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in Abstract Analysis

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A COMBINATORIAL PROPERTY OF FRÉCHET ITERATED FILTERS

by

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The main result we present here is totally elementary in the sense that its statement does not refer to any new notion. However the motivation of this result lies in the study of the following notion that we recall :

Theorem-Definition

For any analytic filter $\mathcal{F}$ on ω there exists a unique countable ordinal $\xi = \mathrm{rk}(\mathcal{F})$ determined by any of the following equivalent (well defined) properties :

1. *The set $\{f = \lim_{\mathcal{F}} f_n; \ \text{ with } f_n : 2^\omega \to \mathbb{R} \text{ continuous } \}$ is exactly the set of all Borel functions $f : 2^\omega \to \mathbb{R}$ of class ξ.*
2. *ξ is the minimal countable ordinal such that $\mathcal{F}$ can be separated from $\mathcal{F}^*$ by a set in $\mathbf{\Sigma}^0_{1+\xi} \cup \mathbf{\Pi}^0_{1+\xi}$.*
3. *ξ is the maximal countable ordinals such that $\mathcal{F}$ cannot be separated from $\mathcal{F}^*$ by a set in $\mathbf{\Sigma}^0_{1+\xi} \cap \mathbf{\Pi}^0_{1+\xi}$.*

Theorem

Any analytic filter can be refined by a Borel filter of the same rank.

In these talks we shall only be concerned by Borel filters.

Theorem

Given any (additive, multiplicative) Baire class Γ there exists a Borel filter of rank 1 which is in the class Γ but not in the dual (multiplicative, additive) Baire class $\check{\Gamma}$.

It follows from condition 3 of Theorem–Definition of the rank that there is no $\mathbf{\Pi}^0_{1+\xi}$ filter of rank ξ.

Question:

What is the minimal complexity of a Borel filter of rank ξ ?

From now on we assume that $\xi = m$ is finite.

Conjecture A:

There is no $\mathbf{\Pi}^0_{2m}$ filter of rank m.

Equivalent form of Conjecture A :

The pointwise limit of a sequence of continuous functions along a $\mathbf{\Pi}^0_{2m}$ filter is of Baire class $m-1$.

The Katětov filters

We recall that the Katětov filters (also called the iterated Fréchet filters) $\mathcal{N}_m$ are defined inductively :

$\mathcal{N}_1$ is the Fréchet filter on ω

$$A \in \mathcal{N}_1 \iff \exists i_0,\ \forall i \geq i_0, i \in A$$

$\mathcal{N}_2$ is the filter on ω^2 defined for $A \subset \omega^2$ by :

$$A \in \mathcal{N}_2 \iff \exists i_0,\ \forall i \geq i_0, \exists j_0,\ \forall j \geq j_0,\ (i,j) \in A$$

$\mathcal{N}_3$ is the filter on ω^3 defined for $A \subset \omega^3$ by :

$$A \in \mathcal{N}_3 \iff \exists i_0,\ \forall i \geq i_0, \exists j_0,\ \forall j \geq j_0, \exists k_0,\ \forall k \geq k_0,\ (i,j,k) \in A$$

Proposition

$\mathcal{N}_m$ is a $\mathbf{\Sigma}^0_{2m}$ filter of rank m.

We shall prove the following weak form of Conjecture A :

Theorem

There is no $\mathbf{\Pi}^0_{2m}$ filter refining $\mathcal{N}_m$.

By this latter result Conjecture A is actually a consequence of the following :

Conjecture B:

$\mathcal{N}_m$ embeds in any filter of rank m.

We recall that if $m = 2$ then Conjucture B is true, hence Conjucture A too.

Theorem

There is no $\mathbf{\Pi}^0_4$ filter of rank 2.

Main result

We shall in fact prove the following more precise result :

Main Theorem

In any $\mathbf{\Pi}^0_{2m}$ *set* $\mathcal{A} \supset \mathcal{N}_m$ *one can find a family of* $m+1$ *elements with empty intersection.*

Remark:

For all $m \geq 1$ *there exists a* $\mathbf{\Pi}^0_{2m}$ *set* $\mathcal{A} \supset \mathcal{N}_m$ *in which the intersection of any family of* m *elements is non empty.*

The case $m = 1$

For $m = 1$ Main Theorem is just the following :

Theorem

In any $\mathbf{G}_\delta$ set $\mathcal{A} \supset \mathcal{N}_1$ one can find two elements with empty intersection.

