Extensions of groups by weighted Steiner loops

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Let L be an extension of a group A by a loop $(S,*)$

$$1 \longrightarrow A \longrightarrow L \longrightarrow S \longrightarrow e \tag{1}$$

which is defined on the set $S \times A$ by the multiplication

$$(x,\xi)(y,\eta) = (x*y, f(x,y)\xi^{T(y)}\eta), \tag{2}$$

where $f : S \times S \to A$ is a map with $f(x,e) = f(e,y) = 1 \in A$ for all $x,y \in S$ and $T : S \to \mathrm{Aut}(A)$ is a function of S into the automorphism group of A with $T(e) = \mathrm{Id}$. The identity element of the loop L is $(e,1)$. We call the loop L Schreier loop.
We give concrete description for all extensions (1) such that the loop S is a weighted Steiner loop and for all $y \in S$ one has $T(y) = Id$.

Definition 1. *A Steiner triple system $\mathfrak{S}$ is an incidence structure consisting of points and blocks such that every two distinct points are contained in precisely one block and any block has precisely three points.*

With a Steiner triple system is associated a loop $(S(\mathfrak{S}),\cdot)$ such that the elements of $S(\mathfrak{S})\setminus\{e\}$ are the points of the Steiner triple system, the product $a \cdot b$ is the third points of the block determined by a, b and $a \cdot a = e$ for all $a \in \mathfrak{S}$. This loop is a Steiner loop, i. e. it has the properties $x \cdot y = y \cdot x$ and $x \cdot (x \cdot y) = y$ for all $x,y \in S(\mathfrak{S})$.

Definition 2. *A weighted Steiner loop (S,h) is a Steiner loop S and a mapping $h : S \setminus \{e\} \to A$, where A is a group.*

We consider extensions L of groups A by weighted Steiner loops (S,h) such that for the factor system $f : S \times S \to A$ one has $f(x,y) = h(x)h(y)$ for all $x \neq y \in S \setminus \{e\}$. Such a loop L is called Steiner-like loop. We describe how weak associativity properties are related to the group D generated by the images of $h(x)$, $x \in S \setminus \{e\}$, in A and we determine D.

Definition 3. *A loop $(L, \cdot)$ is power-associative if $(\langle a \rangle, \cdot)$ is associative for each $a \in L$. A loop L is flexible if $(x \cdot y) \cdot x = x \cdot (y \cdot x)$ holds for all $x, y \in L$.*

Proposition 4. *A Steiner-like loop L is power-associative if and only if the set $\{f(x,x);\ x \in S\}$ is contained in the centre of A.*

Proposition 5. *A Steiner-like loop is flexible precisely if $\{f(x,y); x, y \in S\}$ is contained in the centre of the group A.*

Definition 6. *A loop L is right alternative, respectively left alternative if $(y \cdot x) \cdot x = y \cdot (x \cdot x)$, respectively $x \cdot (x \cdot y) = (x \cdot x) \cdot y$ holds for all $x, y \in L$. A loop L satisfies the left inverse, respectively the right inverse, respectively the cross inverse property if for all $x, y \in L$*

$$x^{\lambda} \cdot (x \cdot y) = y, \ \text{respectively } (y \cdot x) \cdot x^{\rho} = y,$$

$$\text{respectively } (x \cdot y) \cdot x^{\rho} = y$$

holds with $x^{\lambda} = e/x$, $x^{\rho} = x \backslash e$.

Proposition 7. *For a Steiner-like loop L the following properties are equivalent:*
a) L is right alternative,
b) L satisfies the right inverse property,
c) the set $\mathcal{F} = \{f(z,z);\ z \in S\}$ is contained in the centre of A and for all $x, y \in S \setminus \{e\}$ with $x \neq y$ the identity

$$h(x)h(y)h(x)h(xy) = f(x,x) \tag{3}$$

holds such that xy is the third element of the block of $S \setminus \{e\}$ determined by x, y.

Proposition 8. *For a Steiner-like loop L the following properties are equivalent:*
a) L is left alternative,
b) L satisfies the left inverse property,
c) the sets $\mathcal{F} = \{f(z,z);\ z \in S\}$ and $\mathcal{K} = \{h(x)h(y);\ x, y \in S \setminus \{e\}, x \neq y\}$ are contained in the centre of A and for all $x, y \in S \setminus \{e\}$ with $x \neq y$ the identity (3) holds.

