Curvature measures of singular sets

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45th Winter School in Abstract Analysis, Svratka 2017

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Curvatures of sets with smooth boundaries

$X \subset \mathbb{R}^d$ - compact C^2-domain (locally epigraph of a C^2-function)
$\nu : \partial X \to S^{d-1}$ - Gauss map
($\nu(x)$ - unit outer normal vector to X at x)
$D\nu(x) : \mathrm{Tan}\,(\partial X, x) \cong \nu(x)^\perp \to \mathrm{Tan}\,(S^{d-1}, \nu(x)) \cong \nu(x)^\perp$
- Weingarten mapping, selfadjoint

$\kappa_1(x), \dots, \kappa_{d-1}(x)$ - eigenvalues - principal curvatures
$b_1(x), \dots, b_{d-1}(x)$ - eigenvectors - principal directions
$S_k(x) = \sum_{|I|=d-1-k} \prod_{i \in I} \kappa_i(x)$ - symmetric functions of principal curvatures
$\mathbf{C}_k(X) = \frac{1}{(d-k)\omega_{d-k}} \int_{\partial X} S_k(x)\, \mathcal{H}^{d-1}(dx)$ - total curvature integrals
$\omega_j := \pi^{j/2}/\Gamma(1 + \frac{j}{2})$

- $\mathbf{C}_{d-1}(X) = \frac{1}{2}\mathcal{H}^{d-1}(\partial X)$
- $\mathbf{C}_0(X) = \chi(X)$ (Gauss-Bonnet formula)

Curvatures of convex bodies

$K \subset \mathbb{R}^d$ convex, compact
$K_r := K + rB$ - parallel r-neighbourhood ($r \geq 0$, $B = B(0,1)$)
Steiner formula:

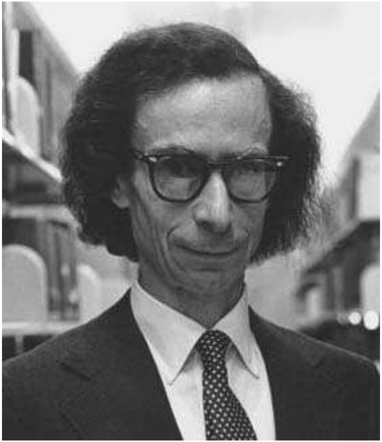
$$\mathcal{L}^d(K_r) = \sum_{k=0}^{d} \omega_k r^k V_{d-k}(K)$$

$V_k(\cdot)$ - intrinsic volumes - can be interpreted as "quermassintegrals":

$$V_k(K) = \binom{d}{k} \frac{\omega_d}{\omega_k \omega_{d-k}} \int_{G(d,k)} \mathcal{L}^k(\pi_L K)\, \nu_k^d(dL)$$

- K C^2-smooth $\implies$ $V_k(K) = \mathbf{C}_k(K)$

Federer's sets with positive reach



Herbert Federer (1920-2010)

Definition

reach $X := \sup\{r \geq 0 :$ each y with $\mathrm{dist}(y, X) \leq r$ has its unique footpoint $\Pi_X(y) \in X\}$.

- curvature measures can be defined by means of a local Steiner formula
- curvature measures satisfy the Gauss-Bonnet and Principal Kinematic Formulas
- extension of both closed convex sets and sets with C^2 smooth boundary

H. Federer, Curvature measures, 1959 (from the Introduction)

"The results of the theory of area have greatly contributed to the understanding of first order tangential properties of point sets, and one can hope for similar success in dealing with second order differential geometric concepts such as curvature..."

"Neither the definition of the curvature measures nor the statement of any important theorem about them may contain explicit assumptions of differentiability, because arbitrary convex sets are to be admissible objects."

"Of course, in order to be worth while, such a theory must contain natural generalizations of the Principal kinematic formula and of the Gauss-Bonnet Theorem."

"This problem presents a timely challenge to a worker in modern real function theory, which was originally created in large part for the study of geometric questions."

Unit normal bundle

$0 < r < \operatorname{reach} X$

$d_X(\cdot) := \operatorname{dist}(\cdot, X)$ is differentiable on $X_r \setminus X$, with gradient

$\nabla d_X(z) = \frac{z - \Pi_X(z)}{d_X(z)}$

$f : X_r \setminus X \to \partial X \times S^{d-1}$

$$z \mapsto \left(\Pi_X(z), \frac{z - \Pi_X(z)}{d_X(z)} \right)$$

- locally Lipschitz mapping

∂X_s is a $C^{1,1}$ surface $(0 < s < \operatorname{reach} X)$ (Implicit function theorem)

$\mathrm{nor}\, X := \operatorname{im} f$ - unit normal bundle of X

$$f^{(s)} := f \restriction_{\partial X_s} : \partial X_s \to \mathrm{nor}\, X$$

bi-Lipschitz mapping

Unit normal bundle - cont.

