Changepoints in a nonlinear expectile model: Theoretical and computational issues

COMSTAT 2024 | The 26th International Conference on Computational Statistics | Giessen, Germany | 27 – 30 August, 2024

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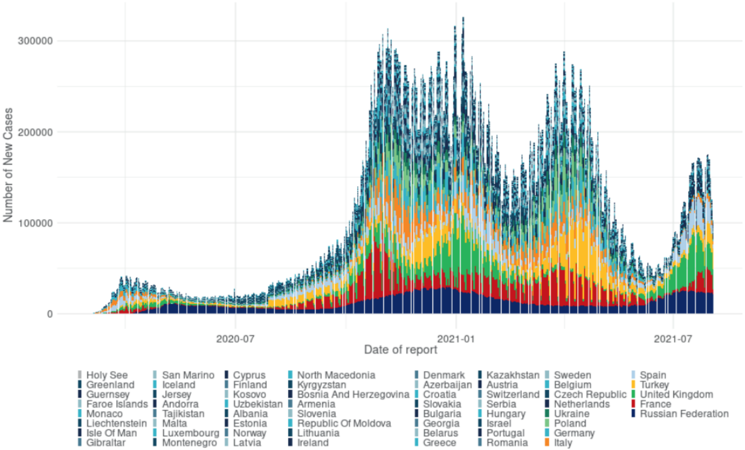

Nathan et al.(2021)

Outline

- **Underlying stochastic model**

  Nonlinear conditional expectile regression *(risk modeling)*

- **Statistical changepoint test**

  Real-time changepoint detection procedure *(online regime)*

- **Theoretical properties**

  Asymptotic guarantees of the estimation and detection *(validity)*

- **Empirical performance**

  Real data illustrations and simulation results *(utilization)*

Theoretical pivots

- **Relatively flexible but (still) fully parametric model**
  ↪ nonlinear regression frsmework, irregularities with respect to parameter changes
- **Robust and complex (distributional) estimation**
  ↪ conditional expectile estimation with additional complexity of the model
- **Theoretical justification and statistical consistency**
  ↪ stochastically valid decisions based on a consistent statistical test
- **Online regime for (some) structural break detection**
  ↪ online detection of various model instabilities in terms of parameter changes
- **Data-driven algorithm (no nuisance parameters)**
  ↪ simple computational approach, free of the specific analytical model form

Starting with a simple location model ...

- a **sparse location model** proposed by Harchaoui and Lévy-Leduc (2010)

$$Y_i = \mu_i + \varepsilon_i, \quad \text{for } i = 1, \ldots, N;$$

(Yao and Au (1989); Mammen and Van De Geer (1997); Massart (2004), Boysen (2009); Frick et al. (2014); Fryzlewicz (2014); Lin et al. (2017); Ciuperca and M. (2020), and others;)

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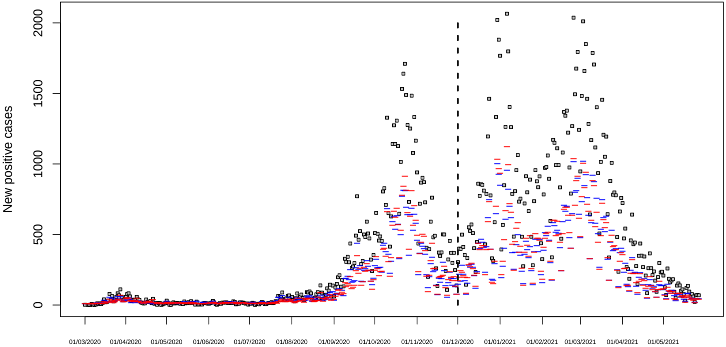
- a general model extension into a **nonlinear regression model**

$$Y_i = f(\boldsymbol{X}_i, \boldsymbol{\beta}_i) + \varepsilon_i, \quad \boldsymbol{\beta}_i \in \mathbb{R}^p, \quad i = 1, \ldots, N;$$

for $f : \mathbb{R}^{q \times p} \to \mathbb{R}$ with an analytic formula that depends on some unknown parameter $\boldsymbol{\beta} \in \mathbb{R}^p$ *(Ciuperca, M., and Pešta, 2024)*

