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Classification of nonassociative Moufang loops of odd order pq^3, $p \neq 3$

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Preliminaries

<u>Definitions</u>

- *Moufang loop*: a loop that satisfies any of the following identities:

$$(xy)(zx) = [x(yz)]x,$$

$$(xy)(zx) = x[(yz)x],$$

$$x[y(xz)] = [(xy)x]z,$$

$$[(zx)y]x = z[x(yx)].$$

Research Objective

- For a fixed positive integer n, does there exist a nonassociative Moufang loop of order n?
  - If Yes, find the product rule(s) of that loop.
  - If No, show that all Moufang loops of order n are groups.

Known Results

Results on existence:

Nonassociative Moufang loops of order

- $2m$, iff a nonabelian group of order m exists [Chein & Rajah, 2000];
- 81 [Bol, 1937];
- p^5 for any prime $p > 3$ [Wright, 1965];
- pq^3 for distinct odd primes p and q iff $q \equiv 1 \pmod{p}$ [Rajah, 2001].

Results on classification:

Nonassociative Moufang loops of order

- ≤ 63 [Goodaire, May & Raman, 1999];
- 64 — 4262 nonisomorphic cases [Nagy & Vojtěchovský, 2007];
- 81 — 5 nonisomorphic cases [Nagy & Vojtěchovský, 2007];
- p^5 ($p > 3$) — 4 nonisomorphic cases [Nagy & Valsecchi, 2007].

Nonassociative Moufang loops of odd order pq^3

[Rajah, 2001]: The construction of a nonassociative Moufang loop L of order pq^3 for odd primes p, q with $q \equiv 1 \pmod{p}$, involves the following:

There exist some $x, y, z \in L$ such that

$$k = (x, y, z) \neq 1, \quad x^{-1}kx = k^{\mu}, \quad xyx^{-1} = y^{\mu},$$

$$xzx^{-1} = z^{\mu}, \quad zy = yzk^{\phi},$$

where μ and ϕ are integers satisfying

$$\mu^p \equiv 1 \pmod{q} \text{ but } \mu \not\equiv 1 \pmod{q},$$

$$\phi \text{ is any integer when } p = 3,$$

$$\phi(\mu - 1) \equiv -2 \pmod{q} \text{ when } p \neq 3.$$

Then it was shown that every element in L can be uniquely expressed in the form $x^{\alpha} \cdot y^{\beta} z^{\gamma} k^{\delta}$ where $\alpha \in \mathbb{Z}_p$ and $\beta, \gamma, \delta \in \mathbb{Z}_q$. The product of any two elements is given by

$$(x^{\alpha_1} \cdot y^{\beta_1} z^{\gamma_1} k^{\delta_1}) \cdot (x^{\alpha_2} \cdot y^{\beta_2} z^{\gamma_2} k^{\delta_2}) = x^{\alpha_{(1,2)}} \cdot y^{\beta_{(1,2)}} z^{\gamma_{(1,2)}} k^{\delta_{(1,2)}}$$

where

$\alpha_{(1,2)} \equiv (\alpha_1 + \alpha_2) \pmod{p}$;

$\beta_{(1,2)} \equiv (\beta_1 \mu^{(p-1)\alpha_2} + \beta_2) \pmod{q}$;

$\gamma_{(1,2)} \equiv (\gamma_1 \mu^{(p-1)\alpha_2} + \gamma_2) \pmod{q}$;

$$\delta_{(1,2)} \equiv \{\delta_1 \mu^{\alpha_2} + \delta_2 + \phi \beta_2 \gamma_1 \mu^{(p-1)\alpha_2} + [\beta_1 \gamma_1 (\mu^{\alpha_2} - \mu^{(p-2)\alpha_2}) + (\beta_1 \gamma_2 - \beta_2 \gamma_1)(\mu^{(\alpha_1+\alpha_2)} - \mu^{(p-1)\alpha_2})]/(\mu - 1)\} \pmod{q}.$$

We shall denote this class of Moufang loops by $M_{pq^3}(\mu, \phi)$.

