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MORE PROPERTIES OF MINIMALLY NONASSOCIATIVE MOUFANG LOOPS

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Preliminaries

Definitions

- *Moufang loop*: a loop that satisfies any of the following identities:

$$(xy)(zx) = [x(yz)]x,$$

$$(xy)(zx) = x[(yz)x],$$

$$x[y(xz)] = [(xy)x]z,$$

$$[(zx)y]x = z[x(yx)].$$

- *Minimal nonassociativity*: the property that a structure is not associative but all its proper substructures and proper quotient structures are associative.

* Note: As opposed to standard convention, the associativity of quotient structures is imposed.

Notations

- L : Minimally nonassociative Moufang loop
- M : Maximal normal subloop
- (x, y, z) : Associator of x, y and z
- L_a : Associator subloop
- $N(L)$: Nucleus

Motivation

- Every minimally nonassociative Moufang loop is not a direct product of a nonassociative Moufang loop and a group.

Theorem

If L is of odd order, then L_a *is an elementary abelian group and the unique minimal normal subloop of L.*

Theorems

a) $|L|/|N(L)| \neq 1$, *p or pq where p and q are (not necessarily distinct) primes.*

b) *Suppose* $|L| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ *where* $p_1, p_2, \ldots, p_n$ *are distinct odd primes and* $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{Z}^+$. *Then*

i. $|L_a| = p_i^{\beta_i}$ *for some i and* β_i *such that* $\alpha_i \geq 2$ *and* $0 < \beta_i < \alpha_i$;

ii. $p_i^{\alpha_i} \nmid |N(L)|$ *for all i.*

iii. $\alpha_k \geq 3$ *for some k.*

Examples

- Suppose $|L| = pq^2r^3$.

  $\Rightarrow |L_a| = q, r$ or r^2; and
  $|N(L)| = 1, q, r, r^2, qr$ or qr^2.

- Suppose $|L| = p^2q^4$.

  $\Rightarrow |L_a| = p, q, q^2$ or q^3; and
  $|N(L)| = 1, p, q, q^2, q^3, pq$ or pq^2.

Phillips' problem:

Does there exist a Moufang loop of odd order with trivial nucleus?

Even order case: Solved!

Existence of $M(S_3, 2)$, the nonassociative Moufang loop of order 12 with trivial nucleus.

Relevant results towards solving Phillips' problem for the odd order case.

The following statements are equivalent:

1. $N(L) \neq \{1\}$.
2. $L_a \trianglelefteq N(L) < M < L$.
3. $(L_a, M, L) = \{1\}$.

Corollary: $N(L) = \{1\} \Leftrightarrow L = \langle k, w, \ell \rangle$ *for some* $k \in L_a$, $w \in M - L_a$ *and* $\ell \in L - M$.

Proof of '$(L_a, M, L) = \{1\} \Rightarrow L_a \subseteq N(L)$':

- Take $k \in L_a$, $x \in L - M$ and $\ell \in L$.
- Write $\ell = wx^{\alpha}$ for some $w \in M$ and $\alpha \in \mathbb{Z}^+$.
- $(k, x, \ell) = (k, x, wx^{\alpha}) = (k, x, w) = 1$.
- Since $(L_a, M, L) = (L_a, L - M, L) = \{1\}$, we have $(L_a, L, L) = \{1\}$.

$\therefore\ L_a \subseteq N(L)$.

More Results

- L_a is a Sylow subloop of $N(L) \Leftrightarrow L_a = N(L)$.
- L_a is cyclic $\Rightarrow L_a \subseteq N(L) \Rightarrow N(L) \neq \{1\}$.
- $(k, M, x) = [k, M] = \{1\}$ for some $k \in L_a - \{1\}$ and $x \in L - M \Rightarrow L_a \subseteq N(L) \Rightarrow N(L) \neq \{1\}$.

Proof of 'L_a is a Sylow subloop of $N(L)$ $\Rightarrow L_a = N(L)$':

- Suppose $L_a \neq N$.
- $\exists\, H < N$ such that $|H| = |N|/|L_a|$.
- Since $L_a \triangleright N$, it follows that $HL_a \leq N$ and $|HL_a| = |H||L_a|/|H \cap L_a| = |H||L_a| = |N|$.
- $N = HL_a$.

- Take $h \in H$ and $n \in N$.
- $N = HL_a \Rightarrow n = h_1k$ for some $h_1 \in H$ and $k \in L_a$.
- $n^{-1}hn = (h_1k)^{-1}h(h_1k) = k^{-1}(h_1^{-1}hh_1)k = h_1^{-1}hh_1$ $\in H$ since $L_a \subseteq C_L(N)$.
- $H \lhd N \lhd L$ and H is a Hall subloop of N $\Rightarrow H \lhd L$.
- L/H is associative $\Rightarrow L_a \subseteq H$.
- $\gcd(|L_a|, |H|) = 1 \Rightarrow |L_a| = 1 \quad \Rightarrow\Leftarrow$

$\therefore L_a = N$

Future Direction of Research

- Eliminate the condition "all proper quotient loops are associative" and get similar results.
- Solve Phillips' problem for minimally nonassociative Moufang loops.

Thank You

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