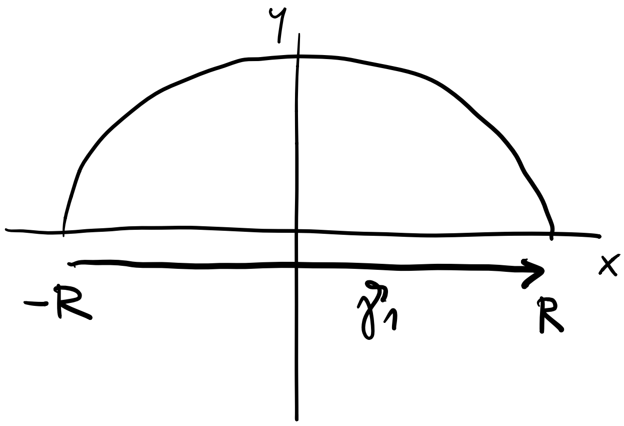
Matematika pro fyziky II 16:30 – 18:00 15. dubna 2021

NAHRÁVAT

(1) $I = \int_{x=-\infty}^{+\infty} \frac{\cosh(ax)}{\cosh(\pi x)}\,dx$, $a>0$ $\qquad e^{(a-\pi)x}$ $\qquad a<\pi$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$$\int_{x=-\infty}^{+\infty} \frac{e^{ax}+e^{-ax}}{e^{\pi x}+e^{-\pi x}}\,dx = \int_{x=-\infty}^{+\infty} \frac{\boxed{e^{ax}}(1+e^{-2ax})}{\boxed{e^{\pi x}}(1+e^{-2\pi x})}\,dx$$



$\int f(z)\,dz \qquad f(z) = \frac{\cosh(az)}{\cosh(\pi x)}$

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} \frac{\cosh(ax)}{\cosh(\pi x)}\,dx$$

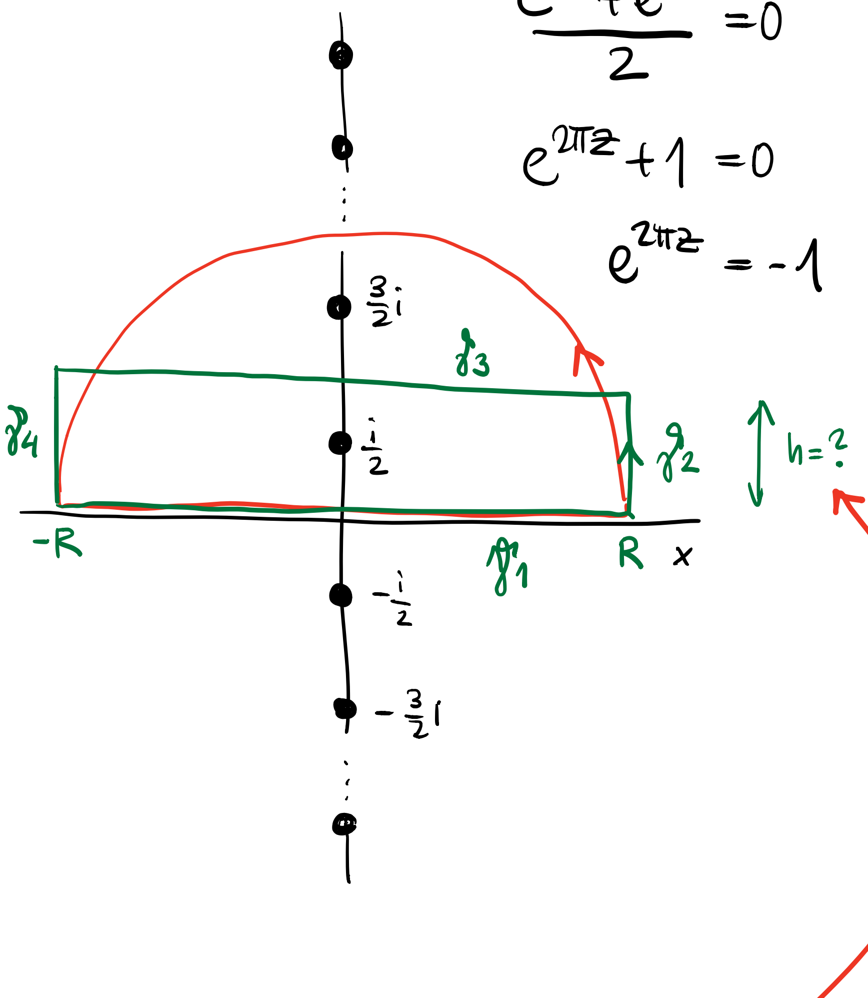
$f(z) = \frac{\cosh(az)}{\cosh(\pi z)}$ $\qquad \cosh(\pi z) \stackrel{?}{=} 0 \qquad \cosh(\pi x) = 0 \quad x \in \mathbb{R}$