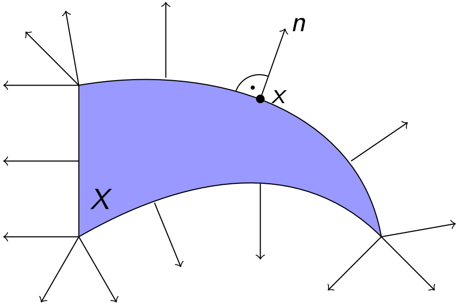


Figure: A set $X \subset \mathbb{R}^2$ with positive reach and its unit normal bundle.

Principal curvatures

∂X_s is $C^{1,1}$; by Rademacher's theorem it is "C^2 almost everywhere"

$z \in \partial X_s$, $f(z) =: (x, n) \in \operatorname{nor} X$

$\kappa_i^{(s)}(x, n)$, $b_i^{(s)}(x, n)$, $i = 1, \dots, d-1$ - principal curvatures and directions of ∂X_s at z

$$\begin{aligned} \kappa_i(x,n) &:= \frac{\kappa_i^{(s)}(x,n)}{1 - s\kappa_i^{(s)}(x,n)} = \lim_{s\to 0_+} \kappa_i^{(s)}(x,n) \in [-\operatorname{reach} X, \infty], \\ b_i(x,n) &:= b_i^{(s)}(x,n), \quad i = 1, \dots, d-1 \end{aligned}$$

- independent of s
- principal curvatures and directions of X at $(x, n) \in \operatorname{nor} X$

We will always assume that $\{b_1, \dots, b_{d-1}, n\}$ is positively oriented

Steiner formula

Jacobian of $f : X_r \setminus X \to \operatorname{nor} X$:

$$J_{d-1}f(z) = \prod_{i=1}^{d-1} \frac{\sqrt{1+\kappa_i(x,n)^2}}{1+s\kappa_i(x,n)}, \quad (x,n) = f(z), s = d_X(z)$$

$$(J_{d-1}f(z))^{-1} = \sum_{k=0}^{d-1} S_k(X;x,n)s^{d-1-k},$$

$$S_k(X;x,n) := \sum_{|I|=d-1-k} \frac{\prod_{i\in I}\kappa_i(x,n)}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i(x,n)^2}}$$

Co-area formula for f implies

$$\mathcal{L}(X_r \setminus X) = \sum_{k=0}^{d-1} \frac{r^{d-k}}{d-k} \int_{\operatorname{nor} X} S_k(X;x,n)\, \mathcal{H}^{d-1}(d(x,n))$$

Steiner formula - cont.

Theorem (Steiner formula for sets with positive reach)

$X \subset \mathbb{R}^d$ *compact*, $\operatorname{reach} X > 0 \implies$

$$\mathcal{L}^d(X_r \setminus X) = \sum_{k=0}^{d-1} \omega_{d-k} r^{d-k} \mathbf{C}_k(X), \quad 0 < r < \operatorname{reach} X,$$

with

$$\mathbf{C}_k(X) := \frac{1}{(d-k)\omega_{d-k}} \int_{\operatorname{nor} X} S_k(X; x, n)\, \mathcal{H}^{d-1}(d(x,n)).$$

(local version - curvature measures $C_k(X, \cdot)$)

Gauss-Bonnet formula

Theorem

$X \subset \mathbb{R}^d$ *compact*, reach $X > 0 \implies$

$$\mathbf{C}_0(X) = \chi(X).$$

Proof.

Let $0 < r <$ reach X.

- the mapping $H : (z, t) \mapsto (1 - t)z + z\Pi_X(z)$ defines a homotopy of X_r onto X. Hence, $\chi(X_r) = \chi(X)$.
- X_r is a "$C^{1,1}$-domain" for which the Gauss-Bonnet formula holds: $\mathbf{C}_0(X_r) = \chi(X_r)$.
- $\mathbf{C}_0(X) = \lim_{r \to 0_+} \mathbf{C}_0(X_r)$ (from the Steiner formula).

$\square$

Principal kinematic formulas

Set $\mathbf{C}_d(X) = \mathcal{L}^d(X)$. Let $\mathcal{G}_d$ be the group of all isometries in $\mathbb{R}^d$ with normalized invariant measure dg.

Theorem

Let $X, Y \subset \mathbb{R}^d$ be two compact sets with positive reach. Then for any $0 \leq k \leq d$,

$$\int_{\mathcal{G}_d} \mathbf{C}_k(X \cap gY)\, dg = \sum_{r+s=k+d} c_{d,r,s} \mathbf{C}_r(X)\mathbf{C}_s(Y).$$

Remark

reach $X > 0$, reach $Y > 0 \not\Longrightarrow$ reach$(X \cap Y) > 0$!! But reach$(X \cap gY) > 0$ for a.a. $g \in \mathcal{G}_d$.

Positive reach and semiconvexity

Definition

$f : U \subset \mathbb{R}^d \to \mathbb{R}$ is semiconvex if $f(x) = g(x) - c\|x\|^2$ for some g convex and $c \geq 0$.

$\operatorname{epi} f := \{(x,t) : t \geq f(x)\}$ (epigraph of f)

Theorem (Fu, 1985)

$f : \mathbb{R}^{d-1} \to \mathbb{R}$ locally Lipschitz. Then

$$\operatorname{reach}(\operatorname{epi} f) > 0 \iff f \text{ is semiconvex.}$$

Theorem (Federer, 1959; R&Zajíček, 2017)

If $X \subset \mathbb{R}^d$ is a Lipschitz manifold of dimension $1 \leq k \leq d-1$ and $\operatorname{reach} X > 0$ then X is a $C^{1,1}$ manifold.

