A few questions on nonlinear embeddings into Banach spaces.

Gilles Lancien

Université Bourgogne Franche-Comté

Svratka, 14-21 January 2017

Overview

1. Isometric embeddings
   - Four fundamental results on isometries
   - Other consequences of Figiel and Godefroy-Kalton
   - Applications of descriptive set theory
2. Lipschitz embeddings
3. Coarse and uniform embeddings
4. Metric invariants
   - Examples of local properties
   - Asymptotic properties

I. ISOMETRIC EMBEDDINGS.

I.1. Four fundamental results on isometries.

Mazur-Ulam (1932)

Let X and Y be two normed spaces. Then any isometry U from X onto Y, such that $U(0) = 0$, is linear.

I.1. Four fundamental results on isometries.

Mazur-Ulam (1932)

Let X and Y be two normed spaces. Then any isometry U from X onto Y, such that $U(0) = 0$, is linear.

Banach-Mazur (1933)

Any separable Banach space is isometric to a subspace of $C([0, 1])$.

I.1. Four fundamental results on isometries.

Mazur-Ulam (1932)

Let X and Y be two normed spaces. Then any isometry U from X onto Y, such that $U(0) = 0$, is linear.

Banach-Mazur (1933)

Any separable Banach space is isometric to a subspace of $C([0, 1])$.

Figiel (1968)

I.1. Four fundamental results on isometries.

Mazur-Ulam (1932)

Let X and Y be two normed spaces. Then any isometry U from X onto Y, such that $U(0) = 0$, is linear.

Banach-Mazur (1933)

Any separable Banach space is isometric to a subspace of $C([0,1])$.

Figiel (1968)

Let X and Y be two Banach spaces and $U : X \to Y$ be an isometry such that $U(0) = 0$.
Then for all $x_1, .., x_n$ in X and all $\lambda_1, .., \lambda_n$ in $\mathbb{R}$:

$$\Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \Big\| \sum_{k=1}^{n} \lambda_k x_k \Big\|_X .$$

I.1. Four fundamental results on isometries.

Mazur-Ulam (1932)

Let X and Y be two normed spaces. Then any isometry U from X onto Y, such that $U(0) = 0$, is linear.

Banach-Mazur (1933)

Any separable Banach space is isometric to a subspace of $C([0,1])$.

Figiel (1968)

Let X and Y be two Banach spaces and $U : X \to Y$ be an isometry such that $U(0) = 0$.
Then for all $x_1, .., x_n$ in X and all $\lambda_1, .., \lambda_n$ in $\mathbb{R}$:

$$\Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \Big\| \sum_{k=1}^{n} \lambda_k x_k \Big\|_X.$$

In other words, there exists a linear quotient map $Q : \overline{sp}(U(X)) \to X$ such that $QU = I_X$ and $\|Q\| = 1$.

Proof of Figiel's Theorem :

Proof of Figiel's Theorem : We may assume that $\dim X < \infty$.

Proof of Figiel's Theorem : We may assume that $\dim X < \infty$.

Lemma

Let $x \in S_X$ such that $\| \, \|_X$ is G-smooth at x and denote $v_x^* \in S_{X^*}$ its differential.

Proof of Figiel's Theorem : We may assume that $\dim X < \infty$.

Lemma

Let $x \in S_X$ such that $\| \ \|_X$ is G-smooth at x and denote $v_x^* \in S_{X^*}$ its differential.
Then v_x^* is the unique 1-Lipschitz function $f : X \to \mathbb{R}$ such that $f(tx) = t$ for all $t \in \mathbb{R}$.

Proof of Figiel's Theorem : We may assume that $\dim X < \infty$.

Lemma

Let $x \in S_X$ such that $\| \ \|_X$ is G-smooth at x and denote $v_x^* \in S_{X^*}$ its differential.
Then v_x^* is the unique 1-Lipschitz function $f : X \to \mathbb{R}$ such that $f(tx) = t$ for all $t \in \mathbb{R}$.

Proof : Let $f : X \to \mathbb{R}$ be such a function, $u \in X$ and $|t| > |f(u)|$.

Proof of Figiel's Theorem : We may assume that $\dim X < \infty$.

Lemma

Let $x \in S_X$ such that $\| \ \|_X$ is G-smooth at x and denote $v_x^* \in S_{X^*}$ its differential.
Then v_x^* is the unique 1-Lipschitz function $f : X \to \mathbb{R}$ such that $f(tx) = t$ for all $t \in \mathbb{R}$.

Proof : Let $f : X \to \mathbb{R}$ be such a function, $u \in X$ and $|t| > |f(u)|$. Then

$$|t - f(u)| \leq \|tx - u\| = |t|\big(1 - v_x^*(u/t) + o(1/t)\big)$$

Proof of Figiel's Theorem : We may assume that $\dim X < \infty$.

Lemma

Let $x \in S_X$ such that $\| \ \|_X$ is G-smooth at x and denote $v_x^* \in S_{X^*}$ its differential.
Then v_x^* is the unique 1-Lipschitz function $f : X \to \mathbb{R}$ such that $f(tx) = t$ for all $t \in \mathbb{R}$.

Proof : Let $f : X \to \mathbb{R}$ be such a function, $u \in X$ and $|t| > |f(u)|$. Then

$$|t - f(u)| \leq \|tx - u\| = |t|\big(1 - v_x^*(u/t) + o(1/t)\big)$$

$$sgn(t)\big(t - f(u)\big) \leq |t| - sgn(t)v_x^*(u) + o(1)$$

Proof of Figiel's Theorem : We may assume that $\dim X < \infty$.

Lemma

Let $x \in S_X$ such that $\| \ \|_X$ is G-smooth at x and denote $v_x^* \in S_{X^*}$ its differential.
Then v_x^* is the unique 1-Lipschitz function $f : X \to \mathbb{R}$ such that $f(tx) = t$ for all $t \in \mathbb{R}$.

Proof : Let $f : X \to \mathbb{R}$ be such a function, $u \in X$ and $|t| > |f(u)|$. Then

$$|t - f(u)| \leq \|tx - u\| = |t|\big(1 - v_x^*(u/t) + o(1/t)\big)$$

$$sgn(t)\big(t - f(u)\big) \leq |t| - sgn(t)v_x^*(u) + o(1)$$

$$sgn(t)v_x^*(u) \leq sgn(t)f(u) + o(1)$$

Proof of Figiel's Theorem : We may assume that $\dim X < \infty$.

Lemma

Let $x \in S_X$ such that $\| \ \|_X$ is G-smooth at x and denote $v_x^* \in S_{X^*}$ its differential.
Then v_x^* is the unique 1-Lipschitz function $f : X \to \mathbb{R}$ such that $f(tx) = t$ for all $t \in \mathbb{R}$.

Proof : Let $f : X \to \mathbb{R}$ be such a function, $u \in X$ and $|t| > |f(u)|$. Then

$$|t - f(u)| \leq \|tx - u\| = |t|\big(1 - v_x^*(u/t) + o(1/t)\big)$$

$$sgn(t)\big(t - f(u)\big) \leq |t| - sgn(t)v_x^*(u) + o(1)$$

$$sgn(t)v_x^*(u) \leq sgn(t)f(u) + o(1)$$

Letting t tend to $+\infty$ and $-\infty$, we get $v_x^*(u) = f(u)$. □

End of proof of Figiel's Theorem : Fix $x \in S_X$.

End of proof of Figiel's Theorem : Fix $x \in S_X$. Then

$$\forall n \in \mathbb{N} \;\; \exists y_n^* \in S_{Y^*} \;\; \left\langle y_n^*, U(nx) - U(-nx) \right\rangle = 2n.$$

End of proof of Figiel's Theorem : Fix $x \in S_X$. Then

$$\forall n \in \mathbb{N} \;\; \exists y_n^* \in S_{Y^*} \;\; \left\langle y_n^*, U(nx) - U(-nx) \right\rangle = 2n.$$

Since U is an isometry and $U(0) = 0$, we get that

$$\forall t \in [-n, n] \;\; \langle y_n^*, U(tx) \rangle = t.$$

End of proof of Figiel's Theorem : Fix $x \in S_X$. Then

$$\forall n \in \mathbb{N} \;\; \exists y_n^* \in S_{Y^*} \;\; \big\langle y_n^*, U(nx) - U(-nx) \big\rangle = 2n.$$

Since U is an isometry and $U(0) = 0$, we get that

$$\forall t \in [-n, n] \;\; \langle y_n^*, U(tx) \rangle = t.$$

Consider now y_x^* a weak*-accumulation point of $(y_n^*)_n$.

End of proof of Figiel's Theorem : Fix $x \in S_X$. Then

$$\forall n \in \mathbb{N} \;\; \exists y_n^* \in S_{Y^*} \;\; \left\langle y_n^*, U(nx) - U(-nx) \right\rangle = 2n.$$

Since U is an isometry and $U(0) = 0$, we get that

$$\forall t \in [-n, n] \;\; \langle y_n^*, U(tx) \rangle = t.$$

Consider now y_x^* a weak*-accumulation point of $(y_n^*)_n$.
We have that for all $t \in \mathbb{R}$, $\left\langle y_x^*, U(tx) \right\rangle = t$.

End of proof of Figiel's Theorem : Fix $x \in S_X$. Then

$$\forall n \in \mathbb{N} \ \ \exists y_n^* \in S_{Y^*} \ \ \left\langle y_n^*, U(nx) - U(-nx) \right\rangle = 2n.$$

Since U is an isometry and $U(0) = 0$, we get that

$$\forall t \in [-n, n] \ \ \langle y_n^*, U(tx) \rangle = t.$$

Consider now y_x^* a weak*-accumulation point of $(y_n^*)_n$.
We have that for all $t \in \mathbb{R}$, $\left\langle y_x^*, U(tx) \right\rangle = t$.
It follows from our Lemma that if $x \in \mathcal{S}$ (the set of points in S_X of G-smoothness of $\| \ \|_X$), then $y_x^* \circ U = v_x^*$.

End of proof of Figiel's Theorem : Fix $x \in S_X$. Then

$$\forall n \in \mathbb{N} \ \ \exists y_n^* \in S_{Y^*} \ \ \left\langle y_n^*, U(nx) - U(-nx) \right\rangle = 2n.$$

Since U is an isometry and $U(0) = 0$, we get that

$$\forall t \in [-n, n] \ \ \langle y_n^*, U(tx) \rangle = t.$$

Consider now y_x^* a weak*-accumulation point of $(y_n^*)_n$.
We have that for all $t \in \mathbb{R}$, $\left\langle y_x^*, U(tx) \right\rangle = t$.
It follows from our Lemma that if $x \in \mathcal{S}$ (the set of points in S_X of G-smoothness of $\| \ \|_X$), then $y_x^* \circ U = v_x^*$.