Covid-19 cases in Prague, Czech Republic



- ❑ Covid-19 positive cases in Prague, Czech Republic
- ❑ Period from the first positive case (March 1, 2020) until end of May 2021
- ❑ Covid-19 restrictions and their role in the overall (global) population

Underlying stochastic (changepoint) model

- **historical data** $\{(Y_i, \boldsymbol{X}_i);\ i = 1, \ldots, m\}$, for q-dimensional $\boldsymbol{X}_i \in \mathbb{R}^q$;
- underlying **nonlinear regression model** of the form

$$Y_i = f(\boldsymbol{X}_i, \beta) + \varepsilon_i, \qquad i = 1, \ldots, m$$

$\hookrightarrow$ for a given nonlinear parametric function $f(\cdot, \beta) : \mathbb{R}^q \to \mathbb{R}$ and some **unknown vector of parameters** $\beta \in \mathbb{R}^p$ (to be estimated);

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- after historical data new **online data** $\{(Y_i, \boldsymbol{X}_i);\ i = m+1, \ldots, m+T_m\}$ are sequentially observed—in a one by one manner (for $T_m \in \mathbb{N}$);
- **analogous model as for the historical data** is supposed to hold, however

$$Y_i = f(\boldsymbol{X}_i, \beta_i) + \varepsilon_i, \qquad i = m+1, \ldots, m+T_m$$

$\hookrightarrow$ again for **unknown vectors of parameters** $\beta_i \in \mathbb{R}^p$ but some of them hypotetically different than $\beta \in \mathbb{R}^p$;

Formal statistical test of no changepoint

- **In the first step** the historical data $\{(Y_i, \boldsymbol{X}_i);\ i = 1, \ldots, m\}$ are used to construct an empirical estimate $\widehat{\beta}_m$ for the unknown vector $\boldsymbol{\beta} \in \mathbb{R}^p$;

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- **In the second step** the online data $\{(Y_i, \boldsymbol{X}_i);\ i = m+1, \ldots, m+T_m\}$ are utilized to run a real-time changepoint test of the null hypothesis

$$H_0 : \quad \boldsymbol{\beta}_i = \boldsymbol{\beta}^0, \qquad i = m+1, \ldots, m+T_m$$

against the alternative hypothesis of the form

$$H_A : \exists k_m^0 \in \{1, \ldots, T_m - 1\}$$

such that

$$\begin{aligned} \boldsymbol{\beta}_i &= \boldsymbol{\beta}^0 \quad i = m+1, \ldots, m+k_m^0 \\ \boldsymbol{\beta}_i &= \boldsymbol{\beta}^1 \quad i = m+k_m^0+1, \ldots, m+T_m \end{aligned}$$

where $\boldsymbol{\beta}^0 \neq \boldsymbol{\beta}^1$;

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where $\boldsymbol{\beta}^0 \neq \boldsymbol{\beta}^1$;

- **Real-time (online)** testing procedures performed as online data arrive;

Step 1: Estimation of the parameter vector β

- **Conditional expectile** estimation of the unknown parameter vector $\boldsymbol{\beta} \in \mathbb{R}^p$ with the true value being denoted as $\boldsymbol{\beta}_0 \in \mathbb{R}^p$
- **Conditional expectiles** provide complex insight into the data-generating mechanism, they are always defined (unlike conditional quantiles) and they are known as coherent and elicitable risk measures (Phillipps, 2022)

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- The estimate for $\boldsymbol{\beta} \in \mathbb{R}^p$ obtained in terms of the minimization problem

$$\widehat{\beta}_m = \underset{\beta \in \mathbb{R}^p}{Argmin} \sum_{i=1}^{m} \rho_\tau\Big(Y_i - f(\boldsymbol{X}_i, \beta)\Big)$$