Proposition 9. *A Steiner-like loop L satisfies the cross inverse property if and only if the group A is commutative and for all $x, y \in S \setminus \{e\}$ with $x \neq y$ the identity (3) holds.*

The group D generated by the set $\{h(x); x \in S \setminus \{e\}\}$ is abelian except the loops satisfying only the right alternative or the right inverse property.

Definition 10. *A (restricted) Fischer group is a pair (G, E) consisting of a group G and a system of generators $E \subset G$ satisfying the following conditions:*
(i) For all $x \in E$ one has $x^2 = 1$.
(ii) For all $x, y \in E$ we have $(xy)^3 = 1$ and $xyx \in E$.
A Steiner triple system $S \setminus \{e\}$ is a Hall system if any three non-collinear points of $S \setminus \{e\}$ generates the affine plane of order 3.

Theorem 11. *Let L be a Steiner-like loop. We assume that the set $\mathcal{F} = \{f(x,x); x \in S\}$ is contained in the centre $Z(A)$ of the group A. Let D be the group generated by the set $\mathcal{H} = \{h(z);\ z \in S \setminus \{e\}\}$. If S has more than 4 elements, then the following properties are equivalent:*
a) For all $x, y \in S \setminus \{e\}$ with $x \neq y$ the identity (3) holds. b) If the group D is abelian, then precisely one of the following cases holds:
(i) For all $x \in S \setminus \{e\}$ one has $h(x) = t$, $f(x,x) = t^4$ and $D = \langle t \rangle$.
(ii) There is a subloop U of the Steiner loop S such that S is the direct product of U and $\mathbb{Z}_2$ and $h(x) = t \neq 1$ for all $x \in U$ whereas $h(y) = t\omega = \omega t$ with a fixed involution ω for all $y \in S \setminus U$. The elements $f(x,x)$ have the form t^4 or $t^4\omega$ depending whether $x \in U$, respectively $x \notin U$. The group D is the direct product of $\langle t \rangle$ and $\langle \omega \rangle$. If the group D is non-abelian and the Steiner loop S is finite, then one has $h(x) = u\omega_x = \omega_x u$, where u is a fixed element of A which centralizes any element of D and for the order $o(\omega_x)$ of the element ω_x of A one has $o(\omega_x) \leq 2$. For every $x \in S \setminus \{e\}$ one has $f(x,x) = u^4$. If Γ is the set of different involutions in the factor group $D\langle u \rangle / \langle u \rangle = \langle \omega_x \langle u \rangle, x \in S \setminus \{e\} \rangle$, then $D\langle u \rangle / \langle u \rangle$ is a restricted Fischer group generated by Γ. If for the mapping $h : x \mapsto u\omega_x$ one has $\omega_x \omega_y \neq \omega_y \omega_x$ for all $x, y \in S \setminus \{e\}$, $x \neq y$, then $S \setminus \{e\}$ is a Hall system.
The Steiner-like loop L is the direct product of A and S precisely if the group D has order ≤ 2.

Proposition 12. *Let L be a Steiner-like loop such that $S = \{e, x, y, xy\}$ is the elementary abelian group of order 4. We assume that the set $\mathcal{F} = \{f(x,x); x \in S\}$ is contained in the centre $Z(A)$ of the group A. Let D be the group generated by the set $\mathcal{H} = \{h(z);\ z \in S \setminus \{e\}\}$. Then the following properties are equivalent:*
a) For all $x, y \in S \setminus \{e\}$ with $x \neq y$ the identity (3) holds.
b) If the group D is non-abelian, then it is a product $K \cdot \langle a \rangle$, where $a = h(x)$ and $t = h(x)h(y)$, such that the abelian group K has the form $\langle s \rangle \times Z(D)$, where the cyclic group $\langle s \rangle$ of order 3 is the commutator subgroup D' of D, the group $Z(D)$ is generated by a^2 and t^3 and a inverts any element of $\langle s \rangle$.

If the group D is abelian, then D is generated by three elements $h(x) = a$, $h(y) = b$ and $h(xy) = c$ such that a, b and c commute and one has

$$f(x,x) = al, \;\; f(y,y) = bl, \;\; f(xy,xy) = cl,$$

where $l = abc$.

Let $x^{-1} := x^\lambda = x^\rho$. A loop L has the automorphic inverse property if the identity $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$ holds for all $x, y \in L$.