Lytchak (2005) showed that it is enough to assume that X is a *topological manifold.*

Positive reach and semiconvexity - cont.

Theorem (R&Zajíček, 2017)

The boundary of a set with positive reach can be locally covered by a finite number of semiconvex graphs.

Definition

$c \in f(\mathbb{R}^d)$ is a *weakly regular value* of a locally Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ if $\liminf_{i\to\infty} |v_i| > 0$ whenever $v_i \in \partial f(x_i)$, $x_i \to x$ and $f(x_i) > 0 = f(x)$.

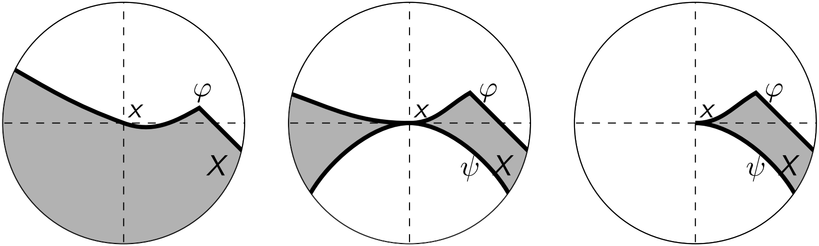
Theorem (Kleinjohann 1981, Bangert 1982)

$X \subset \mathbb{R}^d$ has locally positive reach iff X is a weakly regular sublevel set of a semiconvex function (i.e., $X = f^{-1}((-\infty, c])$ for $f : \mathbb{R}^d \to \mathbb{R}$ semiconvex, c weakly regular value of f).

Characterization in the plane:

Theorem (RZ 2017)

A closed set $X \subset \mathbb{R}^2$ has locally positive reach iff each $x \in X$ is either an isolated point, or an interior point, or one of the following three situations ocurs (φ semiconcave, ψ semiconvex):



In higher dimension - more complicated

Intrinsic geometry of sets with positive reach

Theorem (Lytchak 2005)

Let $X \subset \mathbb{R}^d$ be compact. Then $\operatorname{reach} X > 0$ *iff* $\exists r, L > 0$ *such that any two points $x, y \in X$ of distance less than r can be connected in X by a $C^{1,1}$-curve of length less than $L|y - x|$.*

True even in Riemannian manifolds.

Question

Are $\mathbf{C}_k(X)$ intrinsic characteristics (considering X as a length space)?

II. Normal cycles - curvature measures as currents

1. Motivation, history
2. Multilinear algebra, differential forms
3. Currents, rectifiable currents
4. Normal cycles for sets with positive reach
5. Properties of normal cycles

Why currents?

Curents - generalized (oriented) manifolds

M_k - oriented k-manifold

$\phi \mapsto \int_{M_k} \phi$ - linear form on k-forms - k-current

Plató's minimal problem: Given a simple closed curve γ in $\mathbb{R}^3$, find a surface with boundary γ of minimal sufrace area.

- Not always solvable by classical means
- Solvable in the setting of currents (uses compactness of integral currents of bounded mass)

Idea: Use currents to represent curvature measures and extend curvature measures via approximations.

M. Zähle: Integral and current representations... (1986)

"The first order (local) integral geometric behaviour of rectifiable sets is characterized by Hausdorff type measures or certain variants. Here Federer's co-area theorem as a counterpart to Hausdorff's area formula plays a fundamental role. It has been applied for the last years in many other fields of mathematics. "

"Although curvature measures describe second order properties of the sets, the "first order theory" suffices for deriving integral geometric relations. The key is to consider the unit normal bundle of the sets as a locally $(d-1)$-rectifiable subset of $\mathbb{R}^{2d}$ and to observe that the first order infinitesimal behaviour of the unit normal bundle determines the curvature measures."

Multilinear algebra

$\bigwedge_k \mathbb{R}^d, \bigwedge^k \mathbb{R}^d$ - space of k-vectors, k-covectors in $\mathbb{R}^d$ - linear spaces of dimension $\binom{d}{k}$

Wedge product (antisymmetric): $\xi \wedge \zeta = -\zeta \wedge \xi$

basis of $\bigwedge_k \mathbb{R}^d$: $\{e_{i_1} \wedge \cdots \wedge e_{i_k}, 1 \le i_1 < \cdots < i_k \le d\}$
basis of $\bigwedge^k \mathbb{R}^d$: $\{dx_{i_1} \wedge \cdots \wedge dx_{i_k}, 1 \le i_1 < \cdots < i_k \le d\}$
$\Omega_d := dx_1 \wedge \cdots \wedge dx_d \in \bigwedge_d \mathbb{R}^d$ - volume form in $\mathbb{R}^d$

Pairing: $\langle \xi, \phi \rangle \in \mathbb{R}$, $\xi \in \bigwedge_k \mathbb{R}^d$, $\phi \in \bigwedge^k \mathbb{R}^d$

