Some open problems in Banach Space Theory II

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Outline

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Efimov–Stechkin'1958–1962: approximately compact (in Hilbert and $L_p, \; p > 1$).

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[V. Klee'1961] $C \subset \ell_2$ w-closed Chebyshev, then C convex (true for X uniformly convex and uniformly smooth).

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We gave (with P. and V. Zizler) an alternative, much easier, proof in 2011.

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Theorem (Vlasov'1970)

X such that X^* rotund. C Chebyshev, p_C continuous. Then C convex.

Chebyshev in noncomplete spaces

Theorem (Fletcher–Moors'2015)

$\exists$ X inner product noncomplete space, $C \subset X$ Chebyshev, nonconvex.

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[Fonf, Lindenstrauss] $\exists$ reflexive X tiled by shifts of a single closed convex S with nonempty interior?

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Problem

[Troyanski] X (uncountable) unconditional basis and Gâteaux norm. Has X^* dual rotund renorming?

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Theorem (Pełczyński'1976, Plichko'1977)

X separable, $\varepsilon > 0$. Then $\exists$ $(1 + \varepsilon)$-bounded (countable) M-basis, i.e., $\|x_n\|.\|x_n^*\| < 1 + \varepsilon$ for all $n \in \mathbb{N}$.

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Can the constant be diminished to $2+\varepsilon$, for all $\varepsilon > 0$?

Auerbach bases

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[Pełczyński] X separable. Does X has an Auerbach basis? Does $C[0,1]$ has an Auerbach basis?

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Theorem (Day)

Every infinite-dimensional Banach has an infinite-dimensional subspace with Auerbach basis.

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2. If $x^{**} \in X^{**} \setminus X$ then $\ker x^{**} \subset X^*$ is norming.
3. If $\{e_n; e_n^*\}$ is a Schauder basis, then $\overline{\mathrm{span}}\{e_n^*\}$ is norming.

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Example [Bonet–Cascales (answering Kunze–Arendt)]:
$X := \ell_1[0, 1]$, $Y := C[0, 1]$. $\mu(X, Y)$ non-complete.

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[Davis–Lindenstrauss'72] If X^{**}/X infinite-dimensional, then $\exists$ w^*-dense non-norming subspace.

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Problem [Godefroy–Kalton]

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Every non-reflexive space has a proper closed norming subspace.

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Theorem (Troyanski)

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Problem [Fabian]

Characterize K compact st $C(K)$ hereditary WCG.

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Theorem (Jiménez–Moreno'97)

Under CH, $\exists$ Asplund X without Mazur Intersection Property

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X SSD $\Rightarrow$ Asplund.

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Under CH, $\exists$ Asplund X without Mazur Intersection Property (Godefroy: with no SSD norm).

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Problem [Godefroy]

In ZFC, $\exists$ Asplund with no SSD norm?

SSD norms

Theorem (Godefroy–M–Zizler'94)

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SSD norms

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Problem

X nonseparable non-Asplund. $\exists \, \|\cdot\|$ nowhere SSD?

Norm attaining operators

Theorem (Lindenstrauss'1963)

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Problem

[Ostrovski] Does there exists X infinite-dimensional separable such that every $T : X \to X$ bounded attains its norm?

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Problem

What if $n > 2$?

Support sets

$C \subset X$ convex, closed, is a support set whenever $\forall x_0 \in C$, x_0 is proper support point, i.e., $\exists f \in X^*$
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Problem

[Rolewicz] X nonseparable Banach. Do there exist support sets?

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Theorem (Kutzarova, Lazar, M., Borwein, Vanderwerff)

X has an uncountable biorthogonal system, then X has support sets.

Support sets

Theorem (Todorcevic'2006)

Under (MM), X nonseparable has support set.

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[Todorcevic] X with density $> \aleph_1$ has a support set?

Polyhedral spaces

$x \in B_X$ is preserved extreme if it is extreme of $B_{X^{**}}$.

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X separable polyhedral, then $\exists$ C^∞-smooth (R) norm $|||\cdot|||$ all $x \in S_X$ unpreserved.

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Theorem (Guirao–M–Zizler'2014)

X WCG, $c_0 \subset X$. Then $\exists$ $|||\cdot|||$ all $x \in S_X$ extreme all unpreserved, one-direction-uniformly.

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Problem

X separable with a bump that depends locally on finitely many coordinates. Is X polyhedral?

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