Proposition 13. *A Steiner-like loop L has the automorphic inverse property if and only if the group A is commutative and for all $x, y \in S \setminus \{e\}$ and $x \neq y$ the identity*

$$h(x)^2 h(y)^2 = f(x,x) f(y,y) f(xy,xy)^{-1}$$

holds.

Proposition 14. *Let S be a Steiner loop and A be an abelian group. Let f be a mapping $S \times S \to A$ such that for $x \neq y \in S \setminus \{e\}$ one has $f(x,y) = h(x)h(y)$ for a mapping $h : S \setminus \{e\} \to A$ and $f(x,e) = f(e,x) = 1$ for all $x \in S$. Then the following assertions are equivalent:*
a) For all $x, y \in S \setminus \{e\}$ and $x \neq y$ the identity

$$h(x)^2 h(y)^2 = f(x,x) f(y,y) f(xy,xy)^{-1}$$

holds.
b) If S has more than 4 elements, then the elements $h(x)$ have the form $h(x) = u\omega_x$, where u is a fixed element of A and for the order $o(\omega_x)$ of the element ω_x one has $o(\omega_x) \leq 2$.
The elements $f(x,x)$ have the form $f(x,x) = u^4 \rho_x$ such that for the order $o(\rho_x)$ of the element ρ_x one has $o(\rho_x) \leq 2$ and $\rho_x \rho_y = \rho_{xy}$ for all $x \neq y \in S \setminus \{e\}$. If $S = \{e, x, y, xy\}$ is the elementary abelian group of order 4, then one has

$$h(x) = \alpha, \quad h(y) = \beta, \quad h(xy) = \gamma,$$
$$f(x,x) = \alpha^2 \beta \gamma \rho_x, \quad f(y,y) = \alpha \beta^2 \gamma \rho_y,$$
$$f(xy,xy) = \alpha \beta \gamma^2 \rho_x \rho_y,$$

where α, β, γ are elements of A, whereas the orders of ρ_x, ρ_y are at most 2.
If the Steiner loop S has more than four elements, then the Steiner-like loop L is the direct product of A and S precisely if the group D is the direct product $\langle u \rangle \times \langle \omega \rangle$ such that for the orders $o(u)$ as well as $o(\omega)$ one has $o(u) \leq 2$ and $o(\omega) \leq 2$ and $\rho_x = 1$ for all $x \in S \setminus \{e\}$. If $S = \{e, x, y, xy\}$ is the elementary abelian group of order 4, then L is the direct product of A and S precisely if the group D has order ≤ 2 and $\rho_x = 1$ for all $x \in S \setminus \{e\}$.

Definition 15. *A loop L is a right Bol loop if $z \cdot [(x \cdot y) \cdot x] = [(z \cdot x) \cdot y] \cdot x$ holds for all $x, y, z \in L$.*

Proposition 16. *Let L be a proper Steiner-like loop.*
a) If $S = \{e, x, y, xy\}$ is the elementary abelian group of order 4, then the loop L satisfies the right Bol identity if and only if the range of h is commutative, the set $\{f(x,x);\ x \in S \setminus \{e\}\}$ is contained in the centre of A and identity $h(x)h(y)h(x)h(xy) = f(x,x)$ holds.
b) If the Steiner loop S has more than 4 elements, then the loop L is a right Bol loop precisely if S is an elementary abelian 2-group and for all $x \in S \backslash \{e\}$ one has $h(x) = t$ and $f(x,x) = t^4$ with a fixed element t of A, but $t^2 \notin Z(A)$.

Definition 17. *A loop L is a left Bol loop if $[x \cdot (y \cdot x)] \cdot z = x \cdot [y \cdot (x \cdot z)]$ holds for all $x, y, z \in L$.*

Theorem 18. *Let L be a Steiner-like loop such that S has more than 4 elements. Then the following conditions are equivalent:*
(i) L is a group.
(ii) S is an elementary abelian 2-group and for all $x \in S \backslash \{e\}$ one has $h(x) = t$ and $f(x,x) = t^4$, where t is a fixed element of A such that $t^2 \in Z(A)$.
(iii) L is a left Bol loop.

Theorem 19. *Let L be a Steiner-like loop such that $S = \{e, x, y, xy\}$ is the elementary abelian group of order 4. Then the following conditions are equivalent:*
(i) L is a group.
(ii) L is left alternative or it has the left inverse property.
(iii) L is a left Bol loop.
Moreover, L is an abelian group if and only if L has the cross inverse property.