Classification of nonassociative Moufang loops of odd order pq^3, $p \neq 3$

- Let $3 < p < q$ be odd primes satisfying $q \equiv 1 \pmod{p}$.
- The number of solutions satisfying $\mu^p \equiv 1 \pmod{q}$ and $\mu \not\equiv 1 \pmod{q}$
  $= \gcd(p, q - 1) - 1$
  $= p - 1$.
- Suppose we have one solution μ_0, then we have all the solutions, namely

$$\{\mu_0, \mu_0^2, \dots, \mu_0^{p-1}\}.$$

- For a fixed μ, there exists a unique ϕ satisfying $\phi(\mu - 1) \equiv -2 \pmod{q}$.
- Proof: Suppose $\exists\ \phi = \dfrac{mq-2}{\mu-1}$ and $\phi' = \dfrac{m'q-2}{\mu-1}$.

$$\Rightarrow \phi = \phi' + r$$

$$\Rightarrow \frac{mq-2}{\mu-1} = \frac{m'q-2}{\mu-1} + r$$

$$\Rightarrow (m-m')q = r(\mu-1)$$

$$\Rightarrow q \mid r(\mu-1)$$

$$\Rightarrow q \mid r$$

$$\Rightarrow \phi \equiv \phi' \pmod{q}$$

- Let L be a nonassociative Moufang loop of order pq^3.
- There exist some $x, y, z \in L$ such that

$$k = (x, y, z) \neq 1,$$
$$x^{-1}kx = k^{\mu_0^n},$$
$$xyx^{-1} = y^{\mu_0^n},$$
$$xzx^{-1} = z^{\mu_0^n},$$
$$zy = yzk^{\phi_n}$$

for some $n \in \{1, 2, \ldots, p-1\}$.

- Let θ_n be an integer satisfying $n\theta_n \equiv 1$ (mod p).
- Let $x_0 = x^{\theta_n}$, $y_0 = y$, $z_0 = z$ and $k_0 = k^{\mu_0^{n(\theta_n-1)}+\mu_0^{n(\theta_n-2)}+\cdots+1}$.

$$\Rightarrow k_0 = (x_0, y_0, z_0) \neq 1,$$

$$x_0^{-1}k_0x_0 = k_0^{\mu_0},$$

$$x_0y_0x_0^{-1} = y_0^{\mu_0},$$

$$x_0z_0x_0^{-1} = z_0^{\mu_0}.$$

- $zy = yzk^{\phi_n}$ where $\phi_n(\mu_0^n - 1) \equiv -2 \pmod{q}$.

  $\Rightarrow z_0 y_0 = zy = yzk^{\phi_n} = y_0 z_0 k_0^{\phi_n \tau_n}$ where

  $$\tau_n(\mu_0^{n(\theta_n - 1)} + \mu_0^{n(\theta_n - 2)} + \cdots + 1) \equiv 1 \pmod{q}.$$

  So $$\phi_n \tau_n = \left(\frac{s_n q - 2}{\mu_0^n - 1} \right) \left(\frac{t_n q + 1}{\mu_0^{n(\theta_n - 1)} + \mu_0^{n(\theta_n - 2)} + \cdots + 1} \right)$$

  for some $s_n, t_n \in \mathbb{Z}$

  $$= \frac{(s_n t_n q + s_n - 2t_n)q - 2}{\mu_0 - 1}.$$

  Hence $\phi_1 = \phi_n \tau_n$ is an integer satisfying

  $\phi_1(\mu_0 - 1) \equiv -2 \pmod{q}$.

- Therefore, by using the substitutions

$x_0 = x^{\theta_n}, y_0 = y, z_0 = z$ and

$k_0 = k^{\mu_0^{n(\theta_n - 1)} + \mu_0^{n(\theta_n - 2)} + \cdots + 1}$,

we prove that $M_{pq^3}(\mu_0^n, \phi_n) \cong M_{pq^3}(\mu_0, \phi_1)$

for any $n \in \{2, 3, \ldots, p-1\}$

Conclusion

- There is only one nonassociative Moufang loop of order pq^3 where $3 < p < q$ are odd primes satisfying $q \equiv 1 \pmod{p}$.
- It is a semidirect product of C_p and a nonabelian group of order q^3 and exponent q.

Direction for future research

Nonassociative Moufang loops of odd order $3q^3$

- Main obstacle:
  For a fixed μ, the value for ϕ is not unique:
  $\phi \in \{0, 1, \dots, q-1\}$.

Thank You

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