Cvičení Matematika pro fyziky II – 22. dubna 2021, 16:30–18:00 NAHRÁVAT!

Implikaci jsme dokázali pomocí Cauchyho integrálního vzorce:

$$f\in H(B_R(a)):\quad f(w) = \frac{1}{2\pi i}\int_{\partial B_\varrho(a)} \frac{f(z)}{z-w}\,dz \qquad \text{kde } w\in B_\varrho(a)$$

$$= \frac{1}{2\pi i}\int_{\Gamma} f(z)(z-w)^{-1}\,dz \qquad \text{kde } w\in \text{vnitřek } \Gamma$$

Zobecnění:

$A\in\mathbb{C}^{d\times d}$ matice, f analytické (holomorfní) fce

$$f(A) \overset{\text{def.}}{:=} \frac{1}{2\pi i}\int_{\Gamma} f(z)(z\mathbb{I}-A)^{-1}\,dz \qquad \text{kde } \Gamma\subset\Omega$$

a Γ uzavírá spektrum matice A

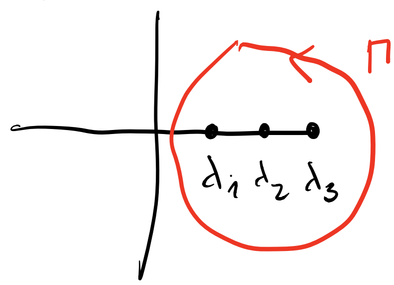
které lze dále zobecnit pro neomezené operátory.

$$A=\begin{bmatrix}6&2&0\\3&7&0\\0&0&1\end{bmatrix}, \qquad \sqrt{A}=? \qquad f(z)=\sqrt{z}$$

$$\sqrt{A}=\frac{1}{2\pi i}\int_{\Gamma}\sqrt{z}\,(z\mathbb{I}-A)^{-1}\,dz$$

Γ – uzavírá spektrum matice A

$$(A-\lambda\mathbb{I})\vec{v}=0$$

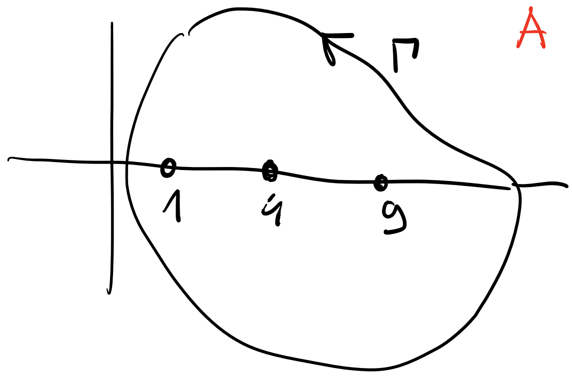


$$(\lambda\mathbb{I}-A)^{-1}=\begin{bmatrix}\lambda-6&-2&0\\-3&\lambda-7&0\\0&0&\lambda-1\end{bmatrix}^{-1}$$

$$\begin{bmatrix}\lambda-6&-2&0\\-3&\lambda-7&0\\0&0&\lambda-1\end{bmatrix}^{-1}=\begin{bmatrix}\frac{\lambda-7}{(\lambda-4)(\lambda-9)}&\frac{2}{(\lambda-4)(\lambda-9)}&0\\ \frac{3}{(\lambda-4)(\lambda-9)}&\frac{\lambda-6}{(\lambda-4)(\lambda-9)}&0\\0&0&\frac{1}{\lambda-1}\end{bmatrix}=(\lambda\mathbb{I}-A)^{-1}$$

$$\sqrt{A} = \frac{1}{2\pi i}\int_\Gamma \sqrt{z}\,(zI - A)^{-1}\,dz = \frac{1}{2\pi i}\int_\Gamma \begin{bmatrix} \frac{\sqrt{z}(z-7)}{(z-4)(z-9)} & \underbrace{\frac{2\sqrt{z}}{(z-4)(z-9)}}_{B} & 0 \\ \underbrace{\frac{3\sqrt{z}}{(z-4)(z-9)}}_{A} & \underbrace{\frac{\sqrt{z}(z-6)}{(z-4)(z-9)}}_{C} & 0 \\ 0 & 0 & \frac{\sqrt{z}}{z-1} \end{bmatrix} dz$$