This is a simple consequence of Baire Theorem. Nevertheless it will be instructive for the proof of the general case to present the proof of this trivial case in the following form :

Sketch of proof for $m = 1$: Consider the game G_1 in which two Players I and II construct by alternate finite extension some element $1_A \in 2^\omega$. Here by “a game" we mean “the rules of a game" without any a priori win condition. Then observe :

(A) *Given any strategy τ for Player II there exists two infinite runs compatible with τ in which the players construct sets A, B such that $A \cap B = \emptyset$.*

(B) *Given any $\mathbf{G}_\delta$ set $\mathcal{A} \supset \mathcal{N}_m$ Player II has a strategy to construct a set $A \in \mathcal{A}$.*

The case $m = 2$

Theorem

In any $\mathbf{\Pi}^0_4 = \mathbf{G}_{\delta\sigma\delta}$ set $\mathcal{A} \supset \mathcal{N}_2$ one can find three elements with empty intersection.

Our plan is to follow the same scheme than in the case $m = 1$, that is to define a game G_2 in each infinite run of which the players “constructs" a set $A \subset \omega^2$, with the same corresponding properties **(A)** and **(B)** :

Notation : for any sets A, B we denote by $\mathrm{Fin}\,(A, B)$ the set of all finite partial mappings from A to B. If $f \in \mathrm{Fin}\,(A, B)$ we set :

$$\mathcal{V}_f = \{g \in B^A : \ f \subset g\}$$

which is a clopen subet of B^A

The case $m = 2$: Definition of the game

- We first define : a set E, a partial ordering R on E, and a monotone mapping :

$$\varepsilon : (E, R) \to \left(\mathrm{Fin}(\omega^2), \subset\right)$$

- Then G will be the game on E defined by :

$$a_0 \ \mathrm{R} \ a_1 \ \mathrm{R} \ a_2 \ \mathrm{R} \ \ldots\ldots\ldots \ \mathrm{R} \ a_n \ \mathrm{R} \ \ldots$$

hence : $\varepsilon(a_0) \subset \varepsilon(a_1) \subset \varepsilon(a_2) \subset \cdots \subset \varepsilon(a_n) \subset \ldots$

- By definition we shall say that the infinite run (a_n) constructs the set $A \subset \omega^2$ if $1_A = \bigcup_n \varepsilon(a_n)$

The case $m = 2$: Definition of the game

Definition of the domain E :

$$E = \mathrm{Fin}\left(\omega\,,\ \{0,1\} \times \mathrm{Fin}\,(\omega, \{0,1\})\right)$$

Let $a \in E$ with $\mathrm{dom}\,(a) = J_a$ finite $\subset \omega$:

$$a \approx \left((a_{(i)})_{i \in J_a}, (a^{(i)})_{i \in J_a}\right)$$

$$(a_{(i)})_{i \in J_a} \ \approx \ \text{labelled partition of } J_a$$

Hence

$$a \approx \begin{cases} (J_a^0, J_a^1) \text{ labelled partition of } J_a \text{ finite } \subset \omega \\ (a^{(i)})_{i \in J_a} \in \mathrm{Fin}\,(\omega, \mathrm{Fin}\,(\omega, \{0,1\})) \end{cases}$$

$$(a^{(i)})_{i \in J_a} \approx \varepsilon(a) \in \mathrm{Fin}\,(\omega^2, \{0,1\})$$

The case $m = 2$: Definition of the game

Definition of the partial ordering R :

$$E = \mathrm{Fin}\left(\omega\,,\,\{0,1\} \times \mathrm{Fin}\,(\omega, \{0,1\})\right)$$

So

$$a \subset b \iff \begin{cases} J_a^0 \subset J_b^0 \text{ and } J_a^1 \subset J_b^1 \\ \forall i \in J_a,\ a^{(i)} \subset b^{(i)} \end{cases}$$

We then set :

$$a \ \mathrm{R}\ b \iff \begin{cases} J_a^0 \subset J_b^0 \text{ and } J_a^1 \subset J_b^1 \\ \forall i \in J_a,\ a^{(i)} \subset b^{(i)} \\ \forall i \in J_a^1,\ b^{(i)} \setminus a^{(i)} \subset 1_\omega \end{cases}$$

Hence

$$a \ \mathrm{R}\ b \Longrightarrow a \subset b \Longrightarrow (a^{(i)})_{i \in J_a} \subset (b^{(i)})_{i \in J_b} \iff \varepsilon(a) \subset \varepsilon(b)$$

The case $m = 2$: Proof of (A)

We first prove property **(A)** :

Lemma 1

Given any strategy τ for Player II in G there exist three infinite runs compatible with τ constructing three sets A, B, C in ω^2 such that $A \cap B \cap C = \emptyset$.

Proof : Construct three runs α, β, γ in G_2 in the following "cyclic" way : Player I makes the first move in α followed by Player II, then similarly two moves in β, followed by two moves in γ, then the players go back to α making two more moves, then again two moves in β, followed by two moves in γ ; and so on. One can show that such a construction can be achieved in such a way that the sets $A, B, C \subset \omega^2$ constructed in these three runs have empty intersection ($A \cap B \cap C = \emptyset$).

