Extension operators and twisted sums III

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Extension operators

For a compact space K, $C(K)$ is the Banach space of real-valued continuous functions on K (with the sup norm).

For a closed $L \subset K$, $C(K|L) = \{f \in C(K) : f|L \equiv 0\}$,
a bounded linear operator $E : C(L) \to C(K)$ is called an extension operator if, for every $f \in C(L)$, Ef is an extension of f.

Fact

For a closed subset L of a compact space K, there exists an extension operator $E : C(L) \to C(K)$, if and only if, for some $r > 0$, there exists a continuous map $\varphi : K \to rM_1(L)$ such that $\varphi(x) = \delta_x$ for every $x \in L$.

Twisted sums

A twisted sum of Banach spaces Y and Z is a short exact sequence

$$0 \to Y \to X \to Z \to 0$$

where X is a Banach space and the maps are bounded linear operators.

Such twisted sum is called trivial if the exact sequence splits, i.e., if the map $Y \to X$ admits a left inverse (equivalently, if the map $X \to Z$ admits a right inverse).

The twisted sum is trivial iff the range of the map $Y \to X$ is complemented in X; in this case, $X \cong Y \oplus Z$.

For a closed subset L of a compact space K, the twisted sum

$$0 \to C(K|L) \to C(K) \to C(L) \to 0$$

is trivial iff there exists an extension operator $E : C(L) \to C(K)$

Problem (Cabello, Castillo, Kalton, Yost)

Let K be a nonmetrizable compact space. Does there exist a nontrivial twisted sum of c_0 and $C(K)$?

A compactification $\gamma\omega$ of the space of natural numbers ω is tame if there is an extension operator $E : C(\gamma\omega \setminus \omega) \to C(\gamma\omega)$, i.e., the twisted sum

$$0 \to C(\gamma\omega|\gamma\omega \setminus \omega) \to C(\gamma\omega) \to C(\gamma\omega \setminus \omega) \to 0$$

is trivial.

If $\gamma\omega$ is non-tame, then $C(\gamma\omega)$ is a nontrivial twisted sum of c_0 and $C(\gamma\omega \setminus \omega)$.

Problem (Castillo, Koszmider, Kubiś)

Characterize tame compactifications $\gamma\omega$.

A compact space K supports a measure if there is $\mu \in P(K)$ such that $\mu(U) > 0$ for each nonempty open subset U of K.

Theorem (Kubiś)

If a compactification $\gamma\omega$ is tame then the remainder $\gamma\omega \setminus \omega$ supports a measure.

Corollary

For every compact space K of weight ω_1 which does not support a measure, there is a non-tame compactification $\gamma\omega$ of ω with the remainder $\gamma\omega \setminus \omega$ homeomorphic to K. Hence there is a nontrivial twisted sum of c_0 and $C(K)$.

Twisted sums of $C(K)$ for separable K

Example (Drygier and Plebanek)

There exists a non-tame compactification $\gamma\omega$ of ω with separable remainder $\gamma\omega \setminus \omega$.

Theorem (Plebanek and M.)

Let K be a separable linearly ordered compact space of weight κ such that $2^\kappa > 2^\omega$. Then there is a nontrivial twisted sum of c_0 and $C(K)$.

Corollary

If K is a separable linearly ordered compact space of weight 2^ω, then there is a nontrivial twisted sum of c_0 and $C(K)$.

Corollary

(CH) *If K is a nonmetrizable linearly ordered compact space, then there is a nontrivial twisted sum of c_0 and $C(K)$.*

Let A be a subset of a closed subset K of the unit interval $I = [0, 1]$. Put

$$K_A = (K \times \{0\}) \cup (A \times \{1\})$$

and equip this set with the order topology given by the lexicographical order (i.e., $(s, i) \prec (t, j)$ if either $s < t$, or $s = t$ and $i < j$).