$\hookrightarrow$ for the **expectile loss function** $\rho_\tau(x) = \Big|\tau - \mathbb{I}_{\{x<0\}}\Big| x^2$ for $x \in \mathbb{R}$ and some expectile level $\tau \in (0,1)$ (NLS for $\tau = 1/2$)

Asymptotic properties

Technical assumptions imposed on the function $f(\cdot, \beta)$, continuous distribution of the error terms (independent), and some regularity conditions (e.g., moment properties)

- **Assymptotic behaviour** of the proposed expectile estimator of $\beta^0 \in \mathbb{R}^p$:

$$\widehat{\beta}_m = \beta^0 + \boldsymbol{\Omega}^{-1} \frac{1}{m} \sum_{i=1}^{m} \nabla f(\boldsymbol{X}_i, \beta^0) g_\tau(\varepsilon_i) + o_{\mathbb{P}}(m^{-1/2})$$

for the sample size of the historical data tending to infinity, thus $m \to \infty$;

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❑ **Notation:**

- $\boldsymbol{\Omega} = \mathbb{E}\Big[h_\tau(\varepsilon)\Big] \boldsymbol{V}(\beta^0)$ and $\boldsymbol{V}(\beta^0) = \lim_{m\to\infty} \boldsymbol{V}_m(\beta^0)$
- $\boldsymbol{V}_m(\beta^0) = \frac{1}{m} \sum_{i=1}^{m} \nabla f(\boldsymbol{X}_i, \widehat{\beta}_m) \nabla^\top f(\boldsymbol{X}_i, \widehat{\beta}_m)$
- $g_\tau(x) = \rho'_\tau(x)$ and $h_\tau(x) = \rho''_\tau(x)$

Some computational issues

- **Optimization problem and the solution**
  - the solution $\widehat{\beta}_m$ can be obtained in terms of a convex minimization problem or a nonconvex optimization problem instead
  - different optimization toolboxes and algorithmic approaches must be used (all depending on the underlying problem) to obtain the final estimate

- **Nuisance parameter estimation**
  - some quantities (e.g., $E\left[h_\tau(\varepsilon)\right] = E\left[\rho''_\tau(\varepsilon)\right]$, or $Var[g_\tau(\varepsilon)] = Var[\rho'_\tau(\varepsilon)]$) must be estimated to perform the test
  - alternative approaches based on various resampling techniques and bootstrap can be used instead $\Longrightarrow$ future ongoing work ...

Step 2: Test of the null hypothesis H_0 against H_A

- **Null hypothesis:** the online data are generated under the same probabilistic model as the model generating the historical data ($\beta^0 \in \mathbb{R}^p$);
- **Alternative:** the online data are generated from a different model than the historical data however, the change is only determined within the paramter vector $\beta^1 \neq \beta^0$;

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- **Test statistic**

$$\mathcal{T}(m) = \sup_{1 \leq k \leq T_m} \frac{\|S(m,k)\|_\infty}{z(m,k,\gamma)}$$

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  - $S(m,k) = \boldsymbol{J}_m^{-1/2}(\widehat{\beta}_m) \sum_{i=m+1}^{m+k} \nabla f(\boldsymbol{X}_i, \widehat{\beta}_m) g_\tau(\widehat{\varepsilon}_i)$
  - $\boldsymbol{J}_m(\widehat{\beta}_m) = \frac{\widehat{Var[g_\tau(\varepsilon)]}}{m} \sum_{i=1}^{m} \nabla f(\boldsymbol{X}_i, \widehat{\beta}_m) \nabla^\top f(\boldsymbol{X}_i, \widehat{\beta}_m)$
  - $z(m,k,\gamma) = m^{1/2}(1 + k/m)(k/(k+m))^\gamma$, for some $\gamma \in [0, \frac{1}{2})$

Changepoint test asymptotics

- Distinguishing for **two differrent scenarios**:
  - **Open-end procedure:** $\lim_{m\to\infty} T_m/m = \infty$
  - **Closed-end procedures:** $\lim_{m\to\infty} T_m/m = T$, for $T \in (0, \infty)$

