullet) \qquad S := \lim_{n\to\infty} S_n = \sum_{n=0}^{\infty} T^n$$

As, for all $n \in \mathbb{N}$,

$$S_n T = T S_n = S_{n+1} - I, \qquad \text{(verify)}$$

letting $n \to \infty$, we get $\left(\|S_n - S\|_{\mathcal{L}(X)} \to 0 \right)$

$$ST = TS = S - I \iff S(I-T) = (I-T)S = I,$$

which implies $S = (I-T)^{-1}$ and by $(\bullet)$: $(I-T)^{-1} = \sum_{n=0}^{\infty} T^n$.

Theorem 9.1 has two valuable consequences:

$\boxed{\text{Consequence 1}}$ If $T \in \mathcal{L}(X)$ and $\|T\|_{\mathcal{L}(X)} < \lambda$, then:

- $T - \lambda I$ is invertible
- $\|(T-\lambda I)^{-1}\|_{\mathcal{L}(X)} \leq \dfrac{1}{\lambda - \|T\|_{\mathcal{L}(X)}}$

Pf Clearly: $\left\|\frac{T}{\lambda}\right\|_{\mathcal{L}(X)} = \frac{1}{|\lambda|}\|T\|_{\mathcal{L}(X)} = \frac{1}{\lambda}\|T\|_{\mathcal{L}(X)} < 1$

By previous Theorem 9.1: $I - \frac{T}{\lambda}$ is invertible, which implies $T - \lambda I$ is invertible (verify! (*)). Finally

$$\|(T-\lambda I)^{-1}\|_{\mathcal{L}(X)} = \left\| \frac{1}{\lambda}\left(\frac{T}{\lambda} - I\right)^{-1} \right\|_{\mathcal{L}(X)} = \frac{1}{\lambda}\left\| \sum_{n=0}^{\infty}\left(\frac{T}{\lambda}\right)^n \right\|_{\mathcal{L}(X)}$$

$$\leq \frac{1}{\lambda}\sum_{n=0}^{\infty}\left\|\frac{T}{\lambda}\right\|^n_{\mathcal{L}(X)} = \frac{1}{\lambda}\,\frac{1}{1 - \frac{\|T\|_{\mathcal{L}(X)}}{\lambda}} = \frac{1}{\lambda - \|T\|_{\mathcal{L}(X)}}$$

(*) $\left(I - \frac{T}{\lambda}\right)S = S\left(I - \frac{T}{\lambda}\right) = I \iff \left(\frac{T}{\lambda} - I\right)\tilde{S} = \tilde{S}\left(\frac{T}{\lambda} - I\right) = I \qquad (\tilde{S} = -S)$

$$\iff (T-\lambda I)\left(\frac{\tilde{S}}{\lambda}\right) = \left(\frac{\tilde{S}}{\lambda}\right)(T-\lambda I) = I$$

Hence $(T-\lambda I)^{-1} = \left(\frac{\tilde{S}}{\lambda}\right) = \frac{1}{\lambda}\left(\frac{T}{\lambda} - I\right)^{-1} = -\frac{1}{\lambda}\left(I - \frac{T}{\lambda}\right)^{-1}$

$\boxed{\text{Consequence 2}}$ The set of invertible operators is open in $\mathcal{L}(X)$.

More precisely: If $T, S \in \mathcal{L}(X)$ and T is invertible and $\|T-S\|_{\mathcal{L}(X)} < \frac{1}{\|T^{-1}\|_{\mathcal{L}(X)}}$, then S is <u>invertible</u>.

(Pf) Let T and S be as above. Consider $T^{-1}S = I - (I - T^{-1}S)$.

By observing

$$\|I - T^{-1}S\| = \|T^{-1}(T-S)\| \le \|T^{-1}\|_{\mathcal{L}(X)} \|T-S\|_{\mathcal{L}(X)} < 1.$$

By Theorem 9.1, $T^{-1}S$ is invertible, i.e.

$$\exists L \in \mathcal{L}(X) \quad (T^{-1}S)L = L(T^{-1}S) = I.$$

Then $(LT^{-1})S = I$ and $S(LT^{-1}) = I$ $\left(\Leftarrow \begin{bmatrix} T^{-1}SL = I \\ \Downarrow\ SL = T \\ \Downarrow\ SLT^{-1} = I \end{bmatrix}\right)$

Hence S is invertible, q.e.d. ☒

$\boxed{\text{Def.}}$ Let $L \in \mathcal{L}(X)$, X Banach. Then we say that

- $\eta \in \mathbb{K}$ belongs to <u>resolvent</u>, denoted $\varrho(L)$,
  $\overset{\text{df.}}{\equiv}$ $\eta I - L$ is invertible $\iff$ $(\eta I - L)^{-1}$ exists*)
  $\boxed{\iff \eta I - L \text{ is bijective}}$
- The complement of $\varrho(L)$, i.e. $\mathbb{C} \setminus \varrho(L)$, is called <u>spectrum</u> of L, denoted $\sigma(L)$.