$$\frac{e^{\pi z} + e^{-\pi z}}{2} = 0$$

$$e^{2\pi z} + 1 = 0$$

$$e^{2\pi z} = -1 \longrightarrow 2\pi z = i\pi + 2k\pi i \quad k \in \mathbb{Z}$$

$$z = \frac{i}{2} + ki \quad k \in \mathbb{Z}$$



$$\int_{\gamma_2} f(z)\,dz = \left| \begin{array}{l} z = R + is\ , \ s\in(0,h) \\ dz = i\,ds \end{array} \right|$$

$z = Re^{i\varphi}$

půlkruh

$$\frac{e^{aRe^{i\varphi}} + e^{-aRe^{i\varphi}}}{\cdots} \quad Re^{i\varphi}$$

$$= \int_{s=0}^{h} i\,\frac{\cosh(a(R+is))}{\cosh(\pi(R+is))}\,ds = \int_{s=0}^{h} i\,\frac{e^{aR+ais} + e^{-aR-ais}}{e^{\pi R+\pi is} + e^{-\pi R-\pi is}}\,ds$$

$$\left|\int_{\gamma_2} f(z)\,dz\right| \leq \int_{s=0}^{h} \left|\frac{e^{aR+ais} + e^{-aR-ais}}{e^{\pi R+\pi is} + e^{-\pi R-\pi is}}\right| ds \leq \int_{s=0}^{h} \frac{e^{aR} + e^{-aR}}{\left|e^{\pi R+\pi is} + e^{-\pi R-\pi is}\right|}\,ds$$

$$\leq \int_{s=0}^{h} \frac{e^{aR} + e^{-aR}}{\left|\,|e^{\pi R+\pi is}| - |e^{-\pi R-\pi is}|\,\right|}\,ds \leq \int_{s=0}^{h} \frac{e^{aR} + e^{-aR}}{e^{\pi R} - e^{-\pi R}}\,ds \xrightarrow[R\to+\infty]{h \text{ je konečné}} 0$$

$a<\pi$

$$\frac{1}{\left||a|-|b|\right|} \leq \frac{1}{|a+b|}$$

$|a+b| \leq |a| + |b|$

$|a| = |a+b-b| \leq |a+b| + |b|$

$|b| = |b-a+a| \leq |a+b| + |a|$

$|a| - |b| \leq |a+b|$

$|b| - |a| \leq |a+b|$

$\left||a|-|b|\right| \leq |a+b|$

$a<\pi$

$$\frac{e^{aR} + e^{-aR}}{e^{\pi R} - e^{-\pi R}} = \frac{e^{aR}\left(1 + e^{-2aR}\right)}{e^{\pi R}\left(1 - e^{-2\pi R}\right)} = e^{\underbrace{(a-\pi)}_{\ominus}R}\,\frac{1 + e^{-2aR}}{1 - e^{-2\pi R}}$$

$$\int_{\gamma_2} f(z)\,dz \xrightarrow{R\to+\infty} 0$$

$$\int_{\gamma_4} f(z)\,dz \xrightarrow{R\to\infty} 0$$

$$-\int_{\gamma_3} f(z)\,dz = \left| \begin{array}{l} z = s + ih \\ dz = ds \end{array} \right| = \int_{s=-R}^{R} \frac{\cosh(a(s+ih))}{\cosh(\pi(s+ih))}\,ds$$

$$= \int_{s=R}^{R} \frac{e^{as+aih} + e^{-as-aih}}{e^{\pi s+\pi ih} + e^{-\pi s - i\pi h}}\, ds \overset{\downarrow}{=}$$ Jak zvolit h ?

$$= \int_{s=-R}^{R} \frac{e^{as}\left(\cos(ah) + i\sin(ah)\right) + e^{-as}\left(\cos(ah) - i\sin(ah)\right)}{e^{\pi s + i\pi h} + e^{-\pi s - i\pi h}}\, ds$$