Vlastní čísla A: $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 9$



$$= \frac{1}{2\pi i}\begin{bmatrix} \int_\Gamma \frac{\sqrt{z}(z-7)}{(z-4)(z-9)}\,dz & \int_\Gamma \big(\; B \;\big)\,dz & 0 \\ \int_\Gamma \big(\; A \;\big)\,dz & \int_\Gamma \big(\; C \;\big)\,dz & 0 \\ 0 & 0 & \int_\Gamma \frac{\sqrt{z}}{z-1}\,dz \end{bmatrix}$$

$$\int_\Gamma \frac{\sqrt{z}(z-7)}{(z-4)(z-9)}\,dz = 2\pi i\left(\operatorname{res}_{z=4}\frac{\sqrt{z}(z-7)}{(z-4)(z-9)} + \operatorname{res}_{z=9}\frac{\sqrt{z}(z-7)}{(z-4)(z-9)}\right)$$

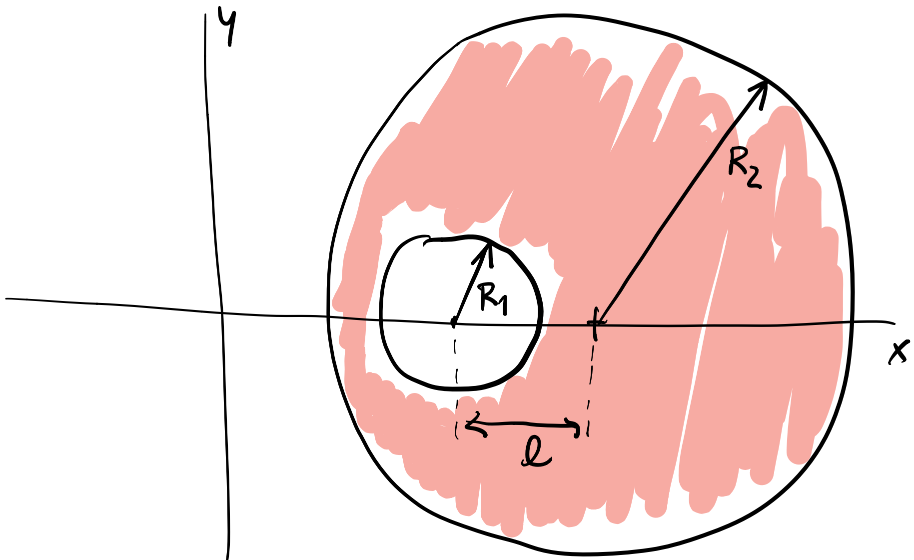
$$= 2\pi i\left(\left.\frac{\sqrt{z}(z-7)}{z-9}\right|_{z=4} + \left.\frac{\sqrt{z}(z-7)}{z-4}\right|_{z=9}\right)$$

$$= 2\pi i\left(\frac{6}{5} + \frac{6}{5}\right)$$

$$= \frac{1}{2\pi i}\begin{bmatrix} 2\pi i\left(\frac{6}{5}+\frac{6}{5}\right) & 2\pi i\left(-\frac{4}{5}+\frac{6}{5}\right) & 0 \\ 2\pi i\left(-\frac{6}{5}+\frac{9}{5}\right) & 2\pi i\left(\frac{4}{5}+\frac{9}{5}\right) & 0 \\ 0 & 0 & 2\pi i\, 1 \end{bmatrix} = \begin{bmatrix} \frac{12}{5} & \frac{2}{5} & 0 \\ \frac{3}{5} & \frac{13}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sqrt{A} = \begin{bmatrix} \frac{12}{5} & \frac{2}{5} & 0 \\ \frac{3}{5} & \frac{13}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \sqrt{A}\cdot\sqrt{A} = \begin{bmatrix} 6 & 2 & 0 \\ 3 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Úloha:



Řeš úlohu:

$$\Delta_{(x,y)} \varphi(x,y) = 0 \quad \text{v mezikruží}$$

\+ okrajové podmínky

$$\varphi(x,y)\Big|_{\text{vnitřní kružnice}} = \varphi_{in}$$

$$\varphi(x,y)\Big|_{\text{vnější kružnice}} = \varphi_{out}$$

reálná čísla, konstanty

Pozorování:

$$f(z) = u(x,y) + i\,v(x,y) \qquad f \text{ je holomorfní}$$

① $\Delta u(x,y) = 0$
$\Delta v(x,y) = 0$

② $u(x,y) = c_1$ pro dané c_1 popisuje $u(x,y)$ nějakou křivku v rovině x,y
$v(x,y) = c_2$ pro dané c_2 popisuje $v(x,y)$ nějakou křivku v rovině x,y