Now

$$\Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \sup_{x \in \mathcal{S}} \Big\langle y_x^*, \sum_{k=1}^{n} \lambda_k U(x_k) \Big\rangle$$

End of proof of Figiel's Theorem : Fix $x \in S_X$. Then

$$\forall n \in \mathbb{N} \ \ \exists y_n^* \in S_{Y^*} \ \ \left\langle y_n^*, U(nx) - U(-nx) \right\rangle = 2n.$$

Since U is an isometry and $U(0) = 0$, we get that

$$\forall t \in [-n, n] \ \ \langle y_n^*, U(tx) \rangle = t.$$

Consider now y_x^* a weak*-accumulation point of $(y_n^*)_n$.
We have that for all $t \in \mathbb{R}$, $\left\langle y_x^*, U(tx) \right\rangle = t$.
It follows from our Lemma that if $x \in \mathcal{S}$ (the set of points in S_X of G-smoothness of $\| \ \|_X$), then $y_x^* \circ U = v_x^*$.

Now

$$\Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \sup_{x \in \mathcal{S}} \Big\langle y_x^*, \sum_{k=1}^{n} \lambda_k U(x_k) \Big\rangle = \sup_{x \in \mathcal{S}} \Big\langle v_x^*, \sum_{k=1}^{n} \lambda_k x_k \Big\rangle$$

End of proof of Figiel's Theorem : Fix $x \in S_X$. Then

$$\forall n \in \mathbb{N} \;\; \exists y_n^* \in S_{Y^*} \;\; \left\langle y_n^*, U(nx) - U(-nx) \right\rangle = 2n.$$

Since U is an isometry and $U(0) = 0$, we get that

$$\forall t \in [-n, n] \;\; \langle y_n^*, U(tx) \rangle = t.$$

Consider now y_x^* a weak*-accumulation point of $(y_n^*)_n$.
We have that for all $t \in \mathbb{R}$, $\left\langle y_x^*, U(tx) \right\rangle = t$.
It follows from our Lemma that if $x \in \mathcal{S}$ (the set of points in S_X of G-smoothness of $\| \, \|_X$), then $y_x^* \circ U = v_x^*$.

Now

$$\Big\| \sum_{k=1}^n \lambda_k U(x_k) \Big\|_Y \geq \sup_{x \in \mathcal{S}} \Big\langle y_x^*, \sum_{k=1}^n \lambda_k U(x_k) \Big\rangle = \sup_{x \in \mathcal{S}} \Big\langle v_x^*, \sum_{k=1}^n \lambda_k x_k \Big\rangle$$

$$= \Big\| \sum_{k=1}^n \lambda_k x_k \Big\|_X,$$

because $\mathcal{S}$ is dense in S_X. □

Godefroy-Kalton lifting theorem (2003)

Godefroy-Kalton lifting theorem (2003)

Let X and Y be two Banach spaces. Assume that X is **separable**, that $Q : Y \to X$ is linear continuous and onto and that there exists a Lipschitz map $L : X \to Y$ so that $QL = Id_X$ (L is a Lipschitz lifting of Q).

Godefroy-Kalton lifting theorem (2003)

Let X and Y be two Banach spaces. Assume that X is **separable**, that $Q : Y \to X$ is linear continuous and onto and that there exists a Lipschitz map $L : X \to Y$ so that $QL = Id_X$ (L is a Lipschitz lifting of Q).
Then there exists $T : X \to Y$ linear so that $QT = Id_X$ and $\|T\| \leq Lip(L)$.

Godefroy-Kalton lifting theorem (2003)

Let X and Y be two Banach spaces. Assume that X is **separable**, that $Q : Y \to X$ is linear continuous and onto and that there exists a Lipschitz map $L : X \to Y$ so that $QL = Id_X$ (L is a Lipschitz lifting of Q).
Then there exists $T : X \to Y$ linear so that $QT = Id_X$ and $\|T\| \leq Lip(L)$.

Corollary

Assume that X is a **separable** *Banach space which is isometric to a* **subset** *of a Banach space Y. Then X is linearly isometric to a* **subspace** *of Y.*

Godefroy-Kalton lifting theorem (2003)

Let X and Y be two Banach spaces. Assume that X is **separable**, that $Q : Y \to X$ is linear continuous and onto and that there exists a Lipschitz map $L : X \to Y$ so that $QL = Id_X$ (L is a Lipschitz lifting of Q).
Then there exists $T : X \to Y$ linear so that $QT = Id_X$ and $\|T\| \leq Lip(L)$.

Corollary

Assume that X is a **separable** *Banach space which is isometric to a* **subset** *of a Banach space Y. Then X is linearly isometric to a* **subspace** *of Y.*

Proof of Corollary. Assume that $U : X \to Y$ is an isometry such that $U(0) = 0$. We may also assume that $Y = \overline{sp}(U(X))$.

Godefroy-Kalton lifting theorem (2003)

Let X and Y be two Banach spaces. Assume that X is **separable**, that $Q : Y \to X$ is linear continuous and onto and that there exists a Lipschitz map $L : X \to Y$ so that $QL = Id_X$ (L is a Lipschitz lifting of Q).
Then there exists $T : X \to Y$ linear so that $QT = Id_X$ and $\|T\| \leq Lip\,(L)$.

Corollary

Assume that X is a **separable** *Banach space which is isometric to a* **subset** *of a Banach space Y. Then X is linearly isometric to a* **subspace** *of Y.*

Proof of Corollary. Assume that $U : X \to Y$ is an isometry such that $U(0) = 0$. We may also assume that $Y = \overline{sp}(U(X))$. Then, by Figiel's Theorem, there exists $Q : Y \to X$ linear such that $QU = I_X$ and $\|Q\| \leq 1$.

Godefroy-Kalton lifting theorem (2003)

Let X and Y be two Banach spaces. Assume that X is **separable**, that $Q : Y \to X$ is linear continuous and onto and that there exists a Lipschitz map $L : X \to Y$ so that $QL = Id_X$ (L is a Lipschitz lifting of Q).
Then there exists $T : X \to Y$ linear so that $QT = Id_X$ and $\|T\| \leq Lip(L)$.

Corollary

Assume that X is a **separable** *Banach space which is isometric to a* **subset** *of a Banach space Y. Then X is linearly isometric to a* **subspace** *of Y.*

Proof of Corollary. Assume that $U : X \to Y$ is an isometry such that $U(0) = 0$. We may also assume that $Y = \overline{sp}(U(X))$. Then, by Figiel's Theorem, there exists $Q : Y \to X$ linear such that $QU = I_X$ and $\|Q\| \leq 1$. Now, by Godefroy-Kalton's Theorem, there exists $T : X \to Y$ linear such that $\|T\| \leq 1$ and $QT = I_X$ and thus T is a linear isometry. $\square$

Godefroy-Kalton lifting theorem (2003)

Let X and Y be two Banach spaces. Assume that X is **separable**, that $Q : Y \to X$ is linear continuous and onto and that there exists a Lipschitz map $L : X \to Y$ so that $QL = Id_X$ (L is a Lipschitz lifting of Q). Then there exists $T : X \to Y$ linear so that $QT = Id_X$ and $\|T\| \leq Lip(L)$.

Corollary

Assume that X is a **separable** *Banach space which is isometric to a* **subset** *of a Banach space Y. Then X is linearly isometric to a* **subspace** *of Y.*

Proof of Corollary. Assume that $U : X \to Y$ is an isometry such that $U(0) = 0$. We may also assume that $Y = \overline{sp}(U(X))$. Then, by Figiel's Theorem, there exists $Q : Y \to X$ linear such that $QU = I_X$ and $\|Q\| \leq 1$. Now, by Godefroy-Kalton's Theorem, there exists $T : X \to Y$ linear such that $\|T\| \leq 1$ and $QT = I_X$ and thus T is a linear isometry. □

More precisely, if $E = T(X)$, then $P = TQ$ is a projection of norm 1 from Y onto E and we can decompose $Y = E \oplus Ker\, P = Y = E \oplus Ker\, Q$ and $\forall x \in X \quad U(x) = (T(x), U(x) - T(x))$.

Godefroy-Kalton lifting theorem (2003)

Let X and Y be two Banach spaces. Assume that X is **separable**, that $Q : Y \to X$ is linear continuous and onto and that there exists a Lipschitz map $L : X \to Y$ so that $QL = Id_X$ (L is a Lipschitz lifting of Q). Then there exists $T : X \to Y$ linear so that $QT = Id_X$ and $\|T\| \leq Lip(L)$.

Corollary

Assume that X is a **separable** *Banach space which is isometric to a* **subset** *of a Banach space Y. Then X is linearly isometric to a* **subspace** *of Y.*

Proof of Corollary. Assume that $U : X \to Y$ is an isometry such that $U(0) = 0$. We may also assume that $Y = \overline{sp}(U(X))$. Then, by Figiel's Theorem, there exists $Q : Y \to X$ linear such that $QU = I_X$ and $\|Q\| \leq 1$. Now, by Godefroy-Kalton's Theorem, there exists $T : X \to Y$ linear such that $\|T\| \leq 1$ and $QT = I_X$ and thus T is a linear isometry. □

More precisely, if $E = T(X)$, then $P = TQ$ is a projection of norm 1 from Y onto E and we can decompose $Y = E \oplus Ker\,P = Y = E \oplus Ker\,Q$ and
$\forall x \in X \quad U(x) = (T(x), U(x) - T(x))$.
Typical example : $U : \mathbb{R} \to \ell^2_\infty$, $U(t) = (t, \sin t)$.

Basics on free spaces.

Basics on free spaces.

Let (M, d) be a metric space with origin 0. Then

$$Lip_0(M) = \{f : M \to \mathbb{R},\ f \text{ Lipschitz},\ f(0) = 0\}.$$

For $f \in Lip_0(M)$, $\|f\|_L = Lip(f)$.

Basics on free spaces.

Let (M,d) be a metric space with origin 0. Then

$$Lip_0(M)=\{f:M\to\mathbb{R},\ f\ \text{Lipschitz},\ f(0)=0\}.$$

For $f\in Lip_0(M)$, $\|f\|_L=Lip\,(f)$.
For $x\in M$ and $f\in Lip_0(M)$, $\langle\delta_M(x),f\rangle=f(x)$.
Then δ_M is an isometry from M into $Lip_0(M)^*$

Basics on free spaces.

Let (M, d) be a metric space with origin 0. Then

$$Lip_0(M) = \{f : M \to \mathbb{R},\ f \text{ Lipschitz},\ f(0) = 0\}.$$

For $f \in Lip_0(M)$, $\|f\|_L = Lip\,(f)$.
For $x \in M$ and $f \in Lip_0(M)$, $\langle \delta_M(x), f \rangle = f(x)$.
Then δ_M is an isometry from M into $Lip_0(M)^*$
and we define $\mathcal{F}(M) = \overline{sp}\{\delta_M(x),\ x \in M\}$.