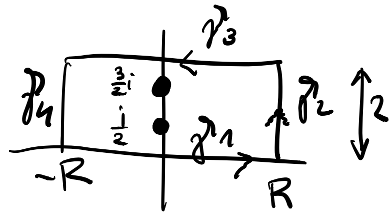
$$I = \int_{x=-\infty}^{+\infty} \frac{e^{ax} + e^{-ax}}{e^{\pi x} + e^{-\pi x}}\, dx$$

$$\overset{h=2}{=} \int_{s=-R}^{R} \frac{e^{as}\left(\cos(2a) + i\sin(2a)\right) + e^{-as}\left(\cos(2a) - i\sin(2a)\right)}{e^{\pi s} + e^{-\pi s}}\, ds$$

$$= \cos(2a) \underbrace{\int_{s=-R}^{R} \frac{e^{as} + e^{-as}}{e^{\pi s} + e^{-\pi s}}\, ds}_{\lim_{R\to+\infty} \int_{s=-R}^{R} (\dots)\, ds = I} + i\sin(2a) \underbrace{\int_{s=-R}^{R} \frac{e^{as} - e^{-as}}{e^{\pi s} + e^{-\pi s}}\, ds}_{\text{Lichá funkce} \longrightarrow \text{nula}}$$

$$\int_{\gamma} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz + \int_{\gamma_4} f(z)dz$$

$$\begin{array}{cccc} \downarrow R\to+\infty & \downarrow R\to+\infty & \downarrow R\to+\infty & \downarrow \\ I & 0 & 0 & -\cos(2a) I \end{array}$$



$$(*) \quad \int_{\gamma} f(z)dz = (1 - \cos(2a)) I$$

$$(**)\quad \int_\gamma f(z)\,dz = 2\pi i\left(\operatorname{Res}_{z=\frac{i}{2}} f(z) + \operatorname{Res}_{z=\frac{3}{2}i} f(z)\right) = 2\left(\cos\left(\frac{a}{2}\right) - \cos\left(\frac{3}{2}a\right)\right)$$

$$\operatorname{Res}_{z=\frac{i}{2}} f(z) = \operatorname{Res}_{z=\frac{i}{2}}\left(\frac{\cosh(az)}{\cosh(\pi z)}\right) = \frac{\cosh(az)}{\pi\sinh(\pi z)}\Bigg|_{z=\frac{i}{2}} = \frac{\cosh\left(\frac{ai}{2}\right)}{\pi\sinh\left(\frac{i\pi}{2}\right)} = \frac{\cos\left(\frac{a}{2}\right)}{\pi i}$$

$$\operatorname{Res}_{z=z_0}\frac{f(z)}{g(z)} = \frac{f(z)}{g'(z)}\Bigg|_{z=z_0}$$

$$\operatorname{Res}_{z=\frac{3}{2}i} f(z) = \text{———}''\text{———} = \frac{\cosh(az)}{\pi\sinh(\pi z)}\Bigg|_{z=\frac{3}{2}i} = \frac{\cosh\left(\frac{3}{2}ai\right)}{\pi\sinh\left(i\frac{3}{2}\pi\right)} = -\frac{\cos\left(\frac{3}{2}a\right)}{\pi i}$$

$(**) + (*)$

$$(1-\cos(2a))\,I = 2\left(\cos\left(\frac{a}{2}\right) - \cos\left(\frac{3}{2}a\right)\right)$$

$$I = \frac{2\left(\cos\left(\frac{a}{2}\right) - \cos\left(\frac{3}{2}a\right)\right)}{1-\cos(2a)}$$

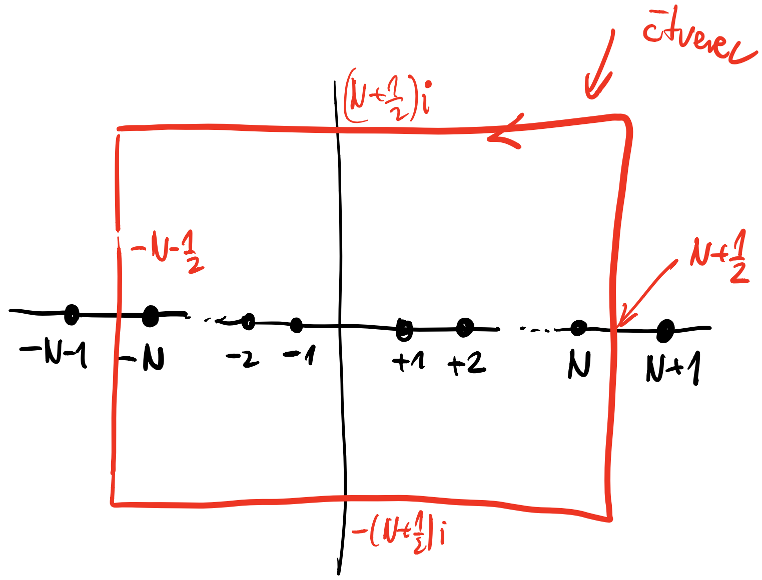
$a \to 0+$ OK

$a \to \pi$ Problém

$$\sum_{n=1}^{+\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Návod:

$$\lim_{N\to+\infty}\int_{\gamma_N}\overbrace{\frac{\cos(\pi z)}{z^2\sin(\pi z)}}^{f(z)}\,dz$$



Proč takhle?