Interior products: $\xi \in \bigwedge_k \mathbb{R}^d, \alpha \in \bigwedge^m \mathbb{R}^d$
$m \le k \implies \xi \llcorner \alpha \in \bigwedge_{k-m} \mathbb{R}^d$: $\langle \xi \llcorner \alpha, \beta \rangle = \langle \xi, \alpha \wedge \beta \rangle$
$m \ge k \implies \xi \lrcorner \alpha \in \bigwedge^{m-k} \mathbb{R}^d$: $\langle \xi, \alpha \lrcorner \beta \rangle = \langle \xi \wedge \alpha, \beta \rangle$

Multivectors and linear subspaces

simple k-vectors: $u_1 \wedge \cdots \wedge u_k, \quad u_1, \dots, u_k \in \mathbb{R}^d$

$\xi \in \bigwedge_k \mathbb{R}^d$ ... $L(\xi) := \{u \in \mathbb{R}^d : u \wedge \xi = 0\}$ - associated linear subspace

$L = L(u_1 \wedge \cdots \wedge u_k) \iff \{u_1, \dots, u_k\}$ is a basis of L
unit simple k-vectors correspond to "oriented linear k-subspaces"

Differential forms

$U \subset \mathbb{R}^d$ open
$\mathcal{D}^k(U) := C_c^\infty(U, \bigwedge^k \mathbb{R}^d)$ (differential k-forms on U with compact support)

$$\phi \in \mathcal{D}^k(U) : \quad \phi(x) = \sum_{1 \le i_1 < \cdots < i_k \le d} \phi_{i_1,\dots,i_k}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$f^\# \phi$ - *pull-back of* ϕ by a smooth mapping $f : G \subseteq \mathbb{R}^n \to U$

$$\langle \xi, f^\# \phi(x) \rangle := \langle (\textstyle\bigwedge_k Df(x))\xi, \phi(f(x)) \rangle.$$

$d\phi \in \mathcal{D}^{k+1}(U)$ - *exterior derivative* of ϕ

$$\langle v_1 \wedge \cdots \wedge v_{k+1}, d\phi(x) \rangle := \sum_{i=1}^{k+1} (-1)^{i-1} \langle v^{(i)}, D\phi(x) v_i \rangle,$$

where $v^{(i)} = v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_{k+1}$

Currents

$\mathcal{D}_k(U) := (\mathcal{D}^k(U))^*$ - *k-currents on U*

- *Boundaries*: $\partial T \in \mathcal{D}_{k-1}(U)$ $(k \geq 1)$, $(\partial T)(\phi) = T(d\phi)$
- *Rectifiable currents*. $T = (\mathcal{H}^k \llcorner W) \wedge \xi \in \mathcal{D}_k(U)$, where $W \subset U$ is locally $\mathcal{H}^k$-rectifiable and $\xi : W \to \bigwedge_k \mathbb{R}^d$ is locally $\mathcal{H}^k$-integrable and for $\mathcal{H}^k$-a.a. $x \in W$, $\xi(x)$ is a simple k-vector associated with $\mathrm{Tan}^k(W, x)$:

$$\left((\mathcal{H}^k \llcorner W) \wedge \xi\right)(\phi) := \int_W \langle \xi(x), \phi(x) \rangle \, d\mathcal{H}^k(x)$$

  Defined even for $\phi : W \to \bigwedge^k \mathbb{R}^d$ locally $\mathcal{H}^k$-integrable.
  $|\xi(x)|$ - *multiplicity* of the current
- *Integral currents*. $T \in \mathcal{D}_k(U)$ is an integral current if both $T, \partial T$ are rectifiable currents with integer multiplicities.

Normal cycles of sets with positive reach

$X \subset \mathbb{R}^d$, $\operatorname{reach} X > 0$
$\operatorname{nor} X \subset \partial X \times S^{d-1}$ - unit normal cycle
$\kappa_i(x,n), b_i(x,n)$, $i = 1, \dots, d-1$ - principal curvatures and directions (defined at a.a. $(x,n) \in \operatorname{nor} X$)

$$a_X := \bigwedge_{i=1}^{d-1} \left(\frac{1}{\sqrt{1+\kappa_i^2}} b_i, \frac{\kappa_i}{\sqrt{1+\kappa_i^2}} b_i \right)$$

$L(a_X(x,n)) = \operatorname{Tan}(\operatorname{nor} X, (x,n))$ for $\mathcal{H}^{d-1}$ a.a. $(x,n) \in \operatorname{nor} X$

$$N_X := (\mathcal{H}^{d-1} \llcorner \operatorname{nor} X) \wedge a_X \text{ - normal cycle of } X$$

-rectifiable current with integer multiplicity

Properties of normal cycles

Proposition

1. *N_X is a cycle:* $\partial N_X = 0$ *(hence, N_X is an integral current)*
2. *N_X is Legendrian:* $N_X \llcorner \alpha = 0$, *where* $\alpha \in \mathcal{D}^1(\mathbb{R}^{2d})$,

$$\langle (u,v), \alpha(x,n)\rangle = u \cdot n, \quad u,v,x,n \in \mathbb{R}^d.$$

Idea of proof.