Basics on free spaces.

Let (M, d) be a metric space with origin 0. Then

$$Lip_0(M) = \{f : M \to \mathbb{R},\ f \text{ Lipschitz},\ f(0) = 0\}.$$

For $f \in Lip_0(M)$, $\|f\|_L = Lip\,(f)$.
For $x \in M$ and $f \in Lip_0(M)$, $\langle \delta_M(x), f\rangle = f(x)$.
Then δ_M is an isometry from M into $Lip_0(M)^*$
and we define $\mathcal{F}(M) = \overline{sp}\{\delta_M(x),\ x \in M\}$.

Fundamental property : If Y is a Banach space and $L : M \to Y$ is Lipschitz with $L(0) = 0$, then there exists $\overline{L} : \mathcal{F}(M) \to Y$ linear such that $\|\overline{L}\| = Lip\,(L)$ and $\overline{L}\delta_M = L$.

Basics on free spaces.

Let (M,d) be a metric space with origin 0. Then

$$Lip_0(M) = \{f : M \to \mathbb{R},\ f \text{ Lipschitz},\ f(0) = 0\}.$$

For $f \in Lip_0(M)$, $\|f\|_L = Lip\,(f)$.
For $x \in M$ and $f \in Lip_0(M)$, $\langle \delta_M(x), f\rangle = f(x)$.
Then δ_M is an isometry from M into $Lip_0(M)^*$
and we define $\mathcal{F}(M) = \overline{sp}\{\delta_M(x),\ x \in M\}$.

Fundamental property : If Y is a Banach space and $L : M \to Y$ is Lipschitz with $L(0) = 0$, then there exists $\overline{L} : \mathcal{F}(M) \to Y$ linear such that $\|\overline{L}\| = Lip\,(L)$ and $\overline{L}\delta_M = L$.
So $B(\mathcal{F}(M), Y) \equiv Lip_0(M, Y)$ and in particular $\mathcal{F}(M)^* \equiv Lip_0(M)$.

Basics on free spaces.

Let (M, d) be a metric space with origin 0. Then

$$Lip_0(M) = \{f : M \to \mathbb{R},\ f \text{ Lipschitz},\ f(0) = 0\}.$$

For $f \in Lip_0(M)$, $\|f\|_L = Lip\,(f)$.
For $x \in M$ and $f \in Lip_0(M)$, $\langle \delta_M(x), f \rangle = f(x)$.
Then δ_M is an isometry from M into $Lip_0(M)^*$
and we define $\mathcal{F}(M) = \overline{sp}\{\delta_M(x),\ x \in M\}$.

<u>**Fundamental property :**</u> If Y is a Banach space and $L : M \to Y$ is Lipschitz with $L(0) = 0$, then there exists $\overline{L} : \mathcal{F}(M) \to Y$ linear such that $\|\overline{L}\| = Lip\,(L)$ and $\overline{L}\delta_M = L$.
So $B(\mathcal{F}(M), Y) \equiv Lip_0(M, Y)$ and in particular $\mathcal{F}(M)^* \equiv Lip_0(M)$.
<u>**Weaver - arXiv 2017.**</u>

Basics on free spaces.

Let (M, d) be a metric space with origin 0. Then

$$Lip_0(M) = \{f : M \to \mathbb{R},\ f \text{ Lipschitz},\ f(0) = 0\}.$$

For $f \in Lip_0(M)$, $\|f\|_L = Lip\,(f)$.
For $x \in M$ and $f \in Lip_0(M)$, $\langle \delta_M(x), f \rangle = f(x)$.
Then δ_M is an isometry from M into $Lip_0(M)^*$
and we define $\mathcal{F}(M) = \overline{sp}\{\delta_M(x),\ x \in M\}$.

__**Fundamental property :**__ If Y is a Banach space and $L : M \to Y$ is Lipschitz with $L(0) = 0$, then there exists $\overline{L} : \mathcal{F}(M) \to Y$ linear such that $\|\overline{L}\| = Lip\,(L)$ and $\overline{L}\delta_M = L$.
So $B(\mathcal{F}(M), Y) \equiv Lip_0(M, Y)$ and in particular $\mathcal{F}(M)^* \equiv Lip_0(M)$.
__**Weaver - arXiv 2017.**__ If M is bounded or is complete and convex, then $Lip_0(M)$ has a unique predual.

Basics on free spaces.

Let (M, d) be a metric space with origin 0. Then

$$Lip_0(M) = \{f : M \to \mathbb{R},\ f \text{ Lipschitz},\ f(0) = 0\}.$$

For $f \in Lip_0(M)$, $\|f\|_L = Lip\,(f)$.
For $x \in M$ and $f \in Lip_0(M)$, $\langle \delta_M(x), f \rangle = f(x)$.
Then δ_M is an isometry from M into $Lip_0(M)^*$
and we define $\mathcal{F}(M) = \overline{sp}\{\delta_M(x),\ x \in M\}$.

Fundamental property : If Y is a Banach space and $L : M \to Y$ is Lipschitz with $L(0) = 0$, then there exists $\overline{L} : \mathcal{F}(M) \to Y$ linear such that $\|\overline{L}\| = Lip\,(L)$ and $\overline{L}\delta_M = L$.
So $B(\mathcal{F}(M), Y) \equiv Lip_0(M, Y)$ and in particular $\mathcal{F}(M)^* \equiv Lip_0(M)$.
Weaver - arXiv 2017. If M is bounded or is complete and convex, then $Lip_0(M)$ has a unique predual.

• If X is a Banach space, then there exists a quotient map $\beta_X : \mathcal{F}(X) \to X$ such that $\|\beta_X\| \leq 1$ and $\beta_X \delta_X = Id_X$.

Basics on free spaces.

Let (M,d) be a metric space with origin 0. Then

$$Lip_0(M) = \{f : M \to \mathbb{R},\ f \text{ Lipschitz},\ f(0) = 0\}.$$

For $f \in Lip_0(M)$, $\|f\|_L = Lip\,(f)$.
For $x \in M$ and $f \in Lip_0(M)$, $\langle \delta_M(x), f\rangle = f(x)$.
Then δ_M is an isometry from M into $Lip_0(M)^*$
and we define $\mathcal{F}(M) = \overline{sp}\{\delta_M(x),\ x \in M\}$.

Fundamental property : If Y is a Banach space and $L : M \to Y$ is Lipschitz with $L(0) = 0$, then there exists $\overline{L} : \mathcal{F}(M) \to Y$ linear such that $\|\overline{L}\| = Lip\,(L)$ and $\overline{L}\delta_M = L$.
So $B(\mathcal{F}(M), Y) \equiv Lip_0(M, Y)$ and in particular $\mathcal{F}(M)^* \equiv Lip_0(M)$.
Weaver - arXiv 2017. If M is bounded or is complete and convex, then $Lip_0(M)$ has a unique predual.

- If X is a Banach space, then there exists a quotient map $\beta_X : \mathcal{F}(X) \to X$ such that $\|\beta_X\| \leq 1$ and $\beta_X \delta_X = Id_X$.

- The map δ_X is an isometric (non linear) lifting of β_X.

Basics on free spaces.

Let (M,d) be a metric space with origin 0. Then

$$Lip_0(M) = \{f : M \to \mathbb{R},\ f \text{ Lipschitz},\ f(0) = 0\}.$$

For $f \in Lip_0(M)$, $\|f\|_L = Lip\,(f)$.
For $x \in M$ and $f \in Lip_0(M)$, $\langle \delta_M(x), f \rangle = f(x)$.
Then δ_M is an isometry from M into $Lip_0(M)^*$
and we define $\mathcal{F}(M) = \overline{sp}\{\delta_M(x),\ x \in M\}$.

Fundamental property : If Y is a Banach space and $L : M \to Y$ is Lipschitz with $L(0) = 0$, then there exists $\overline{L} : \mathcal{F}(M) \to Y$ linear such that $\|\overline{L}\| = Lip\,(L)$ and $\overline{L}\delta_M = L$.
So $B(\mathcal{F}(M), Y) \equiv Lip_0(M, Y)$ and in particular $\mathcal{F}(M)^* \equiv Lip_0(M)$.
Weaver - arXiv 2017. If M is bounded or is complete and convex, then $Lip_0(M)$ has a unique predual.

- If X is a Banach space, then there exists a quotient map $\beta_X : \mathcal{F}(X) \to X$ such that $\|\beta_X\| \leq 1$ and $\beta_X \delta_X = Id_X$.

- The map δ_X is an isometric (non linear) lifting of β_X.

Godefroy and Kalton showed that if X is separable, then β_X admits a linear isometric lifting V.

Basics on free spaces.

Let (M,d) be a metric space with origin 0. Then

$$Lip_0(M) = \{f : M \to \mathbb{R},\ f \text{ Lipschitz},\ f(0) = 0\}.$$

For $f \in Lip_0(M)$, $\|f\|_L = Lip\,(f)$.
For $x \in M$ and $f \in Lip_0(M)$, $\langle \delta_M(x), f\rangle = f(x)$.
Then δ_M is an isometry from M into $Lip_0(M)^*$
and we define $\mathcal{F}(M) = \overline{sp}\{\delta_M(x),\ x \in M\}$.

Fundamental property : If Y is a Banach space and $L : M \to Y$ is Lipschitz with $L(0) = 0$, then there exists $\overline{L} : \mathcal{F}(M) \to Y$ linear such that $\|\overline{L}\| = Lip\,(L)$ and $\overline{L}\delta_M = L$.
So $B(\mathcal{F}(M), Y) \equiv Lip_0(M, Y)$ and in particular $\mathcal{F}(M)^* \equiv Lip_0(M)$.
Weaver - arXiv 2017. If M is bounded or is complete and convex, then $Lip_0(M)$ has a unique predual.

- If X is a Banach space, then there exists a quotient map $\beta_X : \mathcal{F}(X) \to X$ such that $\|\beta_X\| \leq 1$ and $\beta_X\delta_X = Id_X$.

- The map δ_X is an isometric (non linear) lifting of β_X.

Godefroy and Kalton showed that if X is separable, then β_X admits a linear isometric lifting V. Then the general case follows.

Sketch of proof of Godefroy-Kalton.

Sketch of proof of Godefroy-Kalton.

Let $(x_n)_{n=1}^{\infty} \subset X$ be linearly independent, such that $\overline{sp}(x_n) = X$ and $\|x_n\| = 2^{-n}$.
Denote λ_N the Lebesgue measure on $[0,1]^N$, $E_N = sp\{x_1, .., x_N\}$.

Sketch of proof of Godefroy-Kalton.