$\frac{d\vec{\varphi}}{ds}(s)$ ... tečný vektor ke křivce $u(x,y) = c_1$
$\frac{d\vec{\beta}}{ds}(s)$ ... tečný vektor ke křivce $v(x,y) = c_2$

Tvrzení: $\frac{d\vec{\varphi}}{ds} \cdot \frac{d\vec{\beta}}{ds} = 0$

$\vec{\gamma}(s) = \begin{bmatrix} \gamma_x(s) \\ \gamma_y(s) \end{bmatrix}$

$c_1 = u(x,y) = u(\gamma_x(s), \gamma_y(s))$ $\Big\downarrow \frac{d}{ds}$ derivování

$$0 = \frac{\partial u}{\partial x}\frac{d\gamma_x}{ds} + \frac{\partial u}{\partial y}\frac{d\gamma_y}{ds}$$

$$0 = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} \bullet \frac{d\vec{\gamma}}{ds}$$

závěr: $\frac{d\vec{\gamma}}{ds}$ je kolmá na gradient u

Druhá křivka $\vec{\beta}(s)$
(stejná manipulace)

$$0 = \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} \bullet \frac{d\vec{\beta}}{ds}$$

Cauchy–Riemannovy podmínky:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\begin{bmatrix} \frac{\partial v}{\partial y} \\ -\frac{\partial v}{\partial x} \end{bmatrix} \bullet \frac{d\vec{\gamma}}{ds} = 0 \qquad \frac{d\vec{\gamma}}{ds} \perp \begin{bmatrix} \frac{\partial v}{\partial y} \\ -\frac{\partial v}{\partial x} \end{bmatrix}$$

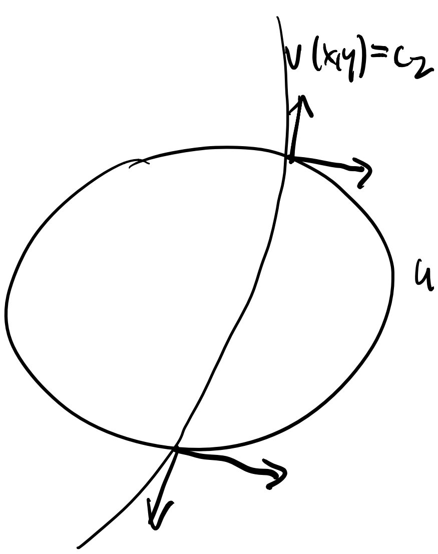
$$\begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} \bullet \begin{bmatrix} \frac{\partial v}{\partial y} \\ -\frac{\partial v}{\partial x} \end{bmatrix} = 0$$

$$\Big\downarrow$$

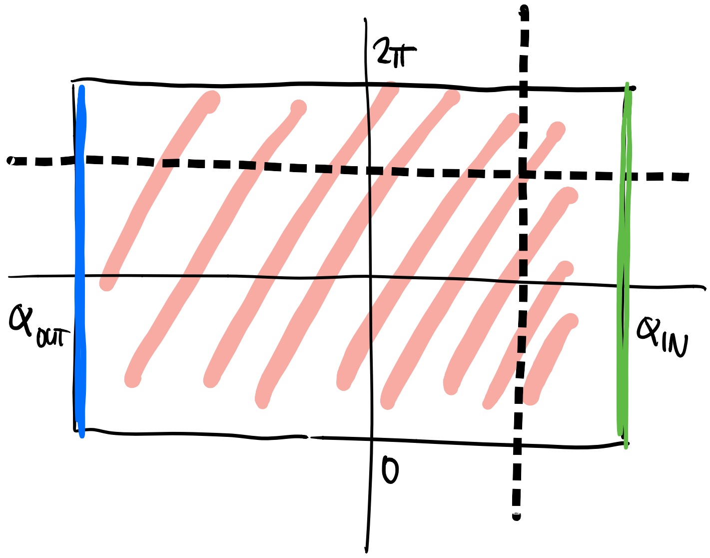
$$\frac{d\vec{\gamma}}{ds} \parallel \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix}$$