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- **Distribution of $\mathcal{T}(m)$ under the null hypothesis:**

  Under some technical assumptions and the null hypothesis validity

$$\mathcal{T}(m) = \sup_{1\leq k\leq T_m} \frac{\|S(m,k)\|_\infty}{z(m,k,\gamma)} \xrightarrow[m\to\infty]{\mathcal{D}} \sup_{0<t<L(T)} \frac{\|\boldsymbol{W}_p(t)\|_\infty}{t^\gamma}$$

  for a p-dimensional Wiener process $\{\boldsymbol{W}_p(t);\ t\in(0,\infty)\}$ and either $L(T)=1$ (open-end) or $L(T)=T/(T+1)$ (closed-end);

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- **Consistency of the test (behaviour of $\mathcal{T}(m)$ under the alternative:**
  Under the alternative hypothesis and $m^{1/2}\|\beta^0 - \beta^1\|_2 \to \infty$ for $m \to \infty$

$$\mathcal{T}(m) \xrightarrow[m\to\infty]{\mathbb{P}} \infty.$$

Step 3: Changepoint time estimate

- **Stopping time** – the first observation for which the null hypothesis is rejected (in favor of the alternative hypothesis)
- The corresponding **changepoint estimate** can be defined as

$$\hat{k}_m = \in \left\{ k \geq 1; \sup_{1 \leq k \leq T_m} \frac{\|S(m,k)\|_\infty}{z(m,k,\gamma)} > c_\alpha(\gamma) \right\},$$

where $c_\alpha(\gamma)$ is the corresponding quantile of the limiting distribution of

$$\underset{0 < t < L(T)}{sup} \frac{\|\boldsymbol{W}_p(t)\|_\infty}{t^\gamma}$$

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- Moreover, it holds that (test consistency)

$$\lim_{m\to\infty} P[\hat{k}_m < \infty | H_0] = \alpha \qquad \text{and} \qquad \lim_{m\to\infty} P[\hat{k}_m < \infty | H_1] = 1$$

Simulation setup

Motivated by Choi, S.H., Kim, H.K., Lee, Y. (2003). Nonlinear asymmetric least squares estimators. Journal of the Korean Statistical Society 32(1), 47 – 64.

- Gompertz curve $f(x, \beta) = \exp\{-\beta_1 e^{-\beta_2 x}\}$, for $\beta_o = (\beta_1, \beta_2)^\top \equiv (10, 5)^\top$
- Estimation of $\beta \in \mathbb{R}^2$ by the iterative grid search algorithm
- Historical data: $m \in \{20, 50, 200\}$; Online data: $T_m \in \{10, m/2, m \log m\}$
- Various changepoint scenarios wrt. to β^1 and the changepoint location

Under the null hypothesis

Distribution	m	$\widehat{\beta}_1$ | Std.Er. $\beta_1 = 10.00$	$\widehat{\beta}_2$ | Std.Er. $\beta_2 = 5.00$	Scenario 1 $T_m = 10$	Scenario 2 $T_m = m/2$	Scenario 3 $T_m = m\log(m)$
$N(0,1)$	20	10.52 *(2.832)*	5.32 *(1.825)*	7.74 %	7.74 %	7.52 %
$\tau = 0.5000$	50	10.32 *(2.857)*	5.20 *(1.382)*	4.92 %	6.08 %	5.64 %
	200	10.24 *(2.766)*	5.03 *(0.808)*	5.08 %	5.58 %	6.54 %
$N(1,1)$	20	10.18 *(2.889)*	5.61 *(2.020)*	5.76 %	5.76 %	6.28 %
$\hat{\tau} = 0.0719$	50	10.27 *(2.869)*	5.39 *(1.638)*	4.08 %	3.26 %	4.46 %
	200	10.30 *(2.833)*	5.08 *(0.977)*	4.40 %	4.26 %	4.88 %
$L(0,1)$	20	10.50 *(2.842)*	5.32 *(1.797)*	7.84 %	7.84 %	9.86 %
$\tau = 0.5000$	50	10.37 *(2.850)*	5.18 *(1.374)*	4.08 %	5.12 %	7.38 %
	200	10.29 *(2.769)*	5.02 *(0.799)*	5.06 %	4.90 %	5.58 %