(i) The bi-Lipschitz homeomorpism $g^{(r)} : \operatorname{nor} X \to \partial X_r$,
$(x,n) \mapsto x + rn$, transforms N_X onto the current corresponding to the oriented closed C^1 surface ∂X_r which is a cycle.
(ii) Follows from:
$(u,v) \in \operatorname{Tan}(\operatorname{nor} X, (x,n)) \implies u \cdot n = v \cdot n = 0.$ □

Definition

$\varphi_k \in C^\infty(\mathbb{R}^{2d}, \bigwedge^k \mathbb{R}^{2d})$ - *Lipschitz-Killing curvature form* of order $0 \leq k \leq d-1$:

$$\langle a_1 \wedge \cdots \wedge a_{d-1}, \varphi_k(x,n)\rangle$$
$$= \frac{1}{(d-k)\omega_{d-k}} \sum_{\substack{j_1+\cdots+j_{d-1}=d-k-1 \\ j_i \in \{0,1\}}} \langle \pi_{j_1} a_1 \wedge \cdots \wedge \pi_{j_{d-1}} a_{d-1} \wedge n, \Omega_d\rangle,$$

$a_1, \ldots, a_{d-1} \in \mathbb{R}^{2d}$, $(x,n) \in \mathbb{R}^{2d}$.

Theorem

If $X \subset \mathbb{R}^d$ is compact and has positive reach then
$\mathbf{C}_k(X) = N_X(\varphi_k)$.

Proof.

Compare the integral representation of $\mathbf{C}_k(X)$ and the definitions of N_X, φ_k. $\square$

Additivity, unions

Additivity property: If all sets $X, Y, X \cup Y, X \cap Y$ have positive reach then

$$\mathbf{C}_k(X \cup Y) + \mathbf{C}_k(X \cap Y) = \mathbf{C}_k(X) + \mathbf{C}_k(Y), \quad k = 0, \ldots, d,$$

$$N_{X \cup Y} + N_{X \cap Y} = N_X + N_Y.$$

Idea - extension to finite unions: if $X, Y, X \cap Y$ have positive reach, define

$$N_{X \cup Y} := N_X + N_Y - N_{X \cap Y}.$$

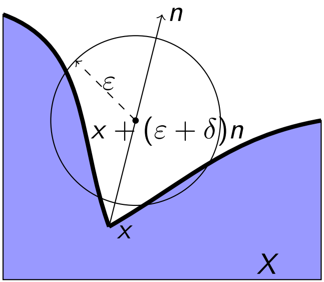
Problems: representation is not unique.

Topological index

Definition

For any $X \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$ and $n \in S^{d-1}$:

$$i_X(x,n) = \mathbf{1}_X(x)\Big(1 - \lim_{\varepsilon\to 0+}\lim_{\delta\to 0+} \chi\big(X \cap B(x+(\varepsilon+\delta)n,\varepsilon)\big)\Big)$$



the *index* of X at x in direction n; provided that the right hand side is determined, where χ denotes the Euler-Poincaré characteristic in the sense of singular homology.

$$i_X(x,n) = 1 - 2 = -1.$$

$\mathcal{U}_{\mathrm{PR}}$ sets

Definition

$X \subset \mathbb{R}^d$ is a *$\mathcal{U}_{\mathrm{PR}}$-set* if it can be represented as locally finite union

$$X = \bigcup_i X^i$$

with $\operatorname{reach}(\bigcap_{i \in I} X^i) > 0$ for any $I \subset \mathbb{N}$ finite.

Proposition

1. *$i_X(x, n)$ is determined for any $X \in \mathcal{U}_{\mathrm{PR}}$, $x \in X$ and $n \in S^{d-1}$.*
2. $\operatorname{reach} X > 0 \implies i_X = \mathbf{1}_{\operatorname{nor} X}$.
3. *$i_{X \cup Y} + i_{X \cap Y} = i_X + i_Y$ whenever $X, Y, X \cap Y, X \cup Y \in \mathcal{U}_{\mathrm{PR}}$.*

Normal cycles of $\mathcal{U}_{\mathrm{PR}}$ sets

For $X \in \mathcal{U}_{\mathrm{PR}}$, define

$$\operatorname{nor} X := \operatorname{spt} i_X = \{(x,n) : i_X(x,n) \neq 0\}$$

- $\operatorname{nor} X$ is locally $(d-1)$-rectifiable (not necessarily closed).
- i_X is locally $\mathcal{H}^{d-1}$-integrable.
- For $\mathcal{H}^{d-1}$-a.a. $(x,n) \in \operatorname{nor} X$, $\operatorname{Tan}(\operatorname{nor} X, (x,n))$ is a $(d-1)$-subspace; let $a_X(x,n)$ be the associated unit simple k vector with orientation given by

$$\lim_{\varepsilon \to 0} \langle \textstyle\bigwedge_{d-1}(\pi_0 + \varepsilon\pi_1) a_X(x,n) \wedge n, \Omega_d \rangle > 0\,.$$

