Monotone assignments in compact and function spaces

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Introduction

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Definition

A function $\phi : [X]^{\leq\omega} \to [Y]^{\leq\omega}$ is ω-**monotone** if satisfy:

1. $A \subset B \in [X]^{\leq\omega}$ imply $\phi(A) \subset \phi(B)$;
2. if $\{A_n\}_{n\in\omega} \subset [X]^{\leq\omega}$ is increasing, then $\phi(\bigcup_{n<\omega} A_n) = \bigcup_{n<\omega} \phi(A_n)$.

Retractional skeletons

Γ will denote an up-directed σ-closed poset.

Definition

An r-**skeleton** in a space X is a family of retractions $\{r_s\}_{s\in\Gamma}$, satisfying the following conditions:

1. $r_s(X)$ has a countable network for each $s \in \Gamma$.
2. $s \leq t$ implies $r_s = r_s \circ r_t = r_t \circ r_s$.
3. For every $x \in X$, $x = \lim_{s\in\Gamma} r_s(x)$.
4. Given $s_0 < s_1 < \cdots$ in Γ, if $t = \sup_{n\in\omega} s_n$, then $r_t(x) = \lim_{n\to\infty} r_{s_n}(x)$ for every $x \in X$.

Theorem

A compact space is Valdivia (Corson) if and only if it admits a commutative (full) r-skeleton.

Definition

Given a space X say that $\{(F_s, \mathcal{B}_s)\}_{s\in\Gamma} \subset \mathcal{CL}(X) \times [\tau(X)]^{\leq\omega}$ is a *c-skeleton* on X if:

1. for each $s \in \Gamma$, $\mathcal{B}_s$ is a base for a topology τ_s on X and there exist a Tychonoff space Z_s and a continuous map $g_s : (X, \tau_s) \to Z_s$ which separates the points of F_s;
2. if $s, t \in \Gamma$ and $s \leq t$, then $F_s \subset F_t$;
3. $X = \overline{\bigcup_{s\in\Gamma} F_s}$;
4. the assignment $s \to \mathcal{B}_s$ is ω-monotone.

In addition, if $X = \bigcup_{s\in\Gamma} F_s$, then we say that the c-skeleton is *full*.

c-skeletons

Theorem

If X is countably compact and has a (full) c-skeleton, then X has a (full) r-skeleton.

Proof. By appliying a closure argument we can find an up-directed and σ-complete partially ordered set Σ such that $g_M(X) = g_M(F_M)$ for each $M \in \Sigma$.

$$\begin{array}{ccc} F_M & \subset & X \\ \downarrow{\scriptstyle g_M\restriction_{F_M}} & & \downarrow \\ Z_M & \xleftarrow[g_M]{} & (X, \tau_M) \end{array}$$

Note that $g_M \restriction_{F_M}$ is a homeomorphism onto its image. Then $r_M = (g_M \restriction_{F_M})^{-1} \circ g_M : X \to F_M$ is a retraction.

Then $\{r_M\}_{M \in \Sigma}$ is an r-skeleton in X.

q-skeletons

In order to get a C_p-dual concept to c-skeleton, we introduce the following notion.

Definition

Let X be a space. Consider a family $\{(q_s, D_s)\}_{s\in\Gamma}$, where $q_s : X \to X_s$ is an $\mathbb{R}$-quotient map and D_s is a countable subset of X for each $s \in \Gamma$. We say that $\{(q_s, D_s)\}_{s\in\Gamma}$ is a *q-skeleton* on X if:

1. the set $q_s(D_s)$ is dense in X_s;
2. if $s, t \in \Gamma$ and $s \leq t$, then there exists a continuous onto map $p_{t,s} : X_t \to X_s$ such that $q_s = p_{t,s} \circ q_t$;
3. the assignment $s \to D_s$ is ω-monotone;
4. $C_p(X) = \overline{\bigcup_{s\in\Gamma} q_s^*(C_p(X_s))}$.

The q-skeleton is *full* whenever $C_p(X) = \bigcup_{s\in\Gamma} q_s^*(C_p(X_s))$.

Duality results

Theorem

If X has a (full) c-skeleton, then $C_p(X)$ has a (full) q-skeleton.

Proof.

Let $\{(F_s, \mathcal{B}_s)\}_{s\in\Gamma}$ be a c-skeleton in X and set $\mathcal{B} = \bigcup\{\mathcal{B}_s\}_{s\in\Gamma}$. For each $W \in \mathcal{W}(\mathcal{B})$ fix $d_W \in W$. For each $s \in \Gamma$ we set $X_s = \pi_{F_s}(C_p(X))$, $q_s = \pi_{F_s} : C_p(X) \to X_s$ and $D_s = \{d_N : N \in \mathcal{W}(\mathcal{B}_s)\}$. Then $\{(q_s, D_s)\}_{s\in\Gamma}$ is a q-skeleton in $C_p(X)$.

$$\begin{array}{ccc} C_p(X) & \xrightarrow{\pi^*_{F_s}} & C_p(F_s) \\ \uparrow & & \uparrow \\ X & \supset & F_s \end{array}$$

[1] ∎

[1] $\mathcal{W}(\mathcal{B})$ consists of all sets $\{f \in C_p(X) : \forall(i \leq n)(f(B_i \subset U_i))\}$, where $B_i \in \mathcal{B}$ and $U_i \in \mathcal{B}_{\mathbb{R}}$.

Duality results

Theorem

If X has a (full) q-skeleton, then $C_p(X)$ has a (full) c-skeleton.

Proof.

Let $\{(q_s, D_s)\}_{s\in\Gamma}$ be a q-skeleton in X. Fix $s \in \Gamma$. The set $F_s = q_s^*(C_p(X_s))$ is closed in $C_p(X)$. Besides, let $\mathcal{B}_s = \mathcal{B}(D_s)$ the family of all canonical open sets with support in D_s. It can ve verified that $\{(F_s, \mathcal{B}_s)\}_{s\in\Gamma}$ is a c-skeleton on $C_p(X)$.

$$\begin{array}{ccc} C_p(X) & \xleftarrow{\;q_s^*\;} & C_p(q_s(X)) \\ \uparrow & & \uparrow \\ X & \xrightarrow{\;q_s\;} & q_s(X) \end{array}$$

□

Generating q-skeletons

Theorem

If X has a (strong) full r-skeleton, then X has a (full) q-skeleton.

Proof.

Let $\{r_s\}_{s\in\Gamma}$ be a strong (full) r-skelton in X. Consider the set $\Sigma = [\mathcal{CL}(X)]^{\leq\omega}$. Construct assignments $\mathcal{F} \to s_{\mathcal{F}}$ and $\mathcal{F} \to D_{\mathcal{F}}$ such that $r_{s_{\mathcal{F}}}(D_{\mathcal{F}})$ is dense in $r_{s_{\mathcal{F}}}(X)$.

$$\mathcal{F} \longrightarrow s_{\mathcal{F}}$$

Let $q_{\mathcal{F}} = r_{s_{\mathcal{F}}}$ for each $\mathcal{F} \in \Sigma$. Then $\{(q_{\mathcal{F}}, D_{\mathcal{F}})\}_{\mathcal{F}\in\Sigma}$ is a (full) q-skeleton in X. □

Corollary

If X has a strong r-skeleton, then X has a full c-skeleton.

Generating q-skeletons

Theorem

Every monotonically ω-stable space has a full q-skeleton.

Proof.

Construct an ω-monotone map $\mathcal{A} : [C_p(X)]^{\leq\omega} \to [C_p(X)]^{\leq\omega}$ such that $\mathcal{A}(A)$ is a dense in $\Delta^*_{\mathcal{A}(A)}(C_p(\Delta_{\mathcal{A}(A)}(X)))$.

$$\begin{array}{ccc} C_p(X) & \xleftarrow{\ \Delta_A^*\ } & C_p(\Delta_A(X)) \\ \uparrow & & \uparrow \\ X & \xrightarrow{\ \Delta_A\ } & \Delta_A(X) \end{array}$$

Given $A \in \Gamma$, the map $q_A = \Delta_{\overline{\mathcal{A}(A)}}$ is an $\mathbb{R}$-quotient map. Let $A \to D_A$ be an ω-monotone assignment such that $q_A(D_A)$ is dense in $q_A(X)$. Then $\{(q_A, D_A)\}_{A\in\Gamma}$ is a full q-skeleton in X. □

Strong r-skeletons

An r-skeleton $\{r_s\}_{s\in\Gamma}$ in a space X is said to be *strong* if whenever $s_0 \in \Gamma$ and F is closed in X^n, for some $n \in \mathbb{N}$, there exists $s \in \Gamma$ such that $s_0 \leq s$ and $r_s^n(F) \subset F$.

Theorem

If X is has a strong r-skeleton and is monotonically ω-monolithic ($monotonically$ ω-stable), then X is Sokolov.

Theorem

A space X has a strong r-skeleton if and only if $C_p(X)$ has a strong r-skeleton.

Theorem

If X is has a strong r-skeleton and is monotonically ω-monolithic, then X is a Lindelöf D-space.

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Thank You!