[NOTE: If $\eta \in \sigma(L)$, then at least L is <u>not</u> <u>one-to-one</u> or L is <u>not</u> <u>onto</u>]

- We say that η belongs to the <u>point spectrum of L</u>, denoted $\sigma_p(L)$, if $\eta I - L$ is not one-to-one.
  This is equivalent to say that
  $\eta \in \sigma_p(L) \iff \exists w \in H, w \neq 0 : Lw = \eta w$
  It means: <u>$\eta \in \sigma_p(L)$</u> $\iff$ <u>η is eigenvalue</u> of L with w being a corresponding eigenvector.

*) and by a consequence of open mapping theorem $(\eta I - L)^{-1} \in \mathcal{L}(X)$.

- The essential spectrum of L, denoted $\sigma_e(L)$, is defined as $$\sigma_e(L) := \sigma(L) - \sigma_p(L);$$ it consists of all $\eta \in \mathbb{K}$ so that $$\eta I - L \text{ is one-to-one, but not onto.}$$

9.1. Spectrum of compact operators on Hilbert spaces

Theorem 9.2 Let $(H, (\cdot,\cdot)_H)$ be (an infinite-dimensional) Hilbert space (over $\mathbb{R}$) and $K \in \mathcal{L}(H)$ be compact. Then

(1) $0 \in \sigma(K)$

(2) $\sigma(K) = \sigma_p(K) \uplus \{0\}$

(3) Either $\sigma_p(K)$ is finite, or else $\sigma_p(K) = \{\lambda_k, k \in \mathbb{N}\}$ and $\lim_{k\to\infty} \lambda_k = 0$

Pf **Ad (1)** By contradiction. If $0 \notin \sigma(K)$, then K has an inverse $K^{-1}: H \to H$ and $K^{-1} \in \mathcal{L}(H)$ (continuous). Then $I = K^{-1} \circ K$.

Since the composition of a continuous and a compact operator is compact, then the identity $I: H \to H$ is compact, which contradicts to the fact that the ball in the infinite-dimensional Hilbert space is not compact.

Ad (2) Let us assume that $\lambda \in \sigma(K)$ with $\lambda \neq 0$.
The goal is to check that $\lambda \in \sigma_p(K)$. If, however, $\lambda \notin \sigma_p(K)$, then $\operatorname{Ker}\{\lambda I - K\} = \{0\}$ which, by Fredholm alternative theorem 8.1, is equivalent to $\operatorname{Im}(\lambda I - K) = H$, but then $\lambda I - K$ is invertible and $\lambda \notin \sigma(K)$, which contradicts to our assumption.

Ad (3) Assume $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of distinct eigenvalues of K and $\lim_{n\to\infty} \lambda_n = \lambda$. Goal is to show that $\lambda = 0$.

Indeed, as $\lambda_n \in \sigma_p(K)$, for each $n \in \mathbb{N}$ there exists w_n (eigenvector) so that $K w_n = \lambda_n w_n$.
Denote

$$H_n := \operatorname{Span}\{w_1, \dots, w_n\}$$

Since w_n are linearly independent (as λ_n are distinct)

$$H_n \subset H_{n+1}$$

Observing that, for every $n \geq 2$, $(K - \lambda_n I) H_n \subseteq H_{n-1}$, we can choose $e_n \in H_n \cap H_{n-1}^{\perp}$ with $\|e_n\|_H = 1$. Then for $m < n$

$$K e_n - \lambda_n e_n \in H_{n-1} \quad \text{and} \quad K e_m - \lambda_m e_m \in H_{m-1} \subset H_{n-1}$$

and $e_n \in H_{n-1}^{\perp}$ with $e_m \in H_m \subseteq H_{n-1}$

Hence

$$\|K e_n - K e_m\|_H = \|\underbrace{(K e_n - \lambda_n e_n) - (K e_m - \lambda_m e_m) - \lambda_m e_m}_{\in H_{n-1}} + \underbrace{\lambda_n e_n}_{\in H_{n-1}^{\perp}}\|_H$$

$$\geq \|\lambda_n e_n\|_H = |\lambda_n|$$

Therefore

$$\liminf_{n,m \to \infty} \|K e_n - K e_m\|_H \geq \lim_{n \to \infty} |\lambda_n| = |\lambda|$$

If $\lambda > 0$, there is no way to select converging subsequence, giving the contradiction. $\square$

Selfadjoint operators

Recall that if $A \in \mathbb{R}^{n \times n}$ is symmetric, then

$$(Ax, y)_m = (x, Ay)_m \quad \text{for all } x, y \in \mathbb{R}^m$$

It means that the operator generated by such A's has the property that $A^* = A$, i.e. A is selfadjoint.