Definition

$$N_X := (\mathcal{H}^{d-1} \llcorner \operatorname{nor} X) \wedge i_X a_X$$

Properties of normal cycles

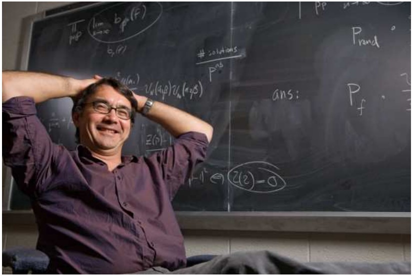
$X \in \mathcal{U}_{\mathrm{PR}}$

- N_X is a cycle
- N_X is Legendrian
- $N_X(\varphi_0)$ (Gauss-Bonnet formula)

Extensions of normal cycles

1. Legendrian cycles
2. Slicing of currents
3. Normal cycle - definition
4. Some easy examples (e.g., convex boundaries)
5. Differential cycles
6. Auras
7. A deformation lemma for auras
8. DC functions as auras
9. WDC sets

Joseph H.G. Fu: Amer. J. Math. (1994)



From the Introduction:
"Given a singular set $X \subset \mathbb{R}^d$, when does a normal cycle N_X exist? How can it be characterized? How can it be constructed? We are able also to give a satisfactory characterization of the normal cycle of an arbitrary compact subset of $\mathbb{R}^d$"

... "we also describe a construction valid for a more general class that contains, for example, the sets with positive reach. Still, this construction lacks formal beauty and seems far from definitive, so we have emphasized instead the subanalytic aspects."

Continuation - 2015+

Delta-convex (DC) functions

- Framework that perfectly fits into Fu's general construction
- Lacks "geometric intuition"
- Still not the most general setting

Legendrian cycle

Definition

A *Legendrian cycle* in $\mathbb{R}^d$ is an integral $(d-1)$-current T in $\mathbb{R}^{2d}$ with the following properties:

- spt $T \subset \mathbb{R}^d \times S^{d-1}$,
- $\partial T = 0$ (T is a cycle),
- $T \llcorner \alpha = 0$ (T is Legendrian),

(α is the contact form in $\mathbb{R}^{2d}$ acting as $\langle (u,v), \alpha(x,n)\rangle = u \cdot n$).

$\pi_0(x,n) \mapsto x$, $\pi_1(x,n) \mapsto n$ - projections
$\sigma : S^{d-1} \to \bigwedge_{d-1}\mathbb{R}^d$ unit orienting vectorfield ($\sigma(u) \wedge u = +1$)

Theorem

For any compactly supported Legendrian cycle T we have

(i) $(\pi_1)_\# T = T(\varphi_0)(\mathcal{H}^{d-1} \llcorner S^{d-1}) \wedge \sigma$,

(ii) $T(\varphi_0) = \sum_{x \in \mathbb{R}^d} \iota_T(x,n)$ *for almost all* $n \in S^{d-1}$.

Legendrian cycle - representation

T - Legendrian cycle: $T = (\mathcal{H}^{d-1} \llcorner W_T) \wedge i_T a_T$

- $W_T \subset \mathbb{R}^d \times S^{d-1}$ locally $\mathcal{H}^{d-1}$-rectifiable ("carrier" of T),
- $a_T(x, n)$ - unit simple $(d-1)$-vector associated with $\operatorname{Tan}^{d-1}(W_T, (x, n))$,
- $i_T(x, n)$ - integer multiplicity

Proposition

$$a_T := \bigwedge_{i=1}^{d-1} \left(\frac{1}{\sqrt{1+\kappa_i^2}} b_i, \frac{\kappa_i}{\sqrt{1+\kappa_i^2}} b_i \right)$$

for some $\kappa_i \equiv \kappa_i(x, n) \in (-\infty, \infty]$ (principal curvatures) and $b_i \equiv b_i(x, n) \perp n$ orthonormal vectors ($i = 1, \ldots, d-1$) (principal directions).

Slices of currents

Theorem (Co-area formula for currents)

Let T be a locally k-rectifiable current in $U \subset \mathbb{R}^d$ and let $f : \operatorname{spt} T \to \mathbb{R}^n$ be locally Lipschitz, $n \leq k$. Then, for $\mathcal{L}^n$-almost all $y \in \mathbb{R}^n$ there exists a locally $(k-n)$-rectifiable current

$$\langle T, f, y \rangle \in \mathcal{D}_{k-n}(U)$$

so that $\operatorname{spt}\langle T, f, y \rangle \subset f^{-1}\{y\} \cap \operatorname{spt} T$ and for any $g \in C^\infty(\mathbb{R}^n)$,

$$T \llcorner f^{\#}(g \wedge \Omega_n)(\cdot) = \int g(y) \langle T, f, y \rangle(\cdot)\, \mathcal{L}^n(dy).$$

Definition

The current $\langle T, f, y \rangle$ from the above theorem is called *slice of T through f at y.*

Slices of currents - cont.