Let $(x_n)_{n=1}^{\infty} \subset X$ be linearly independent, such that $\overline{sp}(x_n) = X$ and $\|x_n\| = 2^{-n}$.
Denote λ_N the Lebesgue measure on $[0,1]^N$, $E_N = sp\{x_1, .., x_N\}$.
$A_N^n = \{k \leq N,\ k \neq n\}$ and λ_N^n the Lebesgue measure on $[0,1]^{A_N^n}$.

Sketch of proof of Godefroy-Kalton.

Let $(x_n)_{n=1}^{\infty} \subset X$ be linearly independent, such that $\overline{sp}(x_n) = X$ and $\|x_n\| = 2^{-n}$.
Denote λ_N the Lebesgue measure on $[0,1]^N$, $E_N = sp\{x_1, .., x_N\}$.
$A_N^n = \{k \leq N,\ k \neq n\}$ and λ_N^n the Lebesgue measure on $[0,1]^{A_N^n}$.

Define $V_N : E_N \to \mathcal{F}(X)$ to be linear and such that

$$\forall n \leq N \quad V_N(x_n) = \int_{[0,1]^{A_N^n}} \Big(\delta_X(x_n + \sum_{k \in A_N^n} t_k x_k) - \delta_X(\sum_{k \in A_N^n} t_k x_k)\Big)\, d\lambda_N^n.$$

Sketch of proof of Godefroy-Kalton.

Let $(x_n)_{n=1}^{\infty} \subset X$ be linearly independent, such that $\overline{sp}(x_n) = X$ and $\|x_n\| = 2^{-n}$.
Denote λ_N the Lebesgue measure on $[0,1]^N$, $E_N = sp\{x_1, .., x_N\}$.
$A_N^n = \{k \leq N,\ k \neq n\}$ and λ_N^n the Lebesgue measure on $[0,1]^{A_N^n}$.

Define $V_N : E_N \to \mathcal{F}(X)$ to be linear and such that

$$\forall n \leq N \quad V_N(x_n) = \int_{[0,1]^{A_N^n}} \Big(\delta_X(x_n + \sum_{k \in A_N^n} t_k x_k) - \delta_X(\sum_{k \in A_N^n} t_k x_k)\Big)\, d\lambda_N^n.$$

Clearly $\beta_X V_N = Id_{E_N}$.

Sketch of proof of Godefroy-Kalton.

Let $(x_n)_{n=1}^\infty \subset X$ be linearly independent, such that $\overline{sp}(x_n) = X$ and $\|x_n\| = 2^{-n}$.
Denote λ_N the Lebesgue measure on $[0,1]^N$, $E_N = sp\{x_1, .., x_N\}$.
$A_N^n = \{k \le N,\ k \neq n\}$ and λ_N^n the Lebesgue measure on $[0,1]^{A_N^n}$.

Define $V_N : E_N \to \mathcal{F}(X)$ to be linear and such that

$$\forall n \le N \quad V_N(x_n) = \int_{[0,1]^{A_N^n}} \Big(\delta_X(x_n + \sum_{k \in A_N^n} t_k x_k) - \delta_X(\sum_{k \in A_N^n} t_k x_k)\Big)\, d\lambda_N^n.$$

Clearly $\beta_X V_N = Id_{E_N}$.

Key step : $\|V_N\| \le 1$ and therefore V_N is a linear isometry.

Sketch of proof of Godefroy-Kalton.

Let $(x_n)_{n=1}^{\infty} \subset X$ be linearly independent, such that $\overline{sp}(x_n) = X$ and $\|x_n\| = 2^{-n}$.
Denote λ_N the Lebesgue measure on $[0,1]^N$, $E_N = sp\{x_1, .., x_N\}$.
$A_N^n = \{k \leq N,\ k \neq n\}$ and λ_N^n the Lebesgue measure on $[0,1]^{A_N^n}$.

Define $V_N : E_N \to \mathcal{F}(X)$ to be linear and such that

$$\forall n \leq N \quad V_N(x_n) = \int_{[0,1]^{A_N^n}} \Big(\delta_X(x_n + \sum_{k \in A_N^n} t_k x_k) - \delta_X(\sum_{k \in A_N^n} t_k x_k)\Big)\, d\lambda_N^n.$$

Clearly $\beta_X V_N = Id_{E_N}$.

Key step : $\|V_N\| \leq 1$ and therefore V_N is a linear isometry.
(Approximation by C^1 functions + fundamental theorem of calculus)

Sketch of proof of Godefroy-Kalton.

Let $(x_n)_{n=1}^{\infty} \subset X$ be linearly independent, such that $\overline{sp}(x_n) = X$ and $\|x_n\| = 2^{-n}$.
Denote λ_N the Lebesgue measure on $[0,1]^N$, $E_N = sp\{x_1, .., x_N\}$.
$A_N^n = \{k \leq N,\ k \neq n\}$ and λ_N^n the Lebesgue measure on $[0,1]^{A_N^n}$.

Define $V_N : E_N \to \mathcal{F}(X)$ to be linear and such that

$$\forall n \leq N \quad V_N(x_n) = \int_{[0,1]^{A_N^n}} \Big(\delta_X(x_n + \sum_{k \in A_N^n} t_k x_k) - \delta_X(\sum_{k \in A_N^n} t_k x_k)\Big)\, d\lambda_N^n.$$

Clearly $\beta_X V_N = Id_{E_N}$.

Key step : $\|V_N\| \leq 1$ and therefore V_N is a linear isometry.
(Approximation by C^1 functions + fundamental theorem of calculus)

Finally : $\forall n \leq N, \quad \|V_{N+1}(x_n) - V_N(x_n)\| \leq 2\|x_{N+1}\| \leq 2^{-N}$.

Sketch of proof of Godefroy-Kalton.

Let $(x_n)_{n=1}^{\infty} \subset X$ be linearly independent, such that $\overline{sp}(x_n) = X$ and $\|x_n\| = 2^{-n}$.
Denote λ_N the Lebesgue measure on $[0,1]^N$, $E_N = sp\{x_1, .., x_N\}$.
$A_N^n = \{k \leq N,\ k \neq n\}$ and λ_N^n the Lebesgue measure on $[0,1]^{A_N^n}$.

Define $V_N : E_N \to \mathcal{F}(X)$ to be linear and such that

$$\forall n \leq N \quad V_N(x_n) = \int_{[0,1]^{A_N^n}} \Big(\delta_X(x_n + \sum_{k \in A_N^n} t_k x_k) - \delta_X(\sum_{k \in A_N^n} t_k x_k)\Big)\, d\lambda_N^n.$$

Clearly $\beta_X V_N = Id_{E_N}$.

Key step : $\|V_N\| \leq 1$ and therefore V_N is a linear isometry.
(Approximation by C^1 functions + fundamental theorem of calculus)

Finally : $\forall n \leq N, \quad \|V_{N+1}(x_n) - V_N(x_n)\| \leq 2\|x_{N+1}\| \leq 2^{-N}$.
So we can define,

$$\forall x \in E_N \quad V(x) := \lim_{M \to \infty, M \geq N} V_M(x).$$

Sketch of proof of Godefroy-Kalton.

Let $(x_n)_{n=1}^{\infty} \subset X$ be linearly independent, such that $\overline{sp}(x_n) = X$ and $\|x_n\| = 2^{-n}$.
Denote λ_N the Lebesgue measure on $[0,1]^N$, $E_N = sp\{x_1, .., x_N\}$.
$A_N^n = \{k \leq N,\ k \neq n\}$ and λ_N^n the Lebesgue measure on $[0,1]^{A_N^n}$.

Define $V_N : E_N \to \mathcal{F}(X)$ to be linear and such that

$$\forall n \leq N \quad V_N(x_n) = \int_{[0,1]^{A_N^n}} \Big(\delta_X(x_n + \sum_{k \in A_N^n} t_k x_k) - \delta_X(\sum_{k \in A_N^n} t_k x_k)\Big)\, d\lambda_N^n.$$

Clearly $\beta_X V_N = Id_{E_N}$.

Key step : $\|V_N\| \leq 1$ and therefore V_N is a linear isometry.
(Approximation by C^1 functions + fundamental theorem of calculus)

Finally : $\forall n \leq N, \quad \|V_{N+1}(x_n) - V_N(x_n)\| \leq 2\|x_{N+1}\| \leq 2^{-N}$.
So we can define,

$$\forall x \in E_N \quad V(x) := \lim_{M \to \infty, M \geq N} V_M(x).$$

Then V extends to a linear isometry from X to Y so that $\beta_X V = Id_X$. $\square$

General case.

Assume that X is a separable Banach space, $Q : Y \to X$ is a quotient map and $L : X \to Y$ is a Lipschitz lifting of Q ($QL = Id_X$).

General case.

Assume that X is a separable Banach space, $Q : Y \to X$ is a quotient map and $L : X \to Y$ is a Lipschitz lifting of Q ($QL = Id_X$).

Recall that there exists $\overline{L} : \mathcal{F}(X) \to Y$ linear with $\|\overline{L}\| = Lip(L)$ so that $\overline{L}\delta_X = L$ and a linear isometry $V : X \to \mathcal{F}(X)$ such that $\beta_X V = Id_X$.

General case.

Assume that X is a separable Banach space, $Q : Y \to X$ is a quotient map and $L : X \to Y$ is a Lipschitz lifting of Q ($QL = Id_X$).

Recall that there exists $\overline{L} : \mathcal{F}(X) \to Y$ linear with $\|\overline{L}\| = Lip(L)$ so that $\overline{L}\delta_X = L$ and a linear isometry $V : X \to \mathcal{F}(X)$ such that $\beta_X V = Id_X$.

Since

$$Q\overline{L}\delta_X = QL = Id_X = \beta_X \delta_X,$$

General case.

Assume that X is a separable Banach space, $Q : Y \to X$ is a quotient map and $L : X \to Y$ is a Lipschitz lifting of Q ($QL = Id_X$).

Recall that there exists $\overline{L} : \mathcal{F}(X) \to Y$ linear with $\|\overline{L}\| = Lip(L)$ so that $\overline{L}\delta_X = L$ and a linear isometry $V : X \to \mathcal{F}(X)$ such that $\beta_X V = Id_X$.

Since

$$Q\overline{L}\delta_X = QL = Id_X = \beta_X \delta_X,$$

we get that

$$Q\overline{L} = \beta_X \ \text{ and } \ Q\overline{L}V = \beta_X V = Id_X.$$

General case.

Assume that X is a separable Banach space, $Q : Y \to X$ is a quotient map and $L : X \to Y$ is a Lipschitz lifting of Q ($QL = Id_X$).

Recall that there exists $\overline{L} : \mathcal{F}(X) \to Y$ linear with $\|\overline{L}\| = Lip(L)$ so that $\overline{L}\delta_X = L$ and a linear isometry $V : X \to \mathcal{F}(X)$ such that $\beta_X V = Id_X$.