Table 1 Simulation results under the null hypothesis (with the theoretical value of $\tau = 0.5$ for the symmetric distributions and the empirical estimate $\hat{\tau} = 0.0719$ in terms of Remark 1 for the asymmetric distribution). The parameter estimates are reported with the corresponding standard errors (in parentheses) over 5000 Monte Carlo simulations. Relative proportions of false rejections are given for three different scenarios for T_m reflecting the open-end and closed-end procedures. The nominal level of all the tests is always set to be $\alpha = 0.05$.

Under the alternative hypothesis

Dist.	m	k_m^0	Scenario 1 $T_m = 10$		Scenario 2 $T_m = m/2$		Scenario 3 $T_m = m\log(m)$	
$N(0,1)$	20	$k_m^{(1)}$	2.79 %	[0.77 | 0.78]	2.79 %	[0.77 | 0.78]	2.44 %	[0.72 | 0.71]
		$k_m^{(2)}$	12.80 %	[0.31 | 0.22]	12.80 %	[0.31 | 0.22]	20.40 %	[0.19 | 0.12]
	50	$k_m^{(1)}$	4.10 %	[0.73 | 0.76]	4.10 %	[0.76 | 0.78]	5.20 %	[0.93 | 0.93]
		$k_m^{(2)}$	13.30 %	[0.33 | 0.11]	14.60 %	[0.44 | 0.46]	35.30 %	[0.32 | 0.25]
	200	$k_m^{(1)}$	6.52 %	[0.87 | 0.89]	8.99 %	[0.80 | 0.82]	21.76 %	[0.83 | 0.85]
		$k_m^{(2)}$	16.70 %	[0.46 | 0.44]	47.60 %	[0.46 | 0.49]	96.40 %	[0.15 | 0.10]
$N(1,1)$	20	$k_m^{(1)}$	2.51 %	[0.71 | 0.67]	2.51 %	[0.71 | 0.67]	2.42 %	[0.62 | 0.55]
		$k_m^{(2)}$	11.20 %	[0.37 | 0.39]	11.20 %	[0.37 | 0.39]	23.30 %	[0.29 | 0.22]
	50	$k_m^{(1)}$	4.20 %	[0.83 | 0.83]	4.30 %	[0.83 | 0.83]	5.00 %	[0.85 | 0.89]
		$k_m^{(2)}$	12.20 %	[0.43 | 0.46]	13.30 %	[0.49 | 0.33]	32.30 %	[0.48 | 0.58]
	200	$k_m^{(1)}$	5.52 %	[0.82 | 0.84]	7.20 %	[0.55 | 0.55]	21.71 %	[0.83 | 0.85]
		$k_m^{(2)}$	14.21 %	[0.42 | 0.44]	38.00 %	[0.48 | 0.49]	93.80 %	[0.18 | 0.15]
$L(0,1)$	20	$k_m^{(1)}$	2.64 %	[0.77 | 0.78]	2.64 %	[0.77 | 0.78]	2.29 %	[0.74 | 0.78]
		$k_m^{(2)}$	11.50 %	[0.36 | 0.22]	11.50 %	[0.36 | 0.22]	21.20 %	[0.24 | 0.16]
	50	$k_m^{(1)}$	3.95 %	[0.75 | 0.78]	4.10 %	[0.78 | 0.79]	4.99 %	[0.81 | 0.86]
		$k_m^{(2)}$	12.90 %	[0.43 | 0.44]	13.20 %	[0.48 | 0.50]	37.70 %	[0.30 | 0.21]
	200	$k_m^{(1)}$	6.70 %	[0.75 | 0.67]	6.93 %	[0.80 | 0.79]	24.04 %	[0.81 | 0.83]
		$k_m^{(2)}$	15.60 %	[0.40 | 0.47]	37.90 %	[0.46 | 0.47]	96.40 %	[0.18 | 0.12]

Table 2 Empirical powers of the proposed real-time changepoint test

Covid-19 positive cases in Prague

Motivated by Chen, D.G., Chen, X., Chen, J.K. (2020.) Reconstructing and forecasting the covid-19 epidemic in the united states using a 5-parameter logistic growth model. Global Health Research and Policy 32(1), 1 – 7.

- Gompertz model $f(x, \beta) = K \exp\{-\beta_1 e^{-\beta_2 x}\}$, for $\boldsymbol{\beta} = (\beta_1, \beta_2, K)^\top \in \mathbb{R}_+^3$
- Historical data: $m = 1275$; Online data: $T_m = 176$
- The null hypothesis rejected on the second day of the online data
  (test statistics $\mathcal{T}(m) = 3.4211$ with the critical value $c_{0.95}(\gamma) = 2.4260$ for $\gamma = 0.1$)

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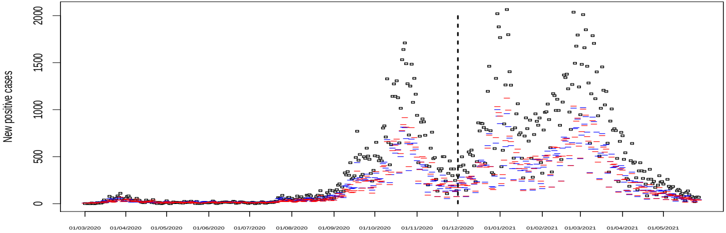
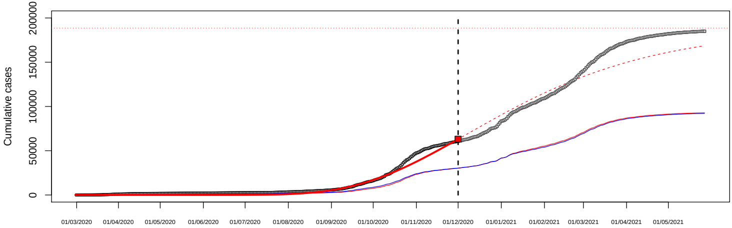
Estimation method & Data	$\boldsymbol{\beta}_1$	$\boldsymbol{\beta}_2$	$\boldsymbol{K}$	**Objective function**
Historical data (until 01/12/2020)				
Symmetric least squares ($\tau = 0.50$)	36.04	0.0129	187 811	3.00×10^6
Expectile method ($\tau = 0.11$)	37.97	0.0129	188 576	0.78×10^6
Re-estimation after the change detection (04/12/2020)				
Symmetric least squares ($\tau = 0.50$)	35.38	0.0130	180 174	3.27×10^6
Expectile method ($\tau = 0.19$)	36.04	0.0129	179 718	1.92×10^6
All available data (until 26/05/2021)				
Symmetric least squares ($\tau = 0.50$)	20.17	0.0096	256 970	10.9×10^6
Expectile method ($\tau = 0.26$)	20.18	0.0095	255 032	8.53×10^6

Covid-19 positive cases in Prague





Overview

- **Nonlinearity and flexibility of the model**
  (relatively high model flexibility while preserving straightforward interpretation in terms of well defined parameters)

- **Complex characterization wrt. conditional expectiles**
  (additional robustness with respect to asymmetric error distributions, or some heavy tails $\rightarrow$ optimal for risk modeling)

- **Online regime for instability detection**
  (distribution of the null hypothesis does not depend on the functional form of the underlying model nor the unknown parameters)

- **Straightforward applicability**
  (relatively mild technical assumptions but some caution is needed when using different functional models)

Thank you for your attention!

Ciuperca, G., MM, and Pešta M. (2024). Real-time detection of a change in a nonlinear model by the expectile method. *Metrika*, 87(2), 105 – 131.DOI:10.1007/s00184-023-00904-6