Theorem

Let T be a locally k-rectifiable current on $U \subseteq \mathbb{R}^d$ of the form

$$T = (\mathcal{H}^k \llcorner A) \wedge \xi,$$

let $f : W \to \mathbb{R}^n$ be locally Lipschitz, $n \leq k$. Then, for $\mathcal{H}^n$-almost all y,

$$\langle T, f, y\rangle = (\mathcal{H}^{k-n} \llcorner f^{-1}\{y\}) \wedge \zeta,$$

where

$$\zeta(x) = \xi(x) \llcorner (f^{\#}\Omega_n)/J_n f(x)$$

whenever $J_n f(x)$ is defined and positive, and 0 otherwise.

Remark

If $n = k$ then $\langle T, f, y\rangle$ is a 0-current, which is, in fact, a signed discrete measure.

Slices of currents - example

T Legendrian cycle, $\pi_1 : (x, v) \mapsto v$, $g \in C_c^\infty(\mathbb{R}^d \times S^{d-1})$

$$\langle T, \pi_0, v\rangle(g) = \sum_{x\in\mathbb{R}^d} \iota_T(x, v) g(x, v) \quad \text{for a.a. } v \in S^{d-1}$$

$\iota_T := i_T(-1)^{\lambda_T}$, $\lambda_T(x, v) := \#\{i : \kappa_i(x, v) < 0\}$

by approximation with smooth functions ($H_{v,t} := \{y : y \cdot v \le t\}$):

$$\langle T, \pi_0, v\rangle(\mathbf{1}_{H_{v,t}\times S^{d-1}}) = \sum_{x\cdot v<t} \iota_T(x, v) \quad \text{for a.a. } (v, t) \in S^{d-1} \times \mathbb{R}$$

X smooth:

$$\chi(X) = \sum_{x:\nu_X(x)=v} \iota_X(x) \quad \text{for a.a. } v \in S^{d-1}$$

$$\chi(X \cap H_{v,t}) = \sum_{x:x\cdot v\le t,\nu_X(x)=v} \iota_X(x) \quad \text{for a.a. } (v, t) \in S^{d-1} \times \mathbb{R}$$

Normal cycle

Notation: $H_{v,t} := \{y \in \mathbb{R}^d : y \cdot v \leq t\}$, $v \in S^{d-1}, t \in \mathbb{R}$ (closed halfspaces)

$\Lambda : (x, v) \mapsto (v, x \cdot v)$, $(x, v) \in \mathbb{R}^d \times S^{d-1}$

Definition

We say that a compact set $X \subset \mathbb{R}^d$ admits a *normal cycle* if there exists a compactly supported Legendrian cycle T such that

(i) $\mathcal{H}^d(\Lambda(\mathrm{spt}\, T)) = 0$,

(ii) $\langle T, \pi_1, v\rangle(\mathbf{1}_{H_{v,t}\times S^{d-1}}) = \chi(X \cap H_{v,t})$ for a.a. $(v, t) \in S^{d-1} \times \mathbb{R}$.

In such a case, we write $N_X := T$ and call N_X the *normal cycle of X*.

Theorem

Any compact set admits at most one normal cycle.

Easy examples

- Sets with positive reach
- $\mathcal{U}_{\mathrm{PR}}$ sets
- Complements of "full-dimensional" sets with positive reach

$$(x,n) \in \operatorname{nor} X \implies (x,-n) \notin \operatorname{nor} X$$

- Boundaries of "full-dimensional" sets with positive reach

Fu's construction: differential cycle

$f : \mathbb{R}^d \to \mathbb{R}$ smooth (C^2)
graph $\nabla f = \{(x, \nabla f(x)) : x \in \mathbb{R}^d\}$ - C^1-surface
orientation $\xi = (e_1, \cdot) \wedge \cdots \wedge (e_d, \cdot)$: follows positive orientation in $\mathbb{R}^d$
$\mathbb{D}f := (\mathcal{H}^d \llcorner \operatorname{graph} \nabla f) \wedge \xi$ - *differential cycle* of f

Definition

$f : \mathbb{R}^d \to \mathbb{R}$ is strongly approximable if $\exists f_k \in C^2(\mathbb{R}^d)$, $f_k \to f$ in L^1_{loc} and

$$\int_K \left| \det \left(\frac{\partial^2 f_k}{\partial u_i \partial u_j} \right)_{i \in I, j \in J} \right| du \leq \mathrm{const}(f, K),$$

K compact, $I, J \subset \{1, \ldots, d\}$, $|I| = |J|$.

Differential cycle - cont.

Theorem (Compactness theorem, Federer-Fleming (1960))

Let (T_i) be a sequence of integral currents with uniformly bounded masses of both T_i and ∂T_i. Then there exists a subsequence converging weakly to an integral current T.