A Steiner-like right Bol loop L does not need to be left alternative as the following examples show. Let A be a finite simple group which is not isomorphic to one of the following groups: $PSL(2, 2^n)$, $n > 1$, $PSL(2, q)$, $q \equiv 3$ or $5 \pmod 8$, $q > 3$, the Janko group of order 175560, a group of Ree type of order $q^3(q-1)(q^3+1)$, where $q = 3^{2k+1}$, $k \geq 1$. Then A contains an element u of order 4. Putting $h(x) = u$ for all $x \in S \setminus \{e\}$ we obtain a right Bol loop L which is not left alternative and has exponent 4 since $f(x,x) = 1$.

Let L be a Steiner-like loop. A left translation $\lambda_{(a,\alpha)} : L \to L$ is the bijection

$$(x, \xi) \longmapsto (a, \alpha)(x, \xi) = (ax, f(a, x)\alpha\xi)$$

and a right translation $\rho_{(a,\alpha)} : L \to L$ is the bijection

$$(x, \xi) \longmapsto (x, \xi)(a, \alpha) = (ax, f(x, a)\xi\alpha).$$

Proposition 20. *Let L be a Steiner-like loop. Then the group G_r generated by all right translations of L is an extension of A by the group Σ generated by the translations of the set $\{\rho_{(a,1)};\ a \in S\}$.*

To obtain an analogous result for the group G_l generated by all left translations of L we must suppose that the set $\{f(x, y);\ x, y \in S\}$ is contained in the centre of A.

Proposition 21. *Let L be a Steiner-like loop. We assume that L has one of the following properties:*
a) L is flexible,
b) L is left alternative or it has the left inverse property,
c) L has the cross inverse property,
d) L has the automorphic inverse property.
Then the group G_l generated by all left translations of L is an extension of A by the group Σ' generated by the translations of the set $\{\lambda_{(a,1)};\ a \in S\}$.

Let $(L_1, \cdot)$ and $(L_2, *)$ be two loops of type (1) realized on the set $S \times A$ by the multiplications

$$(x, \xi) \cdot (y, \eta) = (xy, f_1(x, y)\xi\eta)$$

respectively

$$(x, \xi) * (y, \eta) = (xy, f_2(x, y)\xi\eta).$$

We consider isomorphism $\alpha : L_1 \to L_2$ such that $(e, \xi)^\alpha = (e, \xi^{\alpha''})$, where α'' is an automorphism of A. For such an isomorphism $\alpha : L_1 \to L_2$ one has $(x, 1)^\alpha = (x^{\alpha'}, \rho(x^{\alpha'}))$, where ρ is a map from S onto A.

Proposition 22. *Let $(L_1, \cdot)$ and $(L_2, *)$ be two loops of type (1) belonging to the functions f_1, respectively f_2. The map*

$$\beta : L_1 \to L_2, (x, \xi) \mapsto (x^{\alpha'}, \rho(x^{\alpha'})\xi^{\alpha''})$$

defines an isomorphism of L_1 onto L_2 if and only if α' is an automorphism of the loop S, α'' is an automorphism of A, for the map $\rho : S \to A$ one has the set $\{\rho(x^{\alpha'});\ x \in S, \alpha' \in Aut(S)\}$ is contained in the centre $Z(A)$ of A and for all $x, y \in S$ identity

$$f_1(x, y)^{\alpha''} = \rho((xy)^{\alpha'})^{-1}\rho(x^{\alpha'})\rho(y^{\alpha'})f_2(x^{\alpha'}, y^{\alpha'})$$

holds.

Proposition 23. *Let L_1 and L_2 be two Steiner-like loops with respect to the functions f_1, respectively f_2. The map $\beta : L_1 \to L_2; (x, \xi) \mapsto (x^{\alpha'}, \rho(x^{\alpha'})\xi^{\alpha''})$ is an isomorphism of L if and only if α' is an automorphism of the Steiner loop S, α'' is an automorphism of A, the set $\{\rho(x^{\alpha'});\ x \in S, \alpha' \in Aut(S)\}$ is contained in the centre of A, for all $x \in S$ identities*

$$f_1(x,x)^{\alpha''} f_2(x^{\alpha'}, x^{\alpha'})^{-1} = \rho(x^{\alpha'})^2$$

and for all $x, y \in S \setminus \{e\}$, $x \neq y$ identity

$$h_1(x)^{\alpha''} h_1(y)^{\alpha''} = \rho((xy)^{\alpha'})^{-1} \rho(x^{\alpha'}) \rho(y^{\alpha'}) h_2(x^{\alpha'}) h_2(y^{\alpha'})$$

are satisfied.