The case $m = 2$: Proof of (B)

Lemma 2

Suppose that $\mathcal{A} \supset \mathcal{N}_2$ is $\mathbf{G}_{\delta\sigma\delta}$ and fix open sets $\mathcal{A}_{i,j,k}$ such that $\mathcal{A} = \bigcap_i \bigcup_j \bigcap_k \mathcal{A}_{i,j,k}$. Then :

$$\forall (i,a)\,,\ \exists (j,b) \text{ with } a\ R\ b, \forall (k,c) \text{ with } b\ R\ c,\ \exists d \text{ with } c\ R\ d$$

such that $\mathcal{V}_{\varepsilon(d)} \subset \mathcal{A}_{i,j,k}$.

Proof : If not ... one constructs (i,a) and $(k_j, a_j)_{j \geq 0}$ such that :

$$\begin{cases} (1) \quad a_0 \ \mathrm{R} \ a_1 \ \mathrm{R} \ a_2 \ldots\ldots\ldots\ldots \mathrm{R} \ a_j \ \mathrm{R} \ \ldots \\ (2)\ J^0 a_0 = J^0 a_1 = J^0 a_2 \ \cdots\cdots\cdots \ = J^0 a_j = \cdots = J^0 a \\ (3) \quad \text{If } a_j \ R \ d \text{ then } \mathcal{V}_{\varepsilon(d)} \cap \mathcal{A}^c_{i,j,k_j} \neq \emptyset \\ (4) \quad \bigcup_j \varepsilon(a_j) = 1_A \end{cases}$$

It follows from (1) and (2) that $A \in \mathcal{N}_2$ and from (3) that $A \notin \mathcal{A}$ which is a contradiction.

Proof of (B)

Lemma 3

Given any $\mathbf{G}_{\delta\sigma\delta}$ set $\mathcal{A} \supset \mathcal{N}_2$ Player II has a strategy to construct a set $A \in \mathcal{A}$

Proof : Fix a “good" enumeration of $\omega \cup \omega^2$:

$$\Big(< 0 >, < 0, 0 >, < 1 >, < 0, 1 >, < 1, 0 >, < 2 >, \dots\Big)$$

and define a strategy $(a_0, a_1 \dots, a_{2n}) \mapsto a_{2n+1}$ for Player II by applying Lemma 2 successively :

$$(i, a) = (0, a_0) \mapsto (j, b) = (j_0, a_1) \;\; ; \;\; (k, c) = (0, a_2) \mapsto d = a_3 \; .$$
$$(i, a) = (1, a_4) \mapsto (j, b) = (j_1, a_5) \; .$$
$$(i, a) = (0, a_0) \mapsto (j, b) = (j_0, a_1) \;\; ; \;\; (k, c) = (1, a_6) \mapsto d = a_7 \; .$$
$$(i, a) = (1, a_4) \mapsto (j, b) = (j_1, a_5) \;\; ; \;\; (k, c) = (0, a_8) \mapsto d = a_9 \; .$$
$$(i, a) = (2, a_{10}) \mapsto (j, b) = (j_2, a_{11}) \; .$$

The general case

Main Theorem

In any $\mathbf{\Pi}^0_{2m}$ set $\mathcal{A} \supset \mathcal{N}_m$ one can find a family of $m+1$ elements with empty intersection.

Plan of proof :

1. Define a game G_m in each infinite run of which the players “constructs" a set $A \subset \omega^m$, with the following properties :
2. (A_m) *Given any strategy τ for Player II in G_m there exists a family of $m+1$ infinite runs compatible with τ constructing sets $A_0, A_1, \ldots, A_m$ such that $\bigcap_{k=0}^m A_k = \emptyset$.*
3. (B_m) *Given any $\mathbf{\Pi}^0_{2m}$ set $\mathcal{A} \supset \mathcal{N}_2$ Player II has a strategy in G_m to construct $A \in \mathcal{A}$.*

The general case

- We define a set $E = E_m$ with two partial orderings $S = S_m =\subset R_m = R$ and a monotone mapping :

$$\varepsilon : (E, R) \to \left(\mathrm{Fin}(\omega^{\mathrm{m}}), \subset\right)$$

- The game $G = G_m$ is defined using only the relation $R = R_m$ as in the case $m = 2$:

$$a_0 \ \mathrm{R} \ a_1 \ \mathrm{R} \ a_2 \ \mathrm{R} \ \ldots\ldots\ldots \ \mathrm{R} \ a_n \ \mathrm{R} \ \ldots$$

hence : $\varepsilon(a_0) \subset \varepsilon(a_1) \subset \varepsilon(a_2) \subset \cdots \subset \varepsilon(a_n) \subset \ldots$

- By definition we shall say that the infinite run (a_n) constructs the set $A \subset \omega^m$ if $1_A = \bigcup_n \varepsilon(a_n)$.
- The finer partial ordering S_m is only used for the inductive definition of R_m and has the following property : *Any infinite S_m chain constructs a set in $\mathcal{N}_m$.*

The general case

Precise definitions : For $m = 0$ let :

$E_0 = \mathrm{Fin}\,(\omega\,,\,\{0,1\}$,
R_0 is the extension relation on E_0.
S_0 is the “extension by 1" relation on E_0.

