Monotone assignments in compact and function spaces

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Introduction

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- Firstly, we are ging to deal with monotone ω-stability, a concept which result to be useful to study retractonal skeletons in general and in function spaces.
- Secondly, we will use ω-monotone assignments provide a proof of the characterizations of Corson and Valdivia compact spaces by some special retractional skeletons.

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- ω-monotone assignements are preserved under several estandard operations like composition.

Finitary maps

Let H be a set. An *n-ary* function on H is an $f : H^n \to H$ if $n > 0$, and a an element of H if $n = 0$. f is a *finitary* function if it is n-ary for some $n \in \omega$.

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For a fnitary map f on H and $A \subset H$ we set

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Given a set $C \subset H$ and a fnitary map f on H we sat that C is *closed under F* whenever $f * C \subset C$. For a family F of finitary functions on H and $A \subset H$, the *closure of A under F* is the smallest set, with respect to inclusion, such that $A \subset C \subset H$ and C is closed under all functions from F. Note that there is a least C, namely

$$C = \bigcap\{D : B \subset D \subset A \text{ and } D \text{ is closed under } F\}.$$

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Theorem

If H is a set and F is a countable family of finitary functions on H. Then the map $\mathcal{C} : [H]^{\leq\omega} \to [H]^{\leq\omega}$, which assigns to each $A \in [H]^{\leq\omega}$ the closure $\mathcal{C}(A)$ of A under F, is well defined and ω-monotone.

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Corollary

If $\phi : [X]^{\leq\omega} \to [X]^{\leq\omega}$ is ω-monotone, then the assignment $A \to \overline{\phi}(A)$ is ω-monootne.

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Let θ be a cardinal. If $R \in [H(\theta)]^{\leq\omega}$ then we can find an ω-monotone function $\mathcal{M} : [H(\theta)]^{\leq\omega} \to [H(\theta)]^{\leq\omega}$ such that $\mathcal{M}(A)$ is an elementary submodel of $H(\theta)$ and $R \subset \mathcal{M}(A)$ for each $A \in [H(\theta)]^{\leq\omega}$.

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In the practice, the set R from the above corollary will be the set of all relevant objects in a given context and θ will be a large enough cardinal. A function $\mathcal{M}$ as in Corollary 4 wil be referenced as an ω-*monotone assignmet of suitable elementary submodels.*

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A space X is **monotonically ω-stable** if there exists an ω-monotone assignment $\mathcal{N} : [C_p(X)]^{\leq\omega} \to [\mathcal{P}(X)]^{\leq\omega}$ such that $\mathcal{N}(A)$ is a network for $\overline{A}$ for all $A \in [X]^{\leq\omega}$.

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Proof. Fix a countable base $\mathcal{B}_{\mathbb{R}}$ for the real line $\mathbb{R}$. Let $A \to \mathcal{M}(A)$ be an ω-monotone assignmet of suitable elementary submodels and for each $A \in [C_p(X)]^{\leq\omega}$ let

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Then the assignment $A \to \mathcal{N}(A)$ is ω-monotone. Given $A \in [C_p(X)]^{\leq\omega}$, we shall prove that $\mathcal{N}(A)$ is a network for $\overline{A}$.

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- Therefore $N = X \setminus \bigcup \mathcal{V} \in \mathcal{N}(A)$ and $x \in N \subset U$. $\square$

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4. Given $s_0 < s_1 < \cdots$ in Γ, if $t = \sup_{n\in\omega} s_n$, then $r_t(x) = \lim_{n\to\infty} r_{s_n}(x)$ for every $x \in X$.

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- We call $\bigcup_{s\in\Gamma} r_s(X)$ *the set induced by* $\{r_s\}_{s\in\Gamma}$.
- The family $\{r_s\}_{s\in\Gamma}$ is a *weak r-skeleton* in X if it only satisfies conditions 1,2 and 4.