Proposition 24. *Let L be a loop of type (1). A mapping $\alpha : L \to L$ defined by $(x, \xi)^\alpha = (x^{\alpha'}, \rho(x^{\alpha'})\xi^{\alpha''})$ is an automorphism of L if and only if α' is an automorphism of the loop S, α'' is an automorphism of the group A, for the map $\rho : S \to A$ with $\rho(e) = 1$ one has the set $\{\rho(x^{\alpha'});\ x \in S, \alpha' \in Aut(S)\}$ is contained in the centre $Z(A)$ of A and for all $x, y \in S$ identity*

$$f(x,y)^{\alpha''} = \rho((xy)^{\alpha'})^{-1} \rho(x^{\alpha'}) \rho(y^{\alpha'}) f(x^{\alpha'}, y^{\alpha'})$$

holds.

The map $\rho : S \to Z(A)$, $x \mapsto \rho(x)$ is a homomorphism from S into $Z(A)$ if and only if there are $\alpha' \in Aut(S)$, $\alpha'' \in Aut(A)$ such that $f(x,y)^{\alpha''} = f(x^{\alpha'}, y^{\alpha'})$ for all $x, y \in S$. Any homomorphism ρ from S into the centre $Z(A)$ determines an automorphism of L which is the mapping $\beta_\rho : (x, \xi) \mapsto (x, \rho(x)\xi)$. The set of these automorphisms forms the normal subgroup Ψ of the automorphism group Γ of L which consists of automorphisms inducing on S as well as on A the identity. The group Ψ is commutative.

Proposition 25. *Let L be a Steiner-like loop. The map $\beta : L \to L$ defined by $(x, \xi)^\beta = (x^{\alpha'}, \rho(x^{\alpha'})\xi^{\alpha''})$ is an automorphism of L if and only if for all $x \in S$ identity*

$$\rho(x^{\alpha'})^2 = f(x,x)^{\alpha''} f(x^{\alpha'}, x^{\alpha'})^{-1}$$

and for all $x, y \in S \setminus \{e\}$, $x \neq y$ identity

$$\rho(x^{\alpha'}) \rho(y^{\alpha'}) h(x^{\alpha'}) h(y^{\alpha'}) = \rho((xy)^{\alpha'}) h(x)^{\alpha''} h(y)^{\alpha''}$$

hold.

The normal subgroup Ψ of the automorphism group Γ of L is an elementary abelian 2-group.

A loop extension of type (1) with respect to the factor system f may be also defined in such a way that the multiplication on the set $S \times A$ is given by

$$(x, \xi)(y, \eta) = (xy, \xi f(x, y)\eta),$$

respectively

$$(x, \xi)(y, \eta) = (xy, \xi\eta f(x, y)).$$

This yields loops L^*, respectively L^{**} which coincide with the extension given by multiplication (1) if and only if the set $\{f(x, y);\ x, y \in S\}$ is contained in the centre of A.

Proposition 26. *A Steiner-like loop L^{**} is left alternative or it satisfies the left inverse property precisely if the set $\mathcal{F} = \{f(z, z);\ z \in S\}$ is contained in the centre of A and identity $h(x)h(y)h(x)h(xy) = f(x, x)$ holds.*

Proposition 27. *Let L^{**} be a proper Steiner-like loop.*
a) If $S = \{e, x, y, xy\}$ is the elementary abelian group of order 4, then the loop L^{**} satisfies the left Bol identity if and only if the range of h is commutative, the set $\{f(x, x);\ x \in S\}$ is contained in the centre of A and identity $h(x)h(y)h(x)h(xy) = f(x, x)$ holds.
b) If the Steiner loop S has more than 4 elements, then the loop L^{**} is a left Bol loop precisely if S is an elementary abelian 2-group and for all $x \in S \backslash \{e\}$ one has $h(x) = t$ and $f(x, x) = t^4$ with a fixed element t of A, but $t^2 \notin Z(A)$.