We then define inductively :

$$E_{m+1} = \mathrm{Fin}\,(\omega\,,\,\{0,1\} \times E_m\,)$$

$$a\ R_{m+1}\ b \iff \begin{cases} J_a^0 \subset J_b^0 \ \text{ and } \ J_a^1 \subset J_b^1 \\ \forall i \in J_a,\ a^{(i)}\ R_m\ b^{(i)} \\ \forall i \in J_a^1,\ a^{(i)}\ S_m\ b^{(i)} \end{cases}$$

$$a\ S_{m+1}\ b \iff a\ R_{m+1}\ b \ \text{ and } J_a^0 = J_b^0$$

The general case

Unfortunately the proof of properties (A_m) and (B_m) are much more complicated than in the case $m = 2$. Actually the proof goes through two very technical properties (A^*_m) and (B^*_m) which are proved by induction and from which one then derives the original properties (A_m) and (B_m).

The proof relies on a general result concerning games of the form G_m that we state in next section.

ORDERED GAMES

- **General frame :**
  – Ω a fixed countable set.
  – (E, R) a partially ordered set.
  – $\varepsilon : (E, R) \to \big(\mathrm{Fin}(\Omega, \{0, 1\}), \subset\big)$ a monotone mapping.
- G denotes the game (with no win condition) :
  $$a_0 \ \mathrm{R} \ a_1 \ \mathrm{R} \ a_2 \ \mathrm{R} \ \ldots\ldots\ldots \ \mathrm{R} \ a_n \ \mathrm{R} \ \ldots$$
  hence : $\varepsilon(a_0) \subset \varepsilon(a_1) \subset \varepsilon(a_2) \subset \cdots \subset \varepsilon(a_n) \subset \ldots$
- For any $A \subset 2^\Omega$, G_A denotes the game G with the following win condition :

  Player II wins the infinite run (a_n) *if :* $\bigcup_n \varepsilon(a_n) \in A$.

Notation :

If $s = (p_0, p_1, \ldots, p_{n-1}, p_n)$ then $s^* = (p_0, p_1, \ldots, p_{n-1})$

Définition:

We shall say that (Σ, ν) *is an* enumerated semi-linear tree (e.s.l. tree) *if* Σ *is a countable tree,and* $\nu : \Sigma \to \omega$ *is a* one-to-one *mapping satisfying :*

0) $\nu(\emptyset) = 0$.

1) If $|s|$ *is odd then* $\nu(s)$ *is odd and* $\nu(s^*) \leq \nu(s)$.

2) If $|s|$ *is even and* > 0 *then* $\nu(s) = \nu(s^*) + 1$.

Let (E, R, ε) as above.

- For any e.s.l. tree (Σ, ν) we denote by $G^{(\Sigma,\nu)}$ the game (with no win condition) :

$$a_0, a_1, a_2, \dots\dots\dots, a_n, \dots$$

with :

(1) $\varepsilon(a_0) \subset \varepsilon(a_1) \subset \varepsilon(a_2) \subset \cdots \subset \varepsilon(a_n) \subset \dots$

(2) If $n \notin \nu(\Sigma)$ then $a_n = a_{n-1}$.

(3) $a_{\nu(s^*)}\ R\ a_{\nu(s)}$.

- For any $A \subset 2^\Omega$, $G_A^{(\Sigma,\nu)}$ denotes the game $G^{(\Sigma,\nu)}$ with the following win condition :

Player II wins the infinite run (a_n) *if :* $\bigcup_n \varepsilon(a_n) \in A$.

Theorem

For any $A \subset 2^{\Omega}$*, if Player II wins* G_A *then Player II wins* $G_A^{(\Sigma,\nu)}$ *for any e.s.l. tree* (Σ, ν)*.*

Theorem

For any $\mathbf{\Pi}_k^0$ *set* $A \subset 2^{\Omega}$ *if Player II wins* $G_A^{(\Sigma,\nu)}$ *for any e.s.l. tree* (Σ, ν) *with* $\mathrm{ht}(\Sigma) < k$*, then Player II wins* G_A*.*