$$\frac{d\vec{\gamma}}{ds} = k \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix}$$

$$\boxed{\frac{d\vec{\gamma}}{ds} \bullet \frac{d\vec{\beta}}{ds} = 0}$$



$\Delta u = 0$
$\Delta v = 0$



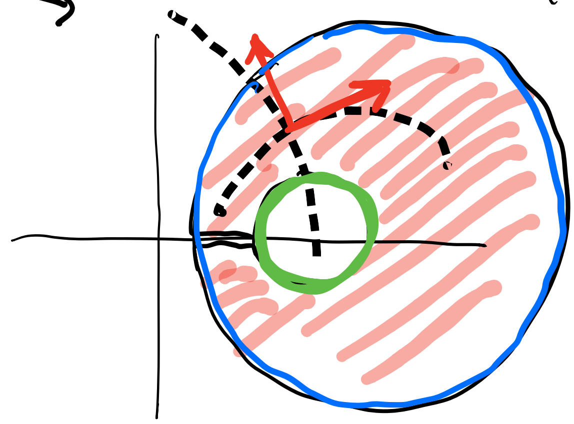
$$z = f(\varsigma)$$

$z = x+iy$

$\varsigma = \alpha + i\beta$

x, y fyzikální prostor

α, β prostor parametrizace



$$\underbrace{\alpha + i\beta}_{\varsigma} = \ln \frac{y+i(x+a)}{y+i(x-a)} \quad \longrightarrow \quad y+ix = ia\,\frac{e^{\alpha+i\beta}+1}{e^{\alpha+i\beta}-1}$$

$z = x+iy$

$$x = \frac{a \sinh\alpha}{\cosh\alpha - \cos\beta}$$

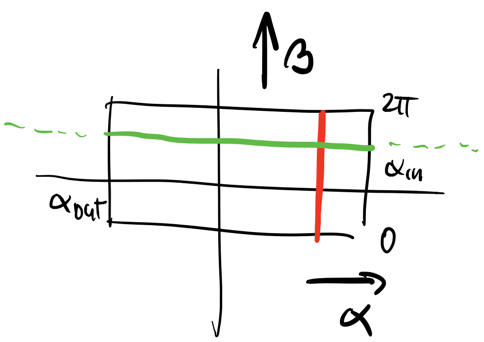
$$y = \frac{a \sin\beta}{\cosh\alpha - \cos\beta}$$

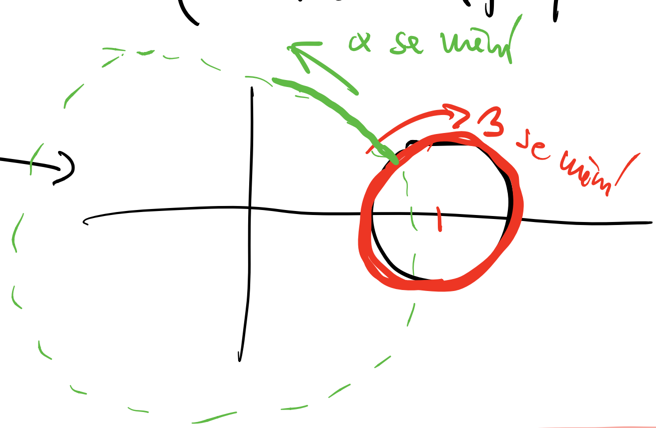
$\xrightarrow{\text{krátká algebraická úprava}}$

$\beta \in (0, 2\pi)$

$$x^2 + (y - a\,\mathrm{cotan}\,\beta)^2 = \frac{a^2}{\sin^2\beta}$$

$$(x - a\,\mathrm{cotanh}\,\alpha)^2 + y^2 = \frac{a^2}{\sinh^2\alpha}$$





střed malé kružnice $\quad a\,\dfrac{\cosh\alpha_{in}}{\sinh\alpha_{in}}$

střed velké kružnice $\quad a\,\dfrac{\cosh\alpha_{out}}{\sinh\alpha_{out}}$

$$e = a\left(\frac{\cosh\alpha_{out}}{\sinh\alpha_{out}} - \frac{\cosh\alpha_{in}}{\sinh\alpha_{in}}\right)$$

