# Extension operators and twisted sums II

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# Extension operators

For a compact space $K$, $C(K)$ is the Banach space of real-valued continuous functions on $K$ (with the sup norm).

For a closed $L \subset K$, $C(K|L) = \{f \in C(K) : f|L \equiv 0\}$,
a bounded linear operator $E : C(L) \to C(K)$ is called an extension operator if, for every $f \in C(L)$, $Ef$ is an extension of $f$.

Such $E$ exists iff the restriction operator $R : C(K) \to C(L)$, defined by $Rf = f|L$ has a right inverse iff $C(K|L)$ is complemented in $C(K)$.
Then $C(K)$ is isomorphic to $C(L) \oplus C(K|L)$

# Twisted sums

A twisted sum of Banach spaces $Y$ and $Z$ is a short exact sequence

$$0 \to Y \to X \to Z \to 0$$

where $X$ is a Banach space and the maps are bounded linear operators.

Such twisted sum is called trivial if the exact sequence splits, i.e., if the map $Y \to X$ admits a left inverse (equivalently, if the map $X \to Z$ admits a right inverse).

The twisted sum is trivial iff the range of the map $Y \to X$ is complemented in $X$; in this case, $X \cong Y \oplus Z$.

For a closed subset $L$ of a compact space $K$, the twisted sum

$$0 \to C(K|L) \to C(K) \to C(L) \to 0$$

is trivial iff there exists an extension operator $E : C(L) \to C(K)$

**Problem (Cabello, Castillo, Kalton, Yost)**

Let $K$ be a nonmetrizable compact space. Does there exist a nontrivial twisted sum of $c_0$ and $C(K)$?

**Theorem (Plebanek and M.)**

**(MA + $\neg$CH)** *The spaces $c_0$ and $C(2^{\omega_1})$ do not have a nontrivial twisted sum.*

**Theorem (Correa-Tausk)**

*If a compact space $K$ contains a copy of $2^{\mathfrak{c}}$, then there exists a nontrivial twisted sum of $c_0$ and $C(K)$*

**Corollary**

*The existence of a nontrivial twisted sum of $c_0$ and $C(2^{\omega_1})$ is independent of* **ZFC**.

If $L$ is a compact space then a compact superspace $L' \supseteq L$ will be called a countable discrete extension of $L$ if $L' \setminus L$ is infinite countable and discrete.

We shall write $L' \in \mathrm{CDE(L)}$ to say that $L'$ is such an extension of $L$.

Typically, when $L' \setminus L$ is dense in $L'$, $L'$ is a compactification of $\omega$ such that its remainder is homeomorphic to $L$.

If $L' \in \mathrm{CDE(L)}$ then we usually identify $L' \setminus L$ with the set of natural numbers $\omega$.

### Remark

*If $L' \in \mathrm{CDE(L)}$ and there is no extension operator $E : C(L) \to C(L')$ then $C(L')$ is a nontrivial twisted sum of $c_0$ and $C(L)$.*

$$0 \to C(L'|L) \to C(L') \to C(L) \to 0$$

For a compact space $K$, by $M(K)$ denote the space of all Radon measures on $K$, which can be identified with the dual space $C(K)^*$.

$M_1(K)$ stands for the unit ball of $M(K)$, equipped with the *weak*$^*$ topology inherited from $C(K)^*$.

$P(K)$ is the subspace of $M_1(K)$ consisting of probability measures.

A compact space $K$ has the property $(\#)$ if for every $L' \in \mathrm{CDE}(\mathrm{M}_1(\mathrm{K}))$ there is a bounded operator $E : C(K) \to C(L')$ such that $Eg(\nu) = \nu(g)$ for every $g \in C(K)$ and $\nu \in M_1(K)$.

