Extension operators and twisted sums

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Winter School in Abstract Analysis
Section Analysis
January 14–21, 2017

Extension operators

For a compact space K, $C(K)$ is the Banach space of real-valued continuous functions on K (with the sup norm).

For a closed $L \subset K$, $C(K|L) = \{f \in C(K) : f|L \equiv 0\}$,
a bounded linear operator $E : C(L) \to C(K)$ is called an extension operator if, for every $f \in C(L)$, Ef is an extension of f.

Such E exists iff the restriction operator $R : C(K) \to C(L)$, defined by $Rf = f|L$ has a right inverse iff $C(K|L)$ is complemented in $C(K)$.
Then $C(K)$ is isomorphic to $C(L) \oplus C(K|L)$

Twisted sums

A twisted sum of Banach spaces Y and Z is a short exact sequence

$$0 \to Y \to X \to Z \to 0$$

where X is a Banach space and the maps are bounded linear operators.

Such twisted sum is called trivial if the exact sequence splits, i.e., if the map $Y \to X$ admits a left inverse (equivalently, if the map $X \to Z$ admits a right inverse).

The twisted sum is trivial iff the range of the map $Y \to X$ is complemented in X; in this case, $X \cong Y \oplus Z$.

For a closed subset L of a compact space K, the twisted sum

$$0 \to C(K|L) \to C(K) \to C(L) \to 0$$

is trivial iff there exists an extension operator $E : C(L) \to C(K)$

Example

$$0 \to c_0 \to \ell_\infty \to \ell_\infty / c_0 \to 0$$

Phillips: c_0 is not complemented in l_∞

$\beta\omega$ is the Čech-Stone compactification of the space of natural numbers ω

$\omega^* = \beta\omega \setminus \omega$

In the sequence

$$0 \to c_0 \to \ell_\infty \to \ell_\infty / c_0 \to 0$$

we can replace all spaces by isometric function spaces obtaining

$$0 \to C(\beta\omega|\omega^*) \to C(\beta\omega) \to C(\omega^*) \to 0$$

This twisted sum is nontrivial because there is no extension operator $E : C(\omega^*) \to C(\beta\omega)$.

A topological space X satisfies the countable chain condition (ccc) if every family of nonempty pairwise disjoint open subsets of X is countable.

Fact

Let L be a closed subset of a separable compact space K, such that L does not satisfy the countable chain condition (ccc). Then $C(L)$ is not isomorphic to a subspace of $C(K)$, hence there is no extension operator $E : C(L) \to C(K)$.

Problem (Cabello, Castillo, Kalton, Yost)

Let K be a nonmetrizable compact space. Does there exist a nontrivial twisted sum of c_0 and $C(K)$?

Remark

If K is a metrizable compact space, then by Sobczyk's theorem, every twisted sum of c_0 and $C(K)$ is trivial.

Known results on twisted sums of c_0 and $C(K)$

(Castillo, Correa-Tausk) For a non-metrizable K, there exists a nontrivial twisted sum of c_0 and $C(K)$ in any of the following cases:

- K is a Gul'ko compact space, in particular if K is an Eberlein compact space;
- **(MA)** K is a Corson compact space;
- K is a Valdivia compact space which does not satisfy ccc;
- the weight $w(K)$ of K is equal to ω_1 and $((C(K))^*, w^*)$ is not separable;
- K has an extension property and does not have ccc;
- $C(K)$ contains an isomorphic copy of ℓ_∞;
- **(CH)** K is a scattered space of finite height;
- K contains a copy of $[0, \omega] \times [0, \mathfrak{c}]$, in particular if K contains a copy of $2^{\mathfrak{c}}$;
- K is an ordinal space, i.e., $K = [0, \kappa]$ for some cardinal κ.

Definitions

A compact space K is an Eberlein compact space if it is homeomorphic to a weakly compact subset of a Banach space.

K is a Gul'ko compact space if $C(K)$ is weakly countably determined, i.e., for some separable metrizable space X, there is an upper semicontinuous map φ from X into the family of compact subsets of $(C(K), w)$ such that the union of all values of φ covers $C(K)$.

For a set Γ, $\Sigma(\Gamma)$ is the Σ-product of real lines indexed by Γ, i.e., the subspace of $\mathbb{R}^\Gamma$ constisting of functions with countable supports.

A compact space K is a Corson compact space if, for some set Γ, there exists an embedding $i : K \to \Sigma(\Gamma)$.

K is a Valdivia compact space if, for some set Γ, there exists an embedding $i : K \to \mathbb{R}^\Gamma$ such that the intersection $i(K) \cap \Sigma(\Gamma)$ is dense in $i(K)$.

$$\text{metrizable} \Rightarrow \text{Eberlein} \Rightarrow \text{Gul'ko} \Rightarrow \text{Corson} \Rightarrow \text{Valdivia}$$

Definitions

A compact space K has the extension property if, for every closed subset L of K there exists an extension operator $E : C(L) \to C(K)$.

A space X is scattered if no nonempty subset $A \subseteq X$ is dense-in-itself.

For an ordinal α, $X^{(\alpha)}$ is the αth Cantor-Bendixson derivative of the space X. For a scattered space X, the scattered height

$$ht(X) = \min\{\alpha : X^{(\alpha)} = \emptyset\}\,.$$

Remark

Let L be a closed subset of a compact space K, such that there is an extension operator $E : C(L) \to C(K)$. If there is a nontrivial twisted sum of c_0 and $C(L)$, then also c_0 and $C(K)$ have a nontrivial twisted sum.

Theorem (Parovičenko)

For every compact space K of weight ω_1 there is a compactification $\gamma\omega$ of ω with the reminder $\gamma\omega \setminus \omega$ homeomorphic to K.

Corollary

For every compact space K with $w(K) = \omega_1$ and without ccc, there is a nontrivial twisted sum of c_0 and $C(K)$.

Fact

Every Valdivia compact space K which does not satisfy ccc has a retract L of of weight ω_1 and without ccc.

Theorem (Plebanek and M.)

(MA + ¬CH) *The spaces* c_0 *and* $C(2^{\omega_1})$ *do not have a nontrivial twisted sum.*

Corollary

The existence of a nontrivial twisted sum of c_0 *and* $C(2^{\omega_1})$ *is independent of* **ZFC**.