$$R_1^2 = \frac{a^2}{\sinh^2\alpha_{in}}$$

$$R_2^2 = \frac{a^2}{\sinh^2\alpha_{out}}$$

Výsledek: Soustava rovnic pro $a, \alpha_{in}, \alpha_{out}$:

$$a = \frac{1}{2e}\left(\left((R_1^2+R_2^2)-e^2\right)^2 - 4R_1^2R_2^2\right)^{1/2}$$

$$\sinh\alpha_{in} = \frac{a}{R_1} \qquad \sinh\alpha_{out} = \frac{a}{R_2}$$

Okrajová podmínka

$$\varphi\Big|_{\text{vnitřní hrana}} = \varphi_{in}$$

$$\|$$

$$\varphi(x(\alpha,\beta), y(\alpha,\beta))\Big|_{\alpha=\alpha_{in},\ \beta\in[0,2\pi)} = \varphi_{in}$$

$$\varphi(x(\alpha,\beta), y(\alpha,\beta))\Big|_{\alpha=\alpha_{out},\ \beta\in[0,2\pi)} = \varphi_{OUT}$$

$$\Delta_{(x,y)}\varphi(x,y) = 0$$

Jak využijeme konformní zobrazení ?

$$\left.\begin{aligned} &\Delta_{(\alpha,\beta)} g(\alpha,\beta) = 0 \\ &g\Big|_{\alpha=\alpha_{in}} = \varphi_{in} \\ &g\Big|_{\alpha=\alpha_{OUT}} = \varphi_{out} \end{aligned}\right\} \quad \left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right) g(\alpha,\beta) = 0$$

Řešení $\Delta g = 0$ + okrajové podmínky

$$g(\alpha,\beta) = \frac{\varphi_{out} - \varphi_{in}}{\alpha_{out} - \alpha_{in}}(\alpha - \alpha_{in}) + \varphi_{in}$$

Viz chování Laplaceova operátoru při konformním zobrazení:

$$\Delta_{(x,y)}\varphi(x,y) = \Delta_{(x,y)} g(\alpha,\beta) = \Delta_{(x,y)} g(\alpha(x,y), \beta(x,y)) = \dots$$

$$= \left(|f'(z)|^2\right)\Delta_{(\alpha,\beta)} g(\alpha,\beta)$$

$$\Delta_{(\alpha,\beta)} g(\alpha,\beta) = 0 \longrightarrow \Delta_{(x,y)} g(\alpha(x,y), \beta(x,y)) = 0$$

$$\downarrow$$

$$\varphi(x,y) =_{\text{def}} g(\alpha(x,y), \beta(x,y))$$

$$\varphi(x,y) = g(\alpha(x,y), \beta(x,y))$$

$$\varphi(x,y) = \frac{\varphi_{out} - \varphi_{in}}{\alpha_{out} - \alpha_{in}}(\alpha - \alpha_{in}) + \varphi_{in} =$$

najdi vzorec pro vztah mezi α a x, y

$$\alpha + i\beta = \ln \frac{y+i(x+a)}{y+i(x-a)}$$

$$\alpha + i\beta = \ln \left| \frac{y+i(x+a)}{y+i(x-a)} \right| + i\left(\cdots\cdots \right)$$

$$= \ln \left(\frac{y^2+(x+a)^2}{y^2+(x-a)^2} \right) + i\left(\cdots\cdots \right) \longrightarrow \alpha = \ln \left(\frac{y^2+(x+a)^2}{y^2+(x-a)^2} \right)$$

dosaď do vzťahu pro φ

$$\varphi(x,y) = \frac{\varphi_{out} - \varphi_{in}}{\alpha_{out} - \alpha_{in}} \left(\ln\left(\frac{y^2+(x+a)^2}{y^2+(x-a)^2} \right) - \alpha_{in} \right) + \varphi_{in}$$

$$a = \frac{1}{2e} \left(\left((R_1^2 + R_2^2) - e^2 \right)^2 - 4R_1^2 R_2^2 \right)^{1/2}$$

$$\sinh \alpha_{in} = \frac{a}{R_1} \qquad \sinh \alpha_{out} = \frac{a}{R_2}$$