It is known (and can be shown as an application of Lagrange multiplier theory concerning constraint optimization) that $m := \min_{|x|=1} (Ax, x)_m$ and $M := \max_{|x|=1} (Ax, x)_m$

are the smallest and the largest eigenvalue of A.

This result can be generalized to an infinite-dimensional setting.

Def. Let $L \in \mathcal{L}(H)$, where $(H, (\cdot,\cdot)_H)$ is a Hilbert space over $\mathbb{R}$. We say that L is <u>symmetric</u> (<u>selfadjoint</u>)

$$\overset{df.}{\equiv} (Lx, y)_H = (x, Ly)_H \quad \forall x, y \in H. \qquad (\circledcirc)$$

If $(\circledcirc)$ holds for $x, y \in H$, where H is over $\mathbb{C}$, then L is called <u>Hermitian</u>

$\boxed{\text{Theorem 9.3}}$ (Bounds on the spectrum of a selfadjoint operator)

Let $L \in \mathcal{L}(H)$ be selfadjoint operator on a real Hilbert space H.

Define
$$m := \inf_{\substack{u\in H\\ \|u\|_H=1}} (Lu,u) \quad \text{and} \quad M := \sup_{\substack{u\in H\\ \|u\|_H=1}} (Lu,u)$$

Then,

(α) $\sigma(L) \subset \langle m, M\rangle$

(β) $m, M \in \sigma(L)$

(γ) $\|L\|_{\mathcal{L}(H)} = \max\{-m, M\}$.

$\boxed{\text{Proof.}}$ $\boxed{\text{Ad}(\alpha)}$ Let $\eta > M$. Then
$$(\eta u - Lu, u) = \eta(u,u) - (Lu,u) \geq (\eta - M)\|u\|_H^2,$$
i.e. $B(u,v) := (\eta u - Lu, v)$ fulfills the assumptions of Lax-Milgram lemma: hence $\eta I - L$ is one-to-one & onto (and, by Open mapping theorem, its inverse is continuous). Hence, every $\eta > M$ belongs to $\varrho(L)$ ... the resolvent set.

Similarly, for $\eta < m$,
$$(Lu - \eta u, u) = (Lu,u) - \eta\|u\|_H^2 \geq (m-\eta)\|u\|_H^2$$
and, by the same arguments, we conclude that all $\eta < m$ belongs to $\varrho(L)$ as well.

$\boxed{\text{Ad}(\gamma)}$ Assume that $\boxed{|m| \leq M}$. Then, for $\forall u, v \in H$,
$$4(Lu,v)_H = (L(u+v), u+v)_H - (L(u-v), u-v)_H$$
$$\leq M\left(\|u+v\|_H^2 + \|u-v\|_H^2\right) = 2M\left(\|u\|_H^2 + \|v\|_H^2\right)$$
If $Lu \neq 0$, setting $v = \frac{\|u\|_H}{\|Lu\|_H} Lu$ we get
$$2\|Lu\|_H\|u\|_H = 2(Lu,v)_H \leq M\left(\|u\|_H^2 + \|v\|_H^2\right) = 2M\|u\|_H^2,$$
which implies

$$\boxed{(*) \quad \|Lu\|_H \leq M\|u\|_H \quad \text{for all } u \in H}$$

Obviously, this holds also for $Lu = 0$.

Since
$$M = \sup_{\|u\|_H=1} (Lu,u) \leq \sup_{\|u\|_H=1} \|Lu\|_H\|u\|_H \leq \|L\|_{\mathcal{L}(H)},$$
this together with $(*)$ gives $\|L\|_{\mathcal{L}(H)} = M$.