### Theorem (Plebanek and M.)

*If a compact space $K$ has the property $(\#)$ then every twisted sum of $c_0$ and $C(K)$ is trivial.*

### Lemma

*Given $K$ and $L' \in \mathrm{CDE}(\mathrm{M}_1(\mathrm{K}))$, the following are equivalent*

(i) *there is $E : C(K) \to C(L')$ such that $Eg(\nu) = \nu(g)$ for every $g \in C(K)$ and $\nu \in M_1(K)$;*

(ii) *there is a bounded sequence $(\nu_n)_n$ in $M(K)$ such that for every $g \in C(K)$, if $\widehat{g} \in C(L')$ is any function extending $g$ on $M_1(K)$ then $\lim_n(\nu_n(g) - \widehat{g}(n)) = 0$.*

# Proof of ($K$ has $(\#)$ $\Rightarrow$ no notrivial twisted sum)

Take any short exact sequence $0 \to c_0 \xrightarrow{i} X \xrightarrow{T} C(K) \to 0$
Put $Z = i(c_0)$,
$e_n \in c_0$, $e_n^* \in (c_0)^*$, $x_n = i(e_n)$
take $x_n^* \in X^*, n \in \omega$, $i^* x_n^* = e_n^*$ and $\|x_n^*\| \leq r_0$
the set $\{x_n^* : n \in \omega\}$ is *weak*$^*$ is discrete
Let

$$L = T^*[r \cdot M_1(K)] \subset X^*,$$

where $r > 0$ is such that $L$ contains $\{x^* \in Z^\perp : ||x^*|| \leq r_0\}$.
Put $L' = L \cup \{x_n^* : n \in \omega\}$ and equip $L'$ with the *weak*$^*$ topology
$L' \in \mathrm{CDE(L)}$
Consider a mapping

$$h : L'' = M_1(K) \cup \omega \to L' = T^*[M_r(K)] \cup \{x_n^* : n \in \omega\},$$

defined by $h(\nu) = T^*(r\nu)$ for $\nu \in M_1(K)$ and $h(n) = x_n^*$ for $n \in \omega$. $h$ is a bijection and we topologize $L''$ so that $h$ becomes a homeomorphism

Since $K$ has property $(\#)$, by Lemma there is a bounded sequence $(\nu_n)_n$ in $M(K)$ satisfying condition (ii)

Let $z_n^* = T^*(r\nu_n)$ for $n \in \omega$.

$(z_n^*)_n$ is a bounded sequence in $X^*$ and the following holds

$z_n^* - x_n^* \to 0$ in the *weak*$^*$ topology of $X^*$

Define

$$P : X \to X, \quad Px = \sum_n (x_n^*(x) - z_n^*(x)) \cdot x_n.$$

Note that $Px_k = x_k$ since $x_n^*(x_k) = 1$ if $n = k$ and is 0 otherwise; moreover, $z_n^*(x_k) = 0$ for every $n$ and $k$.

$P$ is a projection onto $Z$

# Spaces of measures and absolute retracts

## Remark

*For every* $L' \in \mathrm{CDE}(\mathrm{P}(2^{\omega_1}))$ *there is an extension operator* $E : C(P(2^{\omega_1})) \to C(L')$*, since* $P(2^{\omega_1})$ *is an absolute retract.*

A compact space $K$ is an absolute retract f $K$ is a retract of any compact space $L$ containing $K$ (equivalently, of any completely regular space $X$ containing $K$).

$K$ is a Dugundji space if for every compact space $L$ containing $K$ there exists a regular extension operator $E : C(K) \to C(L)$, i.e. an extension operator of the norm 1 preserving constant functions.

A convex compact space $K$ is a Dugundji space if and only if it is an absolute retract.

## Theorem (Ditor and Haydon)

$P(K)$ *is an absolute retract if and only if* $K$ *is a Dugundji space of weight at most* $\omega_1$*.*

## Theorem (Plebanek and M.)

*If $K$ is a nonmetrizable compact space, then the space $M_1(K)$ is not a Dugundji space, in particular, it is not an absolute retract.*

For a surjection $\varphi : L \to K$ between compact spaces $K, L$, $\varphi^* : M_1(L) \to M_1(K)$ denotes the canonical surjection associated with $\varphi$, i.e., the surjection given by the operator conjugate to the isometrical embedding of $C(K)$ into $C(L)$ induced by $\varphi$.

## Proposition

*Let $\varphi : L \to K$ be a surjection of a compact space L onto an infinite space $K$. If $\varphi$ is not injective, then the map $\varphi^* : M_1(L) \to M_1(K)$ is not open.*