Theorem (Ostaszewski)

The space L is a separable compact linearly ordered space iff L is homeomorphic to K_A for some closed set $K \subseteq I$ and a subset $A \subseteq K$.

Lemma

Let L be a separable linearly ordered compact space of uncountable weight κ. Then L contains a topological copy of the space I_B, where B is a dense subset of $(0, 1)$ of the cardinality κ.

Theorem

Let B be a dense subset of $(0, 1)$ of the cardinality κ such that $2^\kappa > 2^\omega$. Then there is a non-tame compactification $\gamma\omega$ which remainder is homeomorphic to I_B.

For a compact space K by $\mathrm{Auth}(K)$ we denote the group of autohomeomorphisms of K.

Theorem (Plebanek and M.)

Let $\delta\omega$ be a compactification of ω such that

(a) $|M(\delta\omega)| = 2^\omega$,

(b) $|\mathrm{Auth}(\delta\omega \setminus \omega)| > 2^\omega$.

Then there exists a non-tame compactification $\gamma\omega$ which remainder is homeomorphic to $\delta\omega$. Hence there is a nontrivial twisted sum of c_0 and $C(\delta\omega)$.

A compact space K is called dyadic if it is a continuous image of some Cantor cube 2^κ.

Theorem (Correa and Tausk)

If a compact space K contains a copy of $2^\mathfrak{c}$, then there exists a nontrivial twisted sum of c_0 and $C(K)$

Corollary

(CH) *For each nonmetrizable dyadic space K, c_0 and $C(K)$ have a nontrivial twisted sum.*

Example

There is a dyadic compactum L of weight 2^ω and a non-tame compactification $\gamma\omega$ with remainder homeomorphic to L.

Remark

Each compactification $\gamma\omega$ with remainder homeomorphic to $2^\mathfrak{c}$ is tame.

Theorem (Castillo)

(CH) *If K is a nonmetrizable scattered compact space of finite height, then there exists a nontrivial twisted sum of c_0 and $C(K)$*

Conjecture

(MA + ¬CH) If K is a separable scattered compact space whose set of accumulation points is the one-point compactification of an discrete space of cardinality ω_1 (K is scattered of weight ω_1 and height 3), then there is no nontrivial twisted sum of c_0 and $C(K)$.

Problem

Does there exist in **ZFC** a compact space K such that there is no nontrivial twisted sum of c_0 and $C(K)$?

Question

Does there exist in **ZFC** a separable compact space K of weight ω_1 such that there exists a nontrivial twisted sum of c_0 and $C(K)$?

Question

Does there exist in **ZFC** a dyadic compact space K of weight ω_1 with a nontrivial twisted sum of c_0 and $C(K)$?

Question

Does there exist in **ZFC** a separable linearly ordered compact space K of weight ω_1 with a nontrivial twisted sum of c_0 and $C(K)$?

Theorem (Correa-Tausk)

(MA) *For each nonmetrizable Corson compact space K there exists a nontrivial twisted sum of c_0 and $C(K)$.*

Question

Can we prove the above theorem in **ZFC**?

Theorem (Castillo)

For each Valdivia compact space K without ccc there exists a nontrivial twisted sum of c_0 and $C(K)$.

Question

Let K be a Valdivia compact space which does not support a measure. Does there exist a nontrivial twisted sum of c_0 and $C(K)$?

Question

Let K be a compact space of weight $> 2^\omega$ which does not support a measure. Does there exist a continuous image of K of weight 2^ω which does not support a measure?

Theorem (Corson-Lindenstrauss)

Let B_H be the unit ball of a nonseparable Hilbert space H, equipped with the weak topology. Then, for any $0 < \lambda < \mu$, the ball λB_H is not a retract of the ball μB_H.

Theorem (Aviles and M.)

Let H be a nonseparable Hilbert space and B_H be the unit ball of H, equipped with the weak topology. Then, for any $0 < \lambda < \mu$, there is no extension operator $T : C(\lambda B_H) \to C(\mu B_H)$.