If $\boxed{M < -m}$, which is the opposite case to $|m| \le M$, is treated similarly, replacing L by $-L$. ⊠

$\boxed{\text{Ad } (\beta)}$ To show that $M \in \sigma_p(L)$, consider a sequence $\{u_n\}_{n=1}^{\infty}$ such that $\|u_n\|_H = 1$ and $(Lu_n, u_n) \to M$

Then $\|(L-MI)u_n\|_H^2 = (\underset{\uparrow \text{op.}}{L}u_n - \underset{\uparrow \text{number}}{M}u_n, Lu_n - Mu_n)$

$$= \|Lu_n\|_H^2 - 2M(Lu_n, u_n) + M^2\|u_n\|_H^2$$

$$\le 2M^2 - 2M(Lu_n, u_n) \xrightarrow[n\to\infty]{} 0.$$

Consequently, $L - MI$ cannot have a bounded inverse.

$\boxed{\text{Theorem 9.4}}$ Let H be separable Hilbert space over $\mathbb{R}$. Let $K \in \mathcal{L}(H)$ be compact and selfadjoint.

Then, there is a countable orthonormal basis of H consisting of eigenvectors of K.

$\boxed{\text{Proof}}$ If $\dim H < \infty$, then see your LA course. Let $\boxed{\dim H = +\infty}$.

Let $\eta_0 = 0$ and $\{\eta_1, \eta_2, \dots\}$ be the set of nonzero eigenvalues of K. Consider

$$H_0 = \operatorname{Ker} K, \quad H_1 = \operatorname{Ker}(K - \eta_1 I), \quad H_2 = \operatorname{Ker}(K - \eta_2 I), \dots$$

Observe that while $\dim H_0$ can be whatever between 0 and $+\infty$, $\dim H_k$ is positive and finite, see Theorem 8.1.

▷ Next, we show that for $m \ne n$: H_m and H_n are orthogonal.

Indeed: if $u \in H_m$, $v \in H_n$. Then

$$\eta_m (u,v)_H = (Ku, v)_H = (u, Kv)_H = \eta_n (u,v)_H,$$

which gives $\underbrace{(\eta_m - \eta_n)}_{\ne 0}(u,v)_H = 0$ implying $(u,v)_H = 0$

▷ Next, we show that H_k (their union) generates the entire space.

Precisely: Let

$$\tilde{H} \equiv \left\{ \sum_{k=1}^{N} \alpha_k u_k, \ N \ge 1, \ u_k \in H_k, \ \alpha_k \in \mathbb{R} \right\}.$$

Then we prove that

$$(✿) \qquad (\tilde{H})^{\perp} \subseteq \operatorname{Ker} K = H_0$$

(Pf) Note that $K(\tilde{H}) \subset \tilde{H}$. Moreover, if $u \in \tilde{H}^\perp$ and $v \in \tilde{H}$, then $Kv \in \tilde{H}$ and $(Ku, v) = (u, Kv) = 0$. This shows that $K(\tilde{H}^\perp) \subseteq \tilde{H}^\perp$.

Furthermore, let $\tilde{K}$ be restriction of K to $\tilde{H}^\perp$. Clearly, $\tilde{K}$ is compact selfadjoint operator. By Theorem 9.3

$$\|\tilde{K}\| = \sup_{\substack{u \in \tilde{H}^\perp \\ \|u\|_H = 1}} |(\tilde{K}u, u)| =: M$$

If $M \neq 0$, then either M or $-M$ belongs to $\sigma(\tilde{K})$. As $\tilde{K}$ is compact, $M \neq 0$, then M or $-M$ — " — $\sigma_p(\tilde{K})$. Hence $\exists\ 0 \neq w \in H^\perp$ so that

$$\tilde{K}w = Kw = \lambda w$$

This however contradicts to the fact that all eigenvectors belong to $\tilde{H}$. We thus conclude that $\|\tilde{K}\| = 0$ proving (B).

(D) For each $k \geq 1$, the finite-dimensional subspaces H_k admits an orthonormal basis $\mathcal{B}_k = \{e_{k,1}, e_{k,2}, \dots, e_{k,N(k)}\}$. Moreover, as H is separable, the closed subspace $H_0 = \operatorname{Ker} K$ admits a countable orthonormal basis $\mathcal{B}_0$. Hence

$$\mathcal{B} := \bigcup_{k \geq 0} \mathcal{B}_k \text{ is an orthonormal basis in } H.$$

∎

$\boxed{\text{Theorem 9.5}}$ (Hellinger-Toeplitz theorem) Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space and $A: H \to H$ be a selfadjoint linear operator. Then A is continuous.

Proof By the Banach closed graph theorem, it is enough to show that A is closed. So, let $x_n \to x$ in H and $Ax_n \to y$ in H

Then $(Ax_n, z)_H \to (y, z)$ and $(Ax_n, z) = (x_n, Az) \xrightarrow[n\to\infty]{} (x, Az) = (Ax, z)$.