Kde má $f(z)$ singularity?

(1) Singularita je v $z=0$. Pól násobnosti 3.

$$z^2\sin(\pi z) = z^3\,\frac{\sin(\pi z)}{z}$$

② $\sin(\pi z) = 0$ Kdy se $\sin(\pi z) = 0$?

$$\frac{e^{i\pi z} - e^{-i\pi z}}{2i} = 0 \longrightarrow e^{2\pi i z} - 1 = 0$$

$$e^{2\pi i z} = 1$$

$$2\pi i z = -i\pi + i2k\pi$$

$$z = k \qquad k \in \mathbb{Z}$$

$$\int_{\gamma_N} f(z)\,dz = 2\pi i \left(\operatorname{res}_{z=0} f(z) + \underbrace{\sum_{k=1}^{N} \operatorname{res}_{z=k} f(z)} + \sum_{k=1}^{N} \operatorname{res}_{z=-k} f(z) \right)$$

- $$\operatorname{res}\Big|_{\substack{z=k\\ k\in\mathbb{Z}\\ k\neq 0}} \frac{\cos(\pi z)}{z^2 \sin(\pi z)} = \frac{\cos \pi z}{z^2}\Big|_{z=k} \frac{1}{\frac{d}{dz}(\sin(\pi z))}\Big|_{z=k} = \frac{\cos \pi z}{z^2}\Big|_{z=k} \frac{1}{\pi \cos(\pi z)}\Big|_{z=k}$$

$$= \frac{1}{k^2\pi}$$

$$\sum_{k=1}^{N} \operatorname{res}_{z=k} f(z) = \sum_{k=1}^{N} \frac{1}{k^2\pi} = \frac{1}{\pi} \sum_{k=1}^{N} \frac{1}{k^2}$$

$$\int_{\gamma_N} f(z)\,dz = 2\pi i \left(\operatorname{res}_{z=0} f(z) + \frac{2}{\pi} \sum_{k=1}^{N} \frac{1}{k^2} \right)$$

- $$\operatorname{res}_{z=0} \frac{\cos(\pi z)}{z^2 \sin(\pi z)}$$

Chceme spočítat

$$\frac{\cos(\pi z)}{z^2 \sin(\pi z)} = \frac{c_{-3}}{z^3} + \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + \dots$$

$$\underbrace{1-\frac{(\pi z)^2}{2!}+\frac{(\pi z)^4}{4!}+\dots}_{\cos(\pi z)} = \left(\frac{c_{-3}}{z^3}+\frac{c_{-2}}{z^2}+\frac{c_{-1}}{z}+c_0+c_1 z+c_2 z^2+\dots\right) z^2\left(\pi z-\frac{(\pi z)^3}{3!}+\frac{(\pi z)^5}{5!}+\dots\right)$$

$$1-\frac{(\pi z)^2}{2!}+\frac{(\pi z)^4}{4!}+\dots = \pi c_{-3} z^0+\pi c_{-2} z^1+\left(\pi c_{-1}-\frac{\pi^3}{3!}c_{-3}\right)z^2+\dots$$

$$1+0\cdot z-\frac{\pi^2}{2!}z^2+\dots =$$

$$\pi c_{-3}=1$$
$$\pi c_{-2}=0$$
$$\pi c_{-1}-\frac{\pi^3}{3!}c_{-3}=-\frac{\pi^2}{2!} \longrightarrow c_{-1}=-\frac{\pi}{3}$$

$$\int_{\gamma_N} f(z)\,dz = \underbrace{2\pi i\left(-\frac{\pi}{3}+\frac{2}{\pi}\sum_{k=1}^{N}\frac{1}{k^2}\right)}_{\downarrow\, N\to+\infty}$$

$$\underbrace{\int_{\gamma_N} f(z)\,dz}_{\xrightarrow{N\to+\infty}\, 0} = 4i\left(-\frac{\pi^2}{6}+\sum_{k=1}^{+\infty}\frac{1}{k^2}\right)$$

$$\left|\int_{\gamma_N}\frac{\cos(\pi z)}{z^2\sin(\pi z)}\,dz\right| \leq M\cdot \overbrace{\text{délka křivky } \gamma_N}^{\sim 8N}$$

$$M=\max_{\gamma_N}\left|\frac{\cos(\pi z)}{z^2\sin(\pi z)}\right| \sim \frac{1}{N^2}\cdot\left|\frac{\cos(\pi z)}{\sin(\pi z)}\right|$$

Domácí úkol: Ukažte, že platí: $\lim_{N \to +\infty} \int_{\gamma_N} f(z)\,dz = 0$