$f : \mathbb{R}^d \to \mathbb{R}$ Lipschitz, $f_i \to f$ strong approximation by smooth functions

$\mathbb{D}f_i$ are integral currents with $\partial \mathbb{D} f_i = 0$ and uniformly bounded masses, hence, a subsequence $(\mathbb{D} f_{i_k})$ converges weakly to an integral current T. Fu showed that there can be only one such limit current and called it $\mathbb{D}f$ - *differential cycle* of f. Hence,

$$\mathbb{D}f_i \to \mathbb{D}f \text{ weakly.}$$

Lemma

$$\operatorname{spt} \mathbb{D}f \subset \operatorname{graph} \partial f := \{(x,u) : u \in \partial f(x), x \in \mathbb{R}^d\}.$$

Auras

Definition

A Lipschitz proper function $f : \mathbb{R}^d \to [0, \infty)$ is an *aura* if

- f is strongly approximable
- 0 is a weakly regular value of f

If f is an aura and $X := \{f = 0\}$ we define

$$N_X := -\nu_{\#}\partial(\mathbb{D}f \llcorner (f \circ \pi_0)^{-1}(0, \infty))$$

$\nu : (x, u) \mapsto (x, \frac{u}{|u|})$

Remark

spt $N_X \subset \{\nu(x, u) : \exists x_i \to x, f(x_i) > 0 = f(x), \partial f(x_i) \ni u_i \to u\}$.
Recall: $\Lambda : (x, v) \mapsto (v, x \cdot v)$; $\Lambda(\text{graph}\, \partial f)$ can be interpreted as the set of all *tangent hyperplanes* to graph f.

Theorem (Fu)

If f is an aura and $\Lambda(\text{graph}\, \partial f)$ has locally finite d-dimensional measure then N_X is the normal cycle of $X = \{f = 0\}$.

Deformation lemma for auras

Lemma

If f is an aura and $X := \{f = 0\}$ compact then there exists an $\varepsilon > 0$ and a deformation retraction

$$H : \{f < \varepsilon\} \times [0, 1] \to \{f < \varepsilon\}$$

such that $\frac{\varepsilon}{2} \operatorname{dist}(z, X) \le f(z) \le (\operatorname{Lip} f) \operatorname{dist}(z, X)$ *whenever* $f(z) < \varepsilon$.

- needed to prove the Gauss-Bonnet formula ($N_X(\varphi_0) = \chi(X)$) and its "local" variant ($\chi(X \cap H_{v,t}) = \dots$) in order to prove the property of normal cycle
- any $r > 0$ small enough is a *regular value* of f and classical methods yield $\chi\{f \le r\} = \nu_{\#}\langle \mathbb{D}f, f \circ \pi_0, r\rangle(\varphi_0)$.
- Basic calculus of currents gives $\nu_{\#}\langle \mathbb{D}f, f \circ \pi_0, r\rangle(\varphi_0) \to N_X(\varphi_0)$, $r \to 0_+$.
- The deformation lemma yields $\chi\{f \le r\} \to \chi\{f = 0\}$.

DC functions come into play

- Crucial point (D. Pokorný, 2013): *Any DC function is strongly approximable.*
- If $f : \mathbb{R}^d \to \mathbb{R}$ is DC then $\Lambda(\operatorname{graph} \partial f)$ has locally finite d-dimensional measure (follows from Pavlica & Zajíček (2007))

Definition

A (closed) set $X \subset \mathbb{R}^d$ is a (locally) *WDC set* if there exists a DC aura $f : \mathbb{R}^d \to [0, \infty)$ such that $X = \{f = 0\}$ (for any $x \in X$ there exists a DC aura f such that X agrees with $\{f = 0\}$ on some neighbourhood of x).

Example

DC manifolds, closed DC domains

Theorem (Pokorný & R. (2013))

Any compact (locally) WDC set admits a normal cycle.

Principal kinematic fomula for WDC sets

Let f be a DC function on $\mathbb{R}^d$. The statement "$\Lambda(\text{graph}\,\partial f)$ has locally finite d-dimensional measure" can be strengthened to

- graph ∂f has locally finite d-dimensional Minkowski content

Theorem (Fu, Pokorný & R. (2015))

If $X \subset \mathbb{R}^d$ is a compact (locally) WDC set then the associated curvature measures $\mathbf{C}_k(X) = N_X(\varphi_k)$ fulfill the principal kinematic formulas.

- proved even in Riemannian manifolds

Properties of WDC sets

Theorem (Pokorný, R. & Zajíček (2016+))

The boundary of a (locally) WDC set can be locally covered by a finite number of DC hypersurfaces. (Hence, it is locally $(d-1)$-rectifiable.)

- follows from a quantitative refinement of a result on singularities of (delta)-convex functions (Zajíček, 1979): "The set of nondifferentiability of a convex function can be covered by countably many DC hypersurfaces"
- uses the Deformation lemma

Theorem (PRZ (2016+))

Any lower-dimensional Lipschitz manifold in $\mathbb{R}^d$ which is locally WDC is a DC manifold.

Some open problems

- Is there some "geometric" characterization of (locally) WDC sets in $\mathbb{R}^d$? (We have one in $\mathbb{R}^2$ only.)
- Is $\operatorname{spt} N_X$ countably $\mathcal{H}^{d-1}$-rectifiable if X is WDC? Is $\operatorname{graph} \partial f$ countably $\mathcal{H}^d$-rectifiable if $f : \mathbb{R}^d \to \mathbb{R}$ is DC?
- Let X be connected and WDC. Can any two points in X be joined by a Lipschitz curve in X of length smaller than a fixed multiple of their distance?
- There are known examples of auras which are not DC (though not many...). Is it possible to describe the maximal family of auras admitting differential cycles, or the maximal family of (compact) sets admitting normal cycles?