Consequently $(Ax, z) = (y, z)$ for all $z \in H$. Hence $\underline{Ax = y}$.

∎

$\boxed{\text{Theorem 9.3*}}$ Let $(H, (\cdot,\cdot)_H)$ be an infinite-dimensional Hilbert space and $A: H \to H$ be compact and selfadjoint linear operator with $\dim(\operatorname{Im} A) = \infty$. Then

$\boxed{(a)}$ $\exists \{\lambda_n\}_{n=1}^{\infty}$ of eigenvalues and $\{v_n\}_{n=1}^{\infty}$ of corresponding eigenvectors satisfying

- $|\lambda_1| = \|A\|$, $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n| \geq \dots$, $|\lambda_n| \neq 0$ for all $n \in \mathbb{N}$,
- $\lim_{n\to\infty} \lambda_n = 0$
- $Av_n = \lambda_n v_n \quad \forall n$; $\quad (v_n, v_m)_H = \delta_{nm}$

$$|\lambda_1| = \frac{|(Av_1, v_1)|}{\|v_1\|_H^2} = \sup_{x\neq 0} \frac{|(Ax,x)|}{\|x\|_H^2}$$

$$|\lambda_n| = \frac{|(Av_n, v_n)|}{\|v_n\|_H^2} = \sup_{\substack{x\neq 0 \\ (x, v_k) = 0 \\ 1 \leq k \leq n-1}} \frac{|(Ax,x)|}{\|x\|_H^2} \quad \text{for all } n \geq 2$$

$\boxed{(b)}$ For any $x \in H$:

$$\boxed{Ax = \sum_{n=1}^{\infty} \lambda_n (x, v_n) v_n}$$

$\boxed{(c)}$ Let $\lambda \neq 0$ be an eigenvalue of A. Then, there exists $m \geq 1$ s.t. $\lambda_m = \lambda$. Besides $I(\lambda) = \{m \geq 1;\ \lambda_m = \lambda\}$ is finite and

$$\{p \in H;\ Ap = \lambda p\} = \operatorname{span}\{p_m\}_{m \in I(\lambda)}$$

$\boxed{(d)}$ The kernel of A is given as

$$\operatorname{Ker} A = \left[\operatorname{span}\{p_m\}_{m=1}^{\infty}\right]^{\perp}$$

$\boxed{\text{Theorem 9.3**}}$

- Let $(H, (\cdot,\cdot)_H)$ be an infinite-dimensional Hilbert space and let $A: X \to X$ be injective, compact, self-adjoint linear operator. Then the eigenvectors $\{v_n\}_{n=1}^{\infty}$ from Theorem 9.3* form a maximal orthonormal family.
- If in addition H is separable Hilbert space, then the eigenvectors $\{v_n\}_{n=1}^{\infty}$ from Theorem 9.3* form a Hilbert basis in H. $\boxed{\forall x \in H: x = \sum_{i=1}^{\infty} (x, v_i) v_i}$

i.e. orthonormal

NOTE If $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of a real Hilbert space H consisting of eigenvectors of a linear, compact, self-adjoint operator K. with $\{\lambda_n\}_{n=1}^{\infty}$ being the corresponding eigenvalues, then: for a given $f \in H$ ~~such that~~ consider

(Eq) $$u - Ku = f.$$

If $1 \notin \sigma(K)$, then (6.9) admits a unique soln.

$$u = \sum c_k e_k \qquad f = \sum b_k e_k$$

then, after inserting into (Eq),

$$\sum (c_k e_k - \lambda_k c_k e_k - b_k e_k) = 0$$

we get

$$u = \sum_{k=1}^{\infty} \frac{b_k}{1-\lambda_k} e_k = \sum_{k=1}^{\infty} \frac{(f, e_k)\, e_k}{(1-\lambda_k)}.$$

NOTE Given ~~a~~ countable set $S = \{u_1, u_2, \dots\}$, a basic pblm is to decide if span S is dense in X. Two important positive cases:

(I) $\boxed{X = C(E)}$ continuous fcts over a compact metric space E. Then $\overline{\text{span } S} = X$ if span S is algebra that contains a constant fctns and separates points.

(II) X is a separable Hilbert space & $\exists\, L: X \to X$ s.t. that
(i) span S contains all eigenvectors of L
(ii) span S — " — the kernel of L
Then, by Hilbert-Schmidt theorem 9.4: $\overline{\text{span } S} = X$.