Since

$$Q\overline{L}\delta_X = QL = Id_X = \beta_X \delta_X,$$

we get that

$$Q\overline{L} = \beta_X \ \text{ and } \ Q\overline{L}V = \beta_X V = Id_X.$$

The operator $T = \overline{L}V$ is a linear lifting of Q with $\|T\| \leq Lip(L)$. □

General case.

Assume that X is a separable Banach space, $Q : Y \to X$ is a quotient map and $L : X \to Y$ is a Lipschitz lifting of Q ($QL = Id_X$).

Recall that there exists $\overline{L} : \mathcal{F}(X) \to Y$ linear with $\|\overline{L}\| = Lip(L)$ so that $\overline{L}\delta_X = L$ and a linear isometry $V : X \to \mathcal{F}(X)$ such that $\beta_X V = Id_X$.

Since

$$Q\overline{L}\delta_X = QL = Id_X = \beta_X \delta_X,$$

we get that

$$Q\overline{L} = \beta_X \ \text{ and } \ Q\overline{L}V = \beta_X V = Id_X.$$

The operator $T = \overline{L}V$ is a linear lifting of Q with $\|T\| \leq Lip(L)$. □

Remarks.

General case.

Assume that X is a separable Banach space, $Q : Y \to X$ is a quotient map and $L : X \to Y$ is a Lipschitz lifting of Q ($QL = Id_X$).

Recall that there exists $\overline{L} : \mathcal{F}(X) \to Y$ linear with $\|\overline{L}\| = Lip\,(L)$ so that $\overline{L}\delta_X = L$ and a linear isometry $V : X \to \mathcal{F}(X)$ such that $\beta_X V = Id_X$.

Since

$$Q\overline{L}\delta_X = QL = Id_X = \beta_X \delta_X,$$

we get that

$$Q\overline{L} = \beta_X \ \text{ and } \ Q\overline{L}V = \beta_X V = Id_X.$$

The operator $T = \overline{L}V$ is a linear lifting of Q with $\|T\| \leq Lip\,(L)$. □

Remarks.

1. Godefroy-Kalton's lifting theorem is false in the non separable setting (Aharoni-Lindenstrauss 1978).

General case.

Assume that X is a separable Banach space, $Q : Y \to X$ is a quotient map and $L : X \to Y$ is a Lipschitz lifting of Q ($QL = Id_X$).

Recall that there exists $\overline{L} : \mathcal{F}(X) \to Y$ linear with $\|\overline{L}\| = Lip(L)$ so that $\overline{L}\delta_X = L$ and a linear isometry $V : X \to \mathcal{F}(X)$ such that $\beta_X V = Id_X$.

Since

$$Q\overline{L}\delta_X = QL = Id_X = \beta_X \delta_X,$$

we get that

$$Q\overline{L} = \beta_X \quad \text{and} \quad Q\overline{L}V = \beta_X V = Id_X.$$

The operator $T = \overline{L}V$ is a linear lifting of Q with $\|T\| \leq Lip(L)$. $\square$

Remarks.

1. Godefroy-Kalton's lifting theorem is false in the non separable setting (Aharoni-Lindenstrauss 1978). Any non separable WCG space fails the lifting property (Godefroy-Kalton).

General case.

Assume that X is a separable Banach space, $Q : Y \to X$ is a quotient map and $L : X \to Y$ is a Lipschitz lifting of Q ($QL = Id_X$).

Recall that there exists $\overline{L} : \mathcal{F}(X) \to Y$ linear with $\|\overline{L}\| = Lip\,(L)$ so that $\overline{L}\delta_X = L$ and a linear isometry $V : X \to \mathcal{F}(X)$ such that $\beta_X V = Id_X$.

Since

$$Q\overline{L}\delta_X = QL = Id_X = \beta_X \delta_X,$$

we get that

$$Q\overline{L} = \beta_X \text{ and } Q\overline{L}V = \beta_X V = Id_X.$$

The operator $T = \overline{L}V$ is a linear lifting of Q with $\|T\| \leq Lip\,(L)$. $\square$

Remarks.

1. Godefroy-Kalton's lifting theorem is false in the non separable setting (Aharoni-Lindenstrauss 1978). Any non separable WCG space fails the lifting property (Godefroy-Kalton).

2. Their Corollary about isometric embeddings is also false in the non separable case (Godefroy-Kalton 2003) :

General case.

Assume that X is a separable Banach space, $Q : Y \to X$ is a quotient map and $L : X \to Y$ is a Lipschitz lifting of Q ($QL = Id_X$).

Recall that there exists $\overline{L} : \mathcal{F}(X) \to Y$ linear with $\|\overline{L}\| = Lip(L)$ so that $\overline{L}\delta_X = L$ and a linear isometry $V : X \to \mathcal{F}(X)$ such that $\beta_X V = Id_X$.

Since

$$Q\overline{L}\delta_X = QL = Id_X = \beta_X \delta_X,$$

we get that

$$Q\overline{L} = \beta_X \ \text{ and } \ Q\overline{L}V = \beta_X V = Id_X.$$

The operator $T = \overline{L}V$ is a linear lifting of Q with $\|T\| \leq Lip(L)$. □

Remarks.

1. Godefroy-Kalton's lifting theorem is false in the non separable setting (Aharoni-Lindenstrauss 1978). Any non separable WCG space fails the lifting property (Godefroy-Kalton).

2. Their Corollary about isometric embeddings is also false in the non separable case (Godefroy-Kalton 2003) :
ℓ_∞ does not linearly embed into $\mathcal{F}(\ell_\infty)$.

I.2. Other consequences of Figiel and Godefroy-Kalton.

I.2. Other consequences of Figiel and Godefroy-Kalton.

Definition

Let X be a Banach space and F be a subset of X. We say that F is an *isometrically representing subset* (IRS) for X if all Banach spaces containing an isometric copy of F contain a subspace (linearly) isometric to X.

I.2. Other consequences of Figiel and Godefroy-Kalton.

Definition

Let X be a Banach space and F be a subset of X. We say that F is an *isometrically representing subset* (IRS) for X if all Banach spaces containing an isometric copy of F contain a subspace (linearly) isometric to X.

Questions.

I.2. Other consequences of Figiel and Godefroy-Kalton.

Definition

Let X be a Banach space and F be a subset of X. We say that F is an *isometrically representing subset* (IRS) for X if all Banach spaces containing an isometric copy of F contain a subspace (linearly) isometric to X.

Questions.

1. Let X be a separable Banach space. Is B_X IRS for X?

I.2. Other consequences of Figiel and Godefroy-Kalton.

Definition

Let X be a Banach space and F be a subset of X. We say that F is an *isometrically representing subset* (IRS) for X if all Banach spaces containing an isometric copy of F contain a subspace (linearly) isometric to X.

Questions.

1. Let X be a separable Banach space. Is B_X IRS for X ?

2. Let X be a separable Banach space. Does X admit a compact IRS subset ?

I.2. Other consequences of Figiel and Godefroy-Kalton.

Definition

Let X be a Banach space and F be a subset of X. We say that F is an *isometrically representing subset* (IRS) for X if all Banach spaces containing an isometric copy of F contain a subspace (linearly) isometric to X.

Questions.

1. Let X be a separable Banach space. Is B_X IRS for X ?

2. Let X be a separable Banach space. Does X admit a compact IRS subset ?

3. Assume that X and Y are separable Banach spaces with the same compact subsets. Do they embed isometrically into each other ?

Proposition

Let X be a separable Banach space and F be a closed convex and total subset of X, with $0 \in F$. Assume that there exists an isometry U from F into a Banach space Y such that $U(0) = 0$ and

$$\forall x_1, .., x_n \in F \;\; \forall \lambda_1, .., \lambda_n \in \mathbb{R} \;\; \Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \Big\| \sum_{k=1}^{n} \lambda_k x_k \Big\|_X . \qquad (*)$$

Then X is linearly isometric to a subspace of Y.

Proposition

Let X be a separable Banach space and F be a closed convex and total subset of X, with $0 \in F$. Assume that there exists an isometry U from F into a Banach space Y such that $U(0) = 0$ and

$$\forall x_1, .., x_n \in F \ \ \forall \lambda_1, .., \lambda_n \in \mathbb{R} \ \ \Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \| \sum_{k=1}^{n} \lambda_k x_k \|_X . \qquad (*)$$

Then X is linearly isometric to a subspace of Y.

Proof. Read carefully the proof of Godefroy-Kalton's Theorem :

Proposition

Let X be a separable Banach space and F be a closed convex and total subset of X, with $0 \in F$. Assume that there exists an isometry U from F into a Banach space Y such that $U(0) = 0$ and

$$\forall x_1, .., x_n \in F \ \ \forall \lambda_1, .., \lambda_n \in \mathbb{R} \ \ \Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \big\| \sum_{k=1}^{n} \lambda_k x_k \big\|_X . \qquad (*)$$

Then X is linearly isometric to a subspace of Y.

Proof. Read carefully the proof of Godefroy-Kalton's Theorem :
There exists a linear isometry $V : X \to \mathcal{F}(F)$, such that $\beta_X V = I_X$.

Proposition

Let X be a separable Banach space and F be a closed convex and total subset of X, with $0 \in F$. Assume that there exists an isometry U from F into a Banach space Y such that $U(0) = 0$ and

$$\forall x_1, .., x_n \in F \ \ \forall \lambda_1, .., \lambda_n \in \mathbb{R} \ \ \Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \Big\| \sum_{k=1}^{n} \lambda_k x_k \Big\|_X . \quad (*)$$

Then X is linearly isometric to a subspace of Y.

Proof. Read carefully the proof of Godefroy-Kalton's Theorem :
There exists a linear isometry $V : X \to \mathcal{F}(F)$, such that $\beta_X V = I_X$.

By $(*)$, $\exists Q : Y \to X$ linear, $\|Q\| \leq 1$ and $QU = I_X$ on F (assume $Y = \overline{sp}(U(X))$.

Proposition

Let X be a separable Banach space and F be a closed convex and total subset of X, with $0 \in F$. Assume that there exists an isometry U from F into a Banach space Y such that $U(0) = 0$ and

$$\forall x_1, .., x_n \in F \ \ \forall \lambda_1, .., \lambda_n \in \mathbb{R} \ \ \Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \big\| \sum_{k=1}^{n} \lambda_k x_k \big\|_X. \qquad (*)$$

Then X is linearly isometric to a subspace of Y.

Proof. Read carefully the proof of Godefroy-Kalton's Theorem :
There exists a linear isometry $V : X \to \mathcal{F}(F)$, such that $\beta_X V = I_X$.

By $(*)$, $\exists Q : Y \to X$ linear, $\|Q\| \leq 1$ and $QU = I_X$ on F (assume $Y = \overline{sp}(U(X))$.

Let $\overline{U} : \mathcal{F}(F) \to Y$ linear with $\overline{U}\delta_F = U$ and $\|\overline{U}\| \leq 1$.

Proposition

Let X be a separable Banach space and F be a closed convex and total subset of X, with $0 \in F$. Assume that there exists an isometry U from F into a Banach space Y such that $U(0) = 0$ and

$$\forall x_1, .., x_n \in F \ \ \forall \lambda_1, .., \lambda_n \in \mathbb{R} \ \ \Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \Big\| \sum_{k=1}^{n} \lambda_k x_k \Big\|_X. \qquad (*)$$

Then X is linearly isometric to a subspace of Y.

Proof. Read carefully the proof of Godefroy-Kalton's Theorem :
There exists a linear isometry $V : X \to \mathcal{F}(F)$, such that $\beta_X V = I_X$.

By $(*)$, $\exists Q : Y \to X$ linear, $\|Q\| \leq 1$ and $QU = I_X$ on F (assume $Y = \overline{sp}(U(X))$.

Let $\overline{U} : \mathcal{F}(F) \to Y$ linear with $\overline{U}\delta_F = U$ and $\|\overline{U}\| \leq 1$.

So $Q\overline{U}\delta_F = QU = \beta_X \delta_F$ and $Q\overline{U} = \beta_X$ on $\mathcal{F}(F)$.

Proposition

Let X be a separable Banach space and F be a closed convex and total subset of X, with $0 \in F$. Assume that there exists an isometry U from F into a Banach space Y such that $U(0) = 0$ and

$$\forall x_1, .., x_n \in F \;\; \forall \lambda_1, .., \lambda_n \in \mathbb{R} \;\; \Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \| \sum_{k=1}^{n} \lambda_k x_k \|_X . \qquad (*)$$

Then X is linearly isometric to a subspace of Y.

Proof. Read carefully the proof of Godefroy-Kalton's Theorem :
There exists a linear isometry $V : X \to \mathcal{F}(F)$, such that $\beta_X V = I_X$.

By $(*)$, $\exists Q : Y \to X$ linear, $\|Q\| \leq 1$ and $QU = I_X$ on F (assume $Y = \overline{sp}(U(X))$.

Let $\overline{U} : \mathcal{F}(F) \to Y$ linear with $\overline{U}\delta_F = U$ and $\|\overline{U}\| \leq 1$.

So $Q\overline{U}\delta_F = QU = \beta_X \delta_F$ and $Q\overline{U} = \beta_X$ on $\mathcal{F}(F)$.

Finally $Q\overline{U}V = \beta_X V = I_X$.

Proposition

Let X be a separable Banach space and F be a closed convex and total subset of X, with $0 \in F$. Assume that there exists an isometry U from F into a Banach space Y such that $U(0) = 0$ and

$$\forall x_1, .., x_n \in F \;\; \forall \lambda_1, .., \lambda_n \in \mathbb{R} \;\; \Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \| \sum_{k=1}^{n} \lambda_k x_k \|_X . \qquad (*)$$

Then X is linearly isometric to a subspace of Y.

Proof. Read carefully the proof of Godefroy-Kalton's Theorem :
There exists a linear isometry $V : X \to \mathcal{F}(F)$, such that $\beta_X V = I_X$.

By $(*)$, $\exists Q : Y \to X$ linear, $\|Q\| \leq 1$ and $QU = I_X$ on F (assume $Y = \overline{sp}(U(X))$.

Let $\overline{U} : \mathcal{F}(F) \to Y$ linear with $\overline{U}\delta_F = U$ and $\|\overline{U}\| \leq 1$.

So $Q\overline{U}\delta_F = QU = \beta_X \delta_F$ and $Q\overline{U} = \beta_X$ on $\mathcal{F}(F)$.

Finally $Q\overline{U}V = \beta_X V = I_X$. So $\overline{U}V$ is a linear isometric embedding from X into Y. □

Definition

Let X be a separable Banach space and F be a closed convex and total subset of X, with $0 \in F$. We say that F has the *Uniform Figiel Property* (UF) if there exists $r \in (0,1]$ such that

$$\forall x_1, .., x_n \in rF \;\; \forall \lambda_1, .., \lambda_n \in \mathbb{R} \;\; \Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \| \sum_{k=1}^{n} \lambda_k x_k \|_X ,$$

Definition

Let X be a separable Banach space and F be a closed convex and total subset of X, with $0 \in F$. We say that F has the *Uniform Figiel Property* (UF) if there exists $r \in (0,1]$ such that

$$\forall x_1, .., x_n \in rF \ \ \forall \lambda_1, .., \lambda_n \in \mathbb{R} \ \ \Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \| \sum_{k=1}^{n} \lambda_k x_k \|_X,$$

whenever $U : F \to Y$ is an isometry such that $U(0) = 0$.

Definition

Let X be a separable Banach space and F be a closed convex and total subset of X, with $0 \in F$. We say that F has the *Uniform Figiel Property* (UF) if there exists $r \in (0,1]$ such that

$$\forall x_1, .., x_n \in rF \;\; \forall \lambda_1, .., \lambda_n \in \mathbb{R} \;\; \Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \| \sum_{k=1}^{n} \lambda_k x_k \|_X,$$

whenever $U : F \to Y$ is an isometry such that $U(0) = 0$.

It follows from the previous Proposition that F is (IRS) for X, whenever it has property (UF).

Definition

Let X be a separable Banach space and F be a closed convex and total subset of X, with $0 \in F$. We say that F has the *Uniform Figiel Property* (UF) if there exists $r \in (0, 1]$ such that

$$\forall x_1, .., x_n \in rF \ \ \forall \lambda_1, .., \lambda_n \in \mathbb{R} \ \ \Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \| \sum_{k=1}^{n} \lambda_k x_k \|_X,$$

whenever $U : F \to Y$ is an isometry such that $U(0) = 0$.

It follows from the previous Proposition that F is (IRS) for X, whenever it has property (UF).

Example 1

Let X be a finite dimensional polyhedral Banach space. Then B_X has property (UF) (and is therefore IRS for X).

Proof. Let $U : B_X \to Y$ be an isometry with $U(0) = 0$.

Proof. Let $U : B_X \to Y$ be an isometry with $U(0) = 0$.
We can write $B_X = \bigcap_{i=1}^{l} \{x \in X,\ |x_i^*(x)| \leq 1\}$, with $x_i^* \in S_{X^*}$.

Proof. Let $U : B_X \to Y$ be an isometry with $U(0) = 0$.
We can write $B_X = \bigcap_{i=1}^{l} \{x \in X,\ |x_i^*(x)| \leq 1\}$, with $x_i^* \in S_{X^*}$.
We may assume that there exists $(u_i)_{i=1}^{l}$ in S_X such that $x_i^*(u_i) = 1$ and for all $i \neq j$, $|x_j^*(u_i)| < 1$.

Proof. Let $U : B_X \to Y$ be an isometry with $U(0) = 0$.
We can write $B_X = \bigcap_{i=1}^{l} \{x \in X,\ |x_i^*(x)| \leq 1\}$, with $x_i^* \in S_{X^*}$.
We may assume that there exists $(u_i)_{i=1}^{l}$ in S_X such that $x_i^*(u_i) = 1$ and for all $i \neq j$, $|x_j^*(u_i)| < 1$. It follows that

$$\exists r \in (0,1] \ \ \forall x \in rB_X \ \ \|u_i - x\| = x_i^*(u_i - x).$$

Proof. Let $U : B_X \to Y$ be an isometry with $U(0) = 0$.
We can write $B_X = \bigcap_{i=1}^{l} \{x \in X,\ |x_i^*(x)| \leq 1\}$, with $x_i^* \in S_{X^*}$.
We may assume that there exists $(u_i)_{i=1}^{l}$ in S_X such that $x_i^*(u_i) = 1$ and for all $i \neq j$, $|x_j^*(u_i)| < 1$. It follows that

$$\exists r \in (0,1] \ \ \forall x \in rB_X \ \ \|u_i - x\| = x_i^*(u_i - x).$$

If $f : B_X \to \mathbb{R}$ is 1-Lipschitz and such that $f(u_i) = -f(-u_i) = 1$, then

$$\forall x \in rB_X \ \ 1 \pm f(x) \leq \|u_i \pm x\| = 1 \pm x_i^*(x), \text{ so } f(x) = x_i^*(x).$$

Proof. Let $U : B_X \to Y$ be an isometry with $U(0) = 0$.
We can write $B_X = \bigcap_{i=1}^{l} \{x \in X,\ |x_i^*(x)| \le 1\}$, with $x_i^* \in S_{X^*}$.
We may assume that there exists $(u_i)_{i=1}^{l}$ in S_X such that $x_i^*(u_i) = 1$ and for all $i \neq j$, $|x_j^*(u_i)| < 1$. It follows that

$$\exists r \in (0,1] \ \ \forall x \in rB_X \ \ \|u_i - x\| = x_i^*(u_i - x).$$

If $f : B_X \to \mathbb{R}$ is 1-Lipschitz and such that $f(u_i) = -f(-u_i) = 1$, then

$$\forall x \in rB_X \ \ 1 \pm f(x) \le \|u_i \pm x\| = 1 \pm x_i^*(x), \text{ so } f(x) = x_i^*(x).$$

Pick now $y_i^* \in S_{Y^*}$ so that $\langle y_i^*, U(u_i) - U(-u_i) \rangle = 2$.

Proof. Let $U : B_X \to Y$ be an isometry with $U(0) = 0$.
We can write $B_X = \bigcap_{i=1}^{l} \{x \in X,\ |x_i^*(x)| \leq 1\}$, with $x_i^* \in S_{X^*}$.
We may assume that there exists $(u_i)_{i=1}^{l}$ in S_X such that $x_i^*(u_i) = 1$ and for all $i \neq j$, $|x_j^*(u_i)| < 1$. It follows that

$$\exists r \in (0,1] \ \ \forall x \in rB_X \ \ \|u_i - x\| = x_i^*(u_i - x).$$

If $f : B_X \to \mathbb{R}$ is 1-Lipschitz and such that $f(u_i) = -f(-u_i) = 1$, then

$$\forall x \in rB_X \ \ 1 \pm f(x) \leq \|u_i \pm x\| = 1 \pm x_i^*(x), \text{ so } f(x) = x_i^*(x).$$

Pick now $y_i^* \in S_{Y^*}$ so that $\langle y_i^*, U(u_i) - U(-u_i) \rangle = 2$.
Then $y_i^* \circ U = x_i^*$ on rB_X.

Proof. Let $U : B_X \to Y$ be an isometry with $U(0) = 0$.
We can write $B_X = \bigcap_{i=1}^{l} \{x \in X,\ |x_i^*(x)| \leq 1\}$, with $x_i^* \in S_{X^*}$.
We may assume that there exists $(u_i)_{i=1}^{l}$ in S_X such that $x_i^*(u_i) = 1$ and for all $i \neq j$, $|x_j^*(u_i)| < 1$. It follows that

$$\exists r \in (0,1] \ \ \forall x \in rB_X \ \ \|u_i - x\| = x_i^*(u_i - x).$$

If $f : B_X \to \mathbb{R}$ is 1-Lipschitz and such that $f(u_i) = -f(-u_i) = 1$, then

$$\forall x \in rB_X \ \ 1 \pm f(x) \leq \|u_i \pm x\| = 1 \pm x_i^*(x), \text{ so } f(x) = x_i^*(x).$$

Pick now $y_i^* \in S_{Y^*}$ so that $\langle y_i^*, U(u_i) - U(-u_i) \rangle = 2$.
Then $y_i^* \circ U = x_i^*$ on rB_X.
We deduce that for all $x_1, .., x_n \in rB_X$ and for all $\lambda_1, .., \lambda_n \in \mathbb{R}$:

$$\Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \sup_{1 \leq i \leq l} \Big\langle y_i^*, \sum_{k=1}^{n} \lambda_k U(x_k) \Big\rangle$$

Proof. Let $U : B_X \to Y$ be an isometry with $U(0) = 0$.
We can write $B_X = \bigcap_{i=1}^{l} \{x \in X,\ |x_i^*(x)| \leq 1\}$, with $x_i^* \in S_{X^*}$.
We may assume that there exists $(u_i)_{i=1}^{l}$ in S_X such that $x_i^*(u_i) = 1$ and for all $i \neq j$, $|x_j^*(u_i)| < 1$. It follows that

$$\exists r \in (0,1] \ \ \forall x \in rB_X \ \ \|u_i - x\| = x_i^*(u_i - x).$$

If $f : B_X \to \mathbb{R}$ is 1-Lipschitz and such that $f(u_i) = -f(-u_i) = 1$, then

$$\forall x \in rB_X \ \ 1 \pm f(x) \leq \|u_i \pm x\| = 1 \pm x_i^*(x), \text{ so } f(x) = x_i^*(x).$$

Pick now $y_i^* \in S_{Y^*}$ so that $\langle y_i^*, U(u_i) - U(-u_i)\rangle = 2$.
Then $y_i^* \circ U = x_i^*$ on rB_X.
We deduce that for all $x_1, .., x_n \in rB_X$ and for all $\lambda_1, .., \lambda_n \in \mathbb{R}$:

$$\Big\| \sum_{k=1}^{n} \lambda_k U(x_k) \Big\|_Y \geq \sup_{1 \leq i \leq l} \Big\langle y_i^*, \sum_{k=1}^{n} \lambda_k U(x_k) \Big\rangle$$

$$= \sup_{1 \leq i \leq l} \Big\langle x_i^*, \sum_{k=1}^{n} \lambda_k x_k \Big\rangle = \Big\| \sum_{k=1}^{n} \lambda_k x_k \Big\|_X.$$

$\square$

Example 2. The same proof shows that B_{c_0} has property (UF).

Example 2. The same proof shows that B_{c_0} has property (UF).

Example 3. Let $K = \{f \in C([0,1]),\ \|f\|_\infty \leq 1 \text{ and } \|f\|_L \leq 1\}$ and assume that $U : K \to Y$ is an isometry so that $U(0) = 0$.

Example 2. The same proof shows that B_{c_0} has property (UF).

Example 3. Let $K = \{f \in C([0,1]),\ \|f\|_\infty \leq 1 \text{ and } \|f\|_L \leq 1\}$ and assume that $U : K \to Y$ is an isometry so that $U(0) = 0$.
For $t \in [0,1]$, we denote $\varphi_t \in K$ the function, which affine with slope 1 on $[0,t]$, affine with slope -1 on $[t,1]$ and such that $\varphi_t(t) = 1$.

Example 2. The same proof shows that B_{c_0} has property (UF).

Example 3. Let $K = \{f \in C([0,1]),\ \|f\|_\infty \leq 1 \text{ and } \|f\|_L \leq 1\}$ and assume that $U : K \to Y$ is an isometry so that $U(0) = 0$.
For $t \in [0,1]$, we denote $\varphi_t \in K$ the function, which affine with slope 1 on $[0,t]$, affine with slope -1 on $[t,1]$ and such that $\varphi_t(t) = 1$.
If $f : K \to \mathbb{R}$ 1-Lipschitz and such that $f(\varphi_t) = -f(-\varphi_t) = 1$, then

$$\forall \psi \in K \quad 1 \pm f(\psi) \leq \|\varphi_t \pm \psi\|_\infty = 1 \pm \psi(t), \text{ so } f(\psi) = \psi(t).$$

Example 2. The same proof shows that B_{c_0} has property (UF).

Example 3. Let $K = \{f \in C([0,1]),\ \|f\|_\infty \leq 1 \text{ and } \|f\|_L \leq 1\}$ and assume that $U : K \to Y$ is an isometry so that $U(0) = 0$.
For $t \in [0,1]$, we denote $\varphi_t \in K$ the function, which affine with slope 1 on $[0,t]$, affine with slope -1 on $[t,1]$ and such that $\varphi_t(t) = 1$.
If $f : K \to \mathbb{R}$ 1-Lipschitz and such that $f(\varphi_t) = -f(-\varphi_t) = 1$, then

$$\forall \psi \in K \ \ 1 \pm f(\psi) \leq \|\varphi_t \pm \psi\|_\infty = 1 \pm \psi(t), \text{ so } f(\psi) = \psi(t).$$

Pick now $y_t^* \in S_{Y^*}$ so that $\langle y_t^*, U(\varphi_t) - U(-\varphi_t)\rangle = 2$.

Example 2. The same proof shows that B_{c_0} has property (UF).

Example 3. Let $K = \{f \in C([0,1]),\ \|f\|_\infty \leq 1 \text{ and } \|f\|_L \leq 1\}$ and assume that $U : K \to Y$ is an isometry so that $U(0) = 0$.
For $t \in [0,1]$, we denote $\varphi_t \in K$ the function, which affine with slope 1 on $[0,t]$, affine with slope -1 on $[t,1]$ and such that $\varphi_t(t) = 1$.
If $f : K \to \mathbb{R}$ 1-Lipschitz and such that $f(\varphi_t) = -f(-\varphi_t) = 1$, then

$$\forall \psi \in K \quad 1 \pm f(\psi) \leq \|\varphi_t \pm \psi\|_\infty = 1 \pm \psi(t), \text{ so } f(\psi) = \psi(t).$$

Pick now $y_t^* \in S_{Y^*}$ so that $\langle y_t^*, U(\varphi_t) - U(-\varphi_t)\rangle = 2$.
Then $y_t^* \circ U = \delta_t$ on K.

Example 2. The same proof shows that B_{c_0} has property (UF).

Example 3. Let $K = \{f \in C([0,1]),\ \|f\|_\infty \leq 1 \text{ and } \|f\|_L \leq 1\}$ and assume that $U : K \to Y$ is an isometry so that $U(0) = 0$.
For $t \in [0,1]$, we denote $\varphi_t \in K$ the function, which affine with slope 1 on $[0,t]$, affine with slope -1 on $[t,1]$ and such that $\varphi_t(t) = 1$.
If $f : K \to \mathbb{R}$ 1-Lipschitz and such that $f(\varphi_t) = -f(-\varphi_t) = 1$, then

$$\forall \psi \in K \ \ 1 \pm f(\psi) \leq \|\varphi_t \pm \psi\|_\infty = 1 \pm \psi(t), \text{ so } f(\psi) = \psi(t).$$

Pick now $y_t^* \in S_{Y^*}$ so that $\langle y_t^*, U(\varphi_t) - U(-\varphi_t)\rangle = 2$.
Then $y_t^* \circ U = \delta_t$ on K.
We deduce that for all $f_1, .., f_n \in K$ and for all $\lambda_1, .., \lambda_n \in \mathbb{R}$:

$$\Big\| \sum_{k=1}^{n} \lambda_k U(f_k) \Big\|_Y \geq \sup_{t \in [0,1]} \Big| \Big\langle y_t^*, \sum_{k=1}^{n} \lambda_k U(f_k) \Big\rangle \Big|$$

<u>**Example 2.**</u> The same proof shows that B_{c_0} has property (UF).

<u>**Example 3.**</u> Let $K = \{f \in C([0,1]),\ \|f\|_\infty \leq 1 \text{ and } \|f\|_L \leq 1\}$ and assume that $U : K \to Y$ is an isometry so that $U(0) = 0$.
For $t \in [0,1]$, we denote $\varphi_t \in K$ the function, which affine with slope 1 on $[0,t]$, affine with slope -1 on $[t,1]$ and such that $\varphi_t(t) = 1$.
If $f : K \to \mathbb{R}$ 1-Lipschitz and such that $f(\varphi_t) = -f(-\varphi_t) = 1$, then

$$\forall \psi \in K \;\; 1 \pm f(\psi) \leq \|\varphi_t \pm \psi\|_\infty = 1 \pm \psi(t), \text{ so } f(\psi) = \psi(t).$$

Pick now $y_t^* \in S_{Y^*}$ so that $\langle y_t^*, U(\varphi_t) - U(-\varphi_t)\rangle = 2$.
Then $y_t^* \circ U = \delta_t$ on K.
We deduce that for all $f_1, .., f_n \in K$ and for all $\lambda_1, .., \lambda_n \in \mathbb{R}$:

$$\Big\|\sum_{k=1}^n \lambda_k U(f_k)\Big\|_Y \geq \sup_{t\in[0,1]} \Big|\Big\langle y_t^*, \sum_{k=1}^n \lambda_k U(f_k)\Big\rangle\Big|$$

$$= \sup_{t\in[0,1]} \Big|\sum_{k=1}^n \lambda_k f_k(t)\Big| = \Big\|\sum_{k=1}^n \lambda_k f_k\Big\|_\infty.$$

Example 2. The same proof shows that B_{c_0} has property (UF).

Example 3. Let $K = \{f \in C([0,1]),\ \|f\|_\infty \leq 1 \text{ and } \|f\|_L \leq 1\}$ and assume that $U : K \to Y$ is an isometry so that $U(0) = 0$.
For $t \in [0,1]$, we denote $\varphi_t \in K$ the function, which affine with slope 1 on $[0,t]$, affine with slope -1 on $[t,1]$ and such that $\varphi_t(t) = 1$.
If $f : K \to \mathbb{R}$ 1-Lipschitz and such that $f(\varphi_t) = -f(-\varphi_t) = 1$, then

$$\forall \psi \in K \ \ 1 \pm f(\psi) \leq \|\varphi_t \pm \psi\|_\infty = 1 \pm \psi(t), \text{ so } f(\psi) = \psi(t).$$

Pick now $y_t^* \in S_{Y^*}$ so that $\langle y_t^*, U(\varphi_t) - U(-\varphi_t)\rangle = 2$.
Then $y_t^* \circ U = \delta_t$ on K.
We deduce that for all $f_1, .., f_n \in K$ and for all $\lambda_1, .., \lambda_n \in \mathbb{R}$:

$$\Big\| \sum_{k=1}^n \lambda_k U(f_k) \Big\|_Y \geq \sup_{t \in [0,1]} \Big| \Big\langle y_t^*, \sum_{k=1}^n \lambda_k U(f_k) \Big\rangle \Big|$$

$$= \sup_{t \in [0,1]} \Big| \sum_{k=1}^n \lambda_k f_k(t) \Big| = \Big\| \sum_{k=1}^n \lambda_k f_k \Big\|_\infty .$$

Thus K has property (UF).

Corollary - Dutrieux-L. (2008)

- The compact space K is IRS for $C([0,1])$.

Corollary - Dutrieux-L. (2008)

• The compact space K is IRS for $C([0,1])$. So, if a Banach space Y is isometrically universal for all compact metric spaces, then it is universal for all separable Banach spaces and isometric linear embeddings.

Corollary - Dutrieux-L. (2008)

• The compact space K is IRS for $C([0,1])$. So, if a Banach space Y is isometrically universal for all compact metric spaces, then it is universal for all separable Banach spaces and isometric linear embeddings.

•. Assume (M,d) is a compact metric space. Then $C(M)$ admits a compact IRS subset

Corollary - Dutrieux-L. (2008)

• The compact space K is IRS for $C([0,1])$. So, if a Banach space Y is isometrically universal for all compact metric spaces, then it is universal for all separable Banach spaces and isometric linear embeddings.

•. Assume (M,d) is a compact metric space. Then $C(M)$ admits a compact IRS subset

Failure of the Figiel property.

Corollary - Dutrieux-L. (2008)

• The compact space K is IRS for $C([0,1])$. So, if a Banach space Y is isometrically universal for all compact metric spaces, then it is universal for all separable Banach spaces and isometric linear embeddings.

•. Assume (M, d) is a compact metric space. Then $C(M)$ admits a compact IRS subset

Failure of the Figiel property.

Example - J. Melleray (unpublished).

Corollary - Dutrieux-L. (2008)

• The compact space K is IRS for $C([0,1])$. So, if a Banach space Y is isometrically universal for all compact metric spaces, then it is universal for all separable Banach spaces and isometric linear embeddings.

•. Assume (M,d) is a compact metric space. Then $C(M)$ admits a compact IRS subset

Failure of the Figiel property.

Example - J. Melleray (unpublished).

There exists a Banach space Y and an isometry $U : B = B_{\ell_2^2} \to Y$ such that $U(0) = 0$ and :

$$\forall r > 0 \ \ \exists x, y \in rB \quad \|U(x) + U(y)\|_Y < \|x + y\|_2.$$

I.3. Applications of descriptive set theory.

Godefroy-Kalton (2006)

If a separable Banach space X contains an isometric copy of every separable strictly convex Banach space, then X contains an isometric copy of every separable Banach space.

I.3. Applications of descriptive set theory.

Godefroy-Kalton (2006)

If a separable Banach space X contains an isometric copy of every separable strictly convex Banach space, then X contains an isometric copy of every separable Banach space.

Kurka (2012)

If a separable Banach space X contains an isometric copy of every separable reflexive Fréchet smooth Banach space, or if it contains an isometric copy of every separable Banach space with Fréchet smooth dual norm, then X contains an isometric copy of every separable Banach space.

I.3. Applications of descriptive set theory.

Godefroy-Kalton (2006)

If a separable Banach space X contains an isometric copy of every separable strictly convex Banach space, then X contains an isometric copy of every separable Banach space.

Kurka (2012)

If a separable Banach space X contains an isometric copy of every separable reflexive Fréchet smooth Banach space, or if it contains an isometric copy of every separable Banach space with Fréchet smooth dual norm, then X contains an isometric copy of every separable Banach space.

Tools. Build Banach spaces $E(T)$ for all subtrees T of $\omega^{<\omega}$ so that if T is well founded, then $E(T)$ is strictly convex and if T is not well founded, $E(T)$ is universal. Equip the set of subspaces of $E(\omega^{<\omega})$ with the Effros-Borel structure. The set A of trees T such that $E(T)$ embeds into X is analytic and contains all well founded trees, but the set of well founded trees is not analytic. So, there is T not well founded such that $E(T)$ embeds into X.

Rolewicz question : Assume that a separable Banach space X contains an isometric copy of every finite dimensional Banach space. Does this imply that X contains an isometric copy of $C([0,1])$?

Rolewicz question : Assume that a separable Banach space X contains an isometric copy of every finite dimensional Banach space. Does this imply that X contains an isometric copy of $C([0,1])$?

Szankowski (1972)

There exists a separable reflexive space which contains a 1-complemented isometric copy of every finite dimensional space.

Rolewicz question : Assume that a separable Banach space X contains an isometric copy of every finite dimensional Banach space. Does this imply that X contains an isometric copy of $C([0,1])$?

Szankowski (1972)

There exists a separable reflexive space which contains a 1-complemented isometric copy of every finite dimensional space.

Kurka (2016)

There exists a separable reflexive space which contains an isometric copy of every separable super-reflexive Banach space.

Rolewicz question : Assume that a separable Banach space X contains an isometric copy of every finite dimensional Banach space. Does this imply that X contains an isometric copy of $C([0,1])$?

Szankowski (1972)

There exists a separable reflexive space which contains a 1-complemented isometric copy of every finite dimensional space.

Kurka (2016)

There exists a separable reflexive space which contains an isometric copy of every separable super-reflexive Banach space.

Open question ? Assume that a separable Banach space X contains an isometric copy of every locally finite metric space. Does this imply that X contains an isometric copy of $C([0,1])$?

II. LIPSCHITZ EMBEDDINGS.

Definition

Let (M, d) and (N, δ) be two metric spaces and $f : M \to N$.
We say that f is a *Lipschitz embedding* if there exist $A, B > 0$ such that

$$\forall x, y \in M \quad Ad(x, y) \leq \delta(f(x), f(y)) \leq Bd(x, y).$$

Definition

Let (M, d) and (N, δ) be two metric spaces and $f : M \to N$.
We say that f is a *Lipschitz embedding* if there exist $A, B > 0$ such that

$$\forall x, y \in M \quad Ad(x, y) \leq \delta(f(x), f(y)) \leq Bd(x, y).$$

Then the distortion of f is $dist(f) = Lip(f)Lip(f^{-1})$.

Definition

Let (M, d) and (N, δ) be two metric spaces and $f : M \to N$.
We say that f is a *Lipschitz embedding* if there exist $A, B > 0$ such that

$$\forall x, y \in M \quad Ad(x, y) \leq \delta(f(x), f(y)) \leq Bd(x, y).$$

Then the distortion of f is $dist(f) = Lip(f)Lip(f^{-1})$.
If there exists a Lipschitz embedding from M into N, we denote $M \underset{L}{\hookrightarrow} N$.

Definition

Let (M, d) and (N, δ) be two metric spaces and $f : M \to N$.
We say that f is a *Lipschitz embedding* if there exist $A, B > 0$ such that

$$\forall x, y \in M \quad Ad(x,y) \leq \delta(f(x), f(y)) \leq Bd(x,y).$$

Then the distortion of f is $dist(f) = Lip(f) Lip(f^{-1})$.
If there exists a Lipschitz embedding from M into N, we denote $M \underset{L}{\hookrightarrow} N$.
If this embedding has distortion $D \geq 1$, we denote $M \overset{D}{\hookrightarrow} N$.

Definition

Let (M, d) and (N, δ) be two metric spaces and $f : M \to N$.
We say that f is a *Lipschitz embedding* if there exist $A, B > 0$ such that

$$\forall x, y \in M \quad Ad(x, y) \leq \delta(f(x), f(y)) \leq Bd(x, y).$$

Then the distortion of f is $dist(f) = Lip(f)Lip(f^{-1})$.
If there exists a Lipschitz embedding from M into N, we denote $M \underset{L}{\hookrightarrow} N$.

If this embedding has distortion $D \geq 1$, we denote $M \overset{D}{\hookrightarrow} N$.
Finally $C_M(N) = \inf\{dist(f), \text{ for } f \text{ Lipschitz embedding from } M \text{ into } N\}$.

Definition

Let (M, d) and (N, δ) be two metric spaces and $f : M \to N$.
We say that f is a *Lipschitz embedding* if there exist $A, B > 0$ such that

$$\forall x, y \in M \quad Ad(x, y) \leq \delta(f(x), f(y)) \leq Bd(x, y).$$

Then the distortion of f is $dist(f) = Lip(f)Lip(f^{-1})$.
If there exists a Lipschitz embedding from M into N, we denote $M \underset{L}{\hookrightarrow} N$.

If this embedding has distortion $D \geq 1$, we denote $M \overset{D}{\hookrightarrow} N$.
Finally $C_M(N) = \inf\{dist(f), \text{ for } f \text{ Lipschitz embedding from } M \text{ into } N\}$.

Aharoni (1974)

Let M be a separable metric space. Then $M \underset{L}{\hookrightarrow} c_0$.

Definition

Let (M, d) and (N, δ) be two metric spaces and $f : M \to N$.
We say that f is a *Lipschitz embedding* if there exist $A, B > 0$ such that

$$\forall x, y \in M \quad Ad(x, y) \leq \delta(f(x), f(y)) \leq Bd(x, y).$$

Then the distortion of f is $dist(f) = Lip(f)Lip(f^{-1})$.
If there exists a Lipschitz embedding from M into N, we denote $M \underset{L}{\hookrightarrow} N$.

If this embedding has distortion $D \geq 1$, we denote $M \overset{D}{\hookrightarrow} N$.
Finally $C_M(N) = \inf\{dist(f), \text{ for } f \text{ Lipschitz embedding from } M \text{ into } N\}$.

Aharoni (1974)

Let M be a separable metric space. Then $M \underset{L}{\hookrightarrow} c_0$.

More precisely, for any $\varepsilon > 0$, $M \overset{6+\varepsilon}{\hookrightarrow} c_0^+$

Definition

Let (M,d) and (N,δ) be two metric spaces and $f : M \to N$.
We say that f is a *Lipschitz embedding* if there exist $A, B > 0$ such that

$$\forall x,y \in M \quad Ad(x,y) \leq \delta(f(x),f(y)) \leq Bd(x,y).$$

Then the distortion of f is $dist(f) = Lip(f)Lip(f^{-1})$.
If there exists a Lipschitz embedding from M into N, we denote $M \underset{L}{\hookrightarrow} N$.

If this embedding has distortion $D \geq 1$, we denote $M \overset{D}{\hookrightarrow} N$.
Finally $C_M(N) = \inf\{dist(f), \text{ for } f \text{ Lipschitz embedