Probability distributions convoluted by quasigroups

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Probabilities on quasigroups

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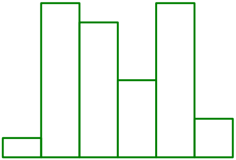
$$u_1 + u_2 + \ldots + u_q = 1$$



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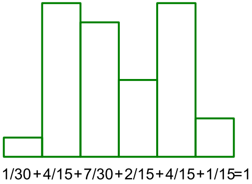

Distribution support: $N(u) = \{i \in Q : u_i > 0\}$.

Probability convolution $u * v$

$$(u * v)_i = \sum_{j=1}^{q} u_j v_{j \setminus i} = \sum_{j=1}^{q} u_{i/j} v_j$$

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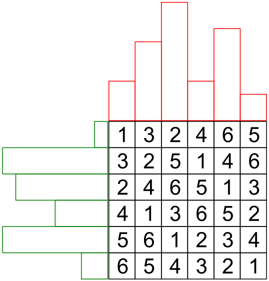

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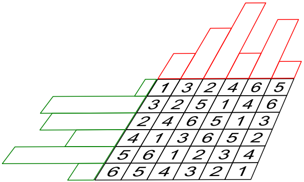

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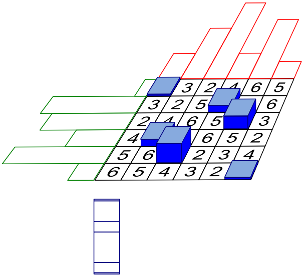

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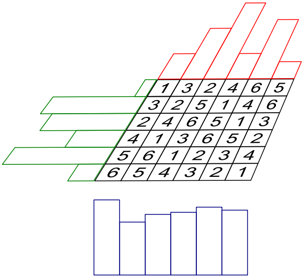
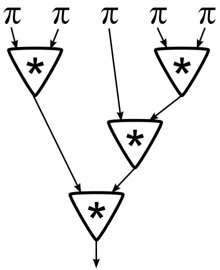


Iterated convolutions

Let π be an initial distribution on Q.

$$(\pi * \pi) * (\pi * (\pi * \pi))$$



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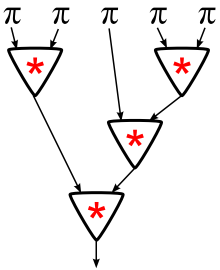

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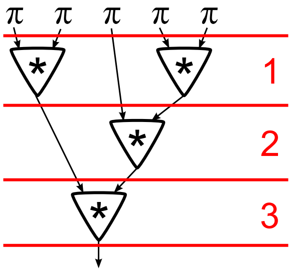

Length $L = 4$. Depth $D = 3$.

Known results

• $(\ldots(\pi * \pi) * \pi)\ldots) * \pi)$	a Markov chain (quasigroup stream filters)

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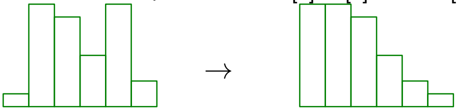
When do quasigroup convolutions converge to the uniform distribution?

Convergence in depth

$$(u_1, u_2, \ldots, u_q) \quad \to \quad (u_{[1]}, u_{[2]}, \ldots, u_{[q]})$$

- 

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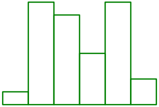
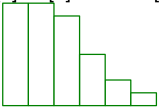
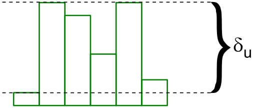
-  $\to$ 
- $\delta_u = u_{[1]} - u_{[q]}$ 

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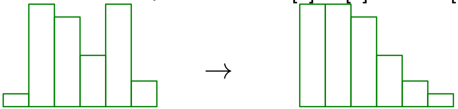
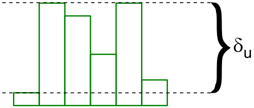
- 
- $\delta_u = u_{[1]} - u_{[q]}$ 

- $\delta_{u*v} \leqslant \min\{(1 - u_{[q]})\delta_v, (1 - v_{[q]})\delta_u\}$

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  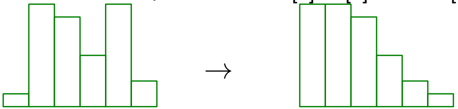
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  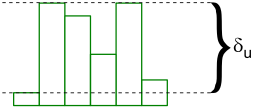

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Theorem

Let w be an iterated convolution of depth k with initial distribution π. Then:

$$\delta_w \leqslant (1 - \pi_{[q]})^k.$$

Theorem (convergence in depth)

Let w be an iterated convolution of depth n with initial distribution π, $|N(\pi)| > \frac{|Q|}{2}$. Then there exists a $d \in (0, 1]$:

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Theorem (convergence in length)

Let w be an iterated convolution of length m with initial distribution π, $|N(\pi)| > \frac{|Q|}{2}$. Then there exists an $\alpha > 0$:

$$\max \left| w_i - \frac{1}{|Q|} \right| \leqslant \frac{1}{m^\alpha}.$$

Generalization failure

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Support properties

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Without loss of generality: $Q' = Q$.

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Define $d^{(n)}$ — the average of distributions with depth n:

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Theorem

There exist $d \in (0;1]$ and n' such that for every $n \geqslant n'$:

$$\delta_{d^{(n)}} \leqslant (1-d)^{n-n'},$$

and consequently

$$\max_{i \in Q} \left| d_i^{(n)} - \frac{1}{|Q|} \right| \leqslant (1-d)^{n-n'}.$$

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There exist r and m' such that for every $m \geqslant m'$ the sets

$$N_m^L, N_{m+1}^L, \dots, N_{m+r-1}^L$$

are pairwise disjoint, while $N_{m+r}^L = N_m^L$.
$N_m^L \cup N_{m+1}^L \cup \ldots \cup N_{m+r-1}^L$ is a subquasigroup $Q' \subseteq Q$, generated by $N(\pi)$.

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Again, without loss of generality: $Q' = Q$.

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3	$((\pi * \pi) * \pi) * \pi, (\pi * (\pi * \pi)) * \pi,$ $\pi * ((\pi * \pi) * \pi), \pi * (\pi * (\pi * \pi)),$ $(\pi * \pi) * (\pi * \pi)$	$\{1,2\}$

Supports are periodic ($r > 1$) iff there exists a homomorphism $\varphi : Q \to \mathbb{Z}_r$ such that $\varphi(N(\pi)) = 1 \in \mathbb{Z}_r$.

Example

	1	2	3	4	5	6
1	3	4	6	5	2	1
2	4	3	5	6	1	2
3	5	6	1	2	4	3
4	6	5	2	1	3	4
5	2	1	3	4	5	6
6	1	2	4	3	6	5

m	L_m	N_m^L
0	π	$\{1,2\}$
1	$\pi * \pi$	$\{3,4\}$
2	$(\pi * \pi) * \pi, \pi * (\pi * \pi)$	$\{5,6\}$
3	$((\pi * \pi) * \pi) * \pi, (\pi * (\pi * \pi)) * \pi,$ $\pi * ((\pi * \pi) * \pi), \pi * (\pi * (\pi * \pi)),$ $(\pi * \pi) * (\pi * \pi)$	$\{1,2\}$

Supports are periodic ($r > 1$) iff there exists a homomorphism $\varphi : Q \to \mathbb{Z}_r$ such that $\varphi(N(\pi)) = 1 \in \mathbb{Z}_r$.
The period r is the biggest value for which such a homomorphism exists.

Averages in length

Define $\ell^{(m)}$ — the average of distributions with length m:

$$\ell^{(m)} = \frac{1}{|L_m|} \sum_{u \in L_m} u.$$

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Theorem

Suppose there exists an m' such that $N_m^L = Q$ for every $m \geqslant m'$. Then there exist $\alpha, \beta > 0$ and $m'' > m'$ such that for $m \geqslant m''$:

$$\max_{i \in Q} \left| \ell_i^{(m)} - \frac{1}{|Q|} \right| \leqslant \frac{\beta}{m^\alpha}.$$

The periodic case

$$\dots \quad N^L_m, \quad N^L_{m+1}, \quad \dots, \quad N^L_{m+r-1}, \quad N^L_{m+r}, \quad N^L_{m+r+1}, \quad \dots$$

The periodic case

$$\begin{array}{cccccccc} \dots & N_m^L, & N_{m+1}^L, & \dots, & N_{m+r-1}^L, & N_{m+r}^L, & N_{m+r+1}^L, & \dots \\ \dots & Q_0, & Q_1, & \dots, & Q_{r-1}, & Q_0, & Q_1, & \dots \end{array}$$

The periodic case

$$\begin{array}{ccccccc} \ldots & N^L_m, & N^L_{m+1}, & \ldots, & N^L_{m+r-1}, & N^L_{m+r}, & N^L_{m+r+1}, & \ldots \\ \ldots & Q_0, & Q_1, & \ldots, & Q_{r-1}, & Q_0, & Q_1, & \ldots \end{array}$$

$$Q_0 \cup Q_1 \cup \ldots \cup Q_{r-1} = Q$$

The periodic case

$$\begin{array}{ccccccc} \ldots & N^L_m, & N^L_{m+1}, & \ldots, & N^L_{m+r-1}, & N^L_{m+r}, & N^L_{m+r+1}, & \ldots \\ \ldots & Q_0, & Q_1, & \ldots, & Q_{r-1}, & Q_0, & Q_1, & \ldots \end{array}$$

$$Q_0 \cup Q_1 \cup \ldots \cup Q_{r-1} = Q$$

$$|Q_0| = |Q_1| = \ldots = |Q_{r-1}| = \frac{|Q|}{r}$$

The periodic case

$$\begin{array}{cccccccc} \dots & N^L_m, & N^L_{m+1}, & \dots, & N^L_{m+r-1}, & N^L_{m+r}, & N^L_{m+r+1}, & \dots \\ \dots & Q_0, & Q_1, & \dots, & Q_{r-1}, & Q_0, & Q_1, & \dots \end{array}$$

$$Q_0 \cup Q_1 \cup \ldots \cup Q_{r-1} = Q$$

$$|Q_0| = |Q_1| = \ldots = |Q_{r-1}| = \frac{|Q|}{r}$$

Theorem

Suppose there exists an m' and sets Q_b, $b = 0, \ldots, r-1$ such that $N^L_{rk+b} = Q_b$ for every $m \geqslant m'$. Then there exist $\alpha, \beta > 0$ and $m'' > m'$ such that for $rk + b \geqslant m''$:

$$\max_{i \in Q_b} \left| \ell_i^{(rk+b)} - \frac{1}{|Q_b|} \right| \leqslant \frac{\beta}{k^\alpha}.$$

Thank You for Your attention!