Closed invariant sets

Lemma

Let X be monotonically ω-stable and let Y be induced by a weak r-skeleton $\{r_s\}_{s\in\Gamma}$ in X. If $n \in \omega$, $F \subset Y$ and $s_0 \in \Gamma$, then there exist $t \in \Gamma$ and $D \in [F]^{\leq\omega}$ such that $s_0 \leq t$ and $r_t(\overline{F}) \subset \overline{D}$.

Proof. Let $\mathcal{N}$ be a monotonically stable operator in X. For each $s \in \Gamma$ fix a countable dense subset A_s of $r_s^*(C_p(r_s(X)))$. Let M be a suitable elementary submodel. Set

$$t = \sup(\Gamma \cap M),\ D = F \cap M \text{ and } A = C_p(X) \cap M.$$

Clearly $s_0 \leq t$. It is enough to show that $r_t^n(F) \subset \overline{D}$.

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- Assume that $r_t(x) \notin \overline{D}$ for some $x \in F$ and choose $U \in \tau(x, X)$ such that $r_t(x) \in U \subset X \setminus D$.

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- From $r_t = \lim_{s \in \Gamma \cap \mathcal{M}} r_s$ we deduce

$$r_t^*(C_p(r_t(X))) \subset \overline{\bigcup_{s \in \Gamma \cap \mathcal{M}} r_s^*(C_p(r_s(X)))} = \overline{\bigcup_{s \in \Gamma \cap \mathcal{M}} A_s} \subset \overline{A}.$$

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- $\mathcal{N}(A)$ is a network for $\overline{A}$, i.e. for $r_t^*(C_p(r_t(X)))$ and r_t.

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- $\mathcal{N}(A)$ is a network for $\overline{A}$, i.e. for $r_t^*(C_p(r_t(X)))$ and r_t.
- Set $N \in \mathcal{N}(A)^n$ such that $x \in N \subset (r_t^n)^{-1}(U)$.

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- $\mathcal{N}(A)$ is a network for $\overline{A}$, i.e. for $r_t^*(C_p(r_t(X)))$ and r_t.
- Set $N \in \mathcal{N}(A)^n$ such that $x \in N \subset (r_t^n)^{-1}(U)$.
- $\mathcal{N}$ is ω-monotone, $N \in \mathcal{N}(E)^n$ for some $E \in [C_p(X) \cap \mathcal{M}]^{<\omega}$ and so $N \in M$.

Proof (continuation)

- Assume that $r_t(x) \notin \overline{D}$ for some $x \in F$ and choose $U \in \tau(x, X)$ such that $r_t(x) \in U \subset X \setminus D$.
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- Pick $y \in (F \cap N) \cap \mathcal{M}$ and $s \in \mathcal{M}$ such that $r_s^n(y) = y$.
- We then have that $y = r_t(y) \in U \cap D$, a contradiction.

$\square$

Some consequences

Corollary

Let X be monotonically ω-stable and let Y be induced by a weak r-skeleton $\{r_s\}_{s\in\Gamma}$ in X. Then:

1. $t(Y) \leq \omega$.
2. $x = \lim_{s\in\Gamma} r_s(x)$ *for each* $x \in \overline{Y}$.

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Proof.

1. Set $A \subset Y$ and $x \in \overline{A}$. Choose $s_0 \in \Gamma$ so that $r_{s_0}(x) = x$. Find $D \in [A]^{\leq\omega}$ and $s \in \Gamma$ such that $s_0 \leq s$ and $r_s(\overline{A}) \subset \overline{D}$. This implies that $x = r_s(x) \in r_s(\overline{A}) \subset \overline{D}$.

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2. Fix $x \in \overline{Y}$ and a neighborhood U of X. Choose an open set V satisfying $x \in V \subset \overline{V} \subset U$. Set $F_1 = V \cap Y$ and $F_2 = (X \setminus U) \cap Y$. Find $s \in \Gamma$ such that $r_s(\overline{F}_i)) \subset \overline{F}_i$ for $i = 1, 2$. Choose $t \in \Gamma$ such that $s \le t$. If $r_t(x) \notin U$, then $r_t(x) \in F_2$ and so $r_s(x) = r_s(r_t(x)) \in \overline{F}_2$, which is not possible. □

We say that an r-skeleton $\{r_s\}_{s\in\Gamma}$ in X is *full* if $X = \bigcup_{s\in\Gamma} r_s(X)$, and *commutative* if $r_s \circ r_t = r_t \circ r_s$ whenever $s, t \in \Gamma$.

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Lemma

Let Y be induced by an r-skeleton $\{r_s\}_{s\in\Gamma}$ in a compact X. Then there is a family of retractions $\{r_A\}_{A\in\mathcal{P}(Y)}$ in X such that for every $A \in \mathcal{P}(Y)$ we have:

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If in adddition the r-skeleton $\{r_s\}_{s\in\Gamma}$ is commutative, then we also can get $r_B \circ r_A = r_A \circ r_B$ for every $A, B \in \mathcal{P}(Y)$.

Proof.
For each $s \in \Gamma$ fix a countable dense subset D_s of $r_s(X)$. Let $A \to \mathcal{M}(A)$ be an ω-monotone assignmet of suitable elementary submodels. For each $A \in [Y]^{\leq\omega}$ set

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Now, choose an arbitrary $A \subset Y$. Let $\Gamma_A = [A]^{\leq \omega}$ and $D_A = \bigcup_{B \in \Gamma_A} D_B$. It happens that $\{r_B \restriction_{\overline{D}_A}\}_{B \in \Gamma_A}$ is an r-skeleton on $\overline{D}_A$ and induces the set $\bigcup_{B \in \Gamma_A} r_B(X) = \bigcup_{B \in \Gamma_A} \overline{D}_B = \overline{D}_A \cap Y$.

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This finish the construction. $\square$

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$$\phi(x)(\alpha) = \begin{cases} \phi_{\alpha+1}(r_{\alpha+1}(x)) - \phi_{\alpha+1}(r_\alpha(x)) & \text{if } \alpha > 0; \\ \phi_0(r_0(x)) & \text{if } \alpha = 0. \end{cases}$$

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Then we can verify that ϕ is an embedding and $Y = \phi^{-1}(\Sigma I^T)$.

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A countably compact space admits a full commutative r-skeleton iff X can be embedded in ΣI^T for some set T.

Proof.

If X admits a full commutative r-skeleton, then it is easy to see that X is induced by a commutative r-skleton in βX. So X is a Σ-subset of βX and hence X can be embedded in ΣI^T for some set T. □

Strong r-skeletons

An r-skeleton $\{r_s\}_{s\in\Gamma}$ in a space X is said to be *strong* if whenever $s_0 \in \Gamma$ and F is closed in X^n, for some $n \in \mathbb{N}$, there exists $s \in \Gamma$ such that $s_0 \leq s$ and $r_s^n(F) \subset F$.

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Theorem

A space X has a strong r-skeleton if and only if $C_p(X)$ has a strong r-skeleton.

Strong r-skeletons

A *neighborhood assignment* for a space (X, τ) is a function $\phi : X \to \tau$ with $x \in \phi(x)$ for every $x \in X$. A *kernel* for ϕ is a subset $D \subset X$ such that $\phi(D) := \bigcup_{x \in D} \phi(x) = X$. X is said to be a *D-space* if every neighborhood assignment ϕ for X has a closed discrete kernel.

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Given a space X let $FS(X) = \bigcup_{n \in \mathbb{N}} X^n$ be the set of all finite sequences in X. For a map $O : FS(X) \to \tau(X)$, an element $x \in X^\omega$ is called an *O-sequence* if $x(n) \in O(x \restriction_n)$ for each $n \in \mathbb{N}$. Besides, we say that a set $H \subset X$ is a *W-set* in X if there is a map $O : FS(X) \to \tau(H, X)$ such that every O-sequence converges to H.

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Theorem

Let X be a countably compact with a full r-skeleton $\{r_s\}_{s \in \Gamma}$. If H is nonempty and closed in X, then H is a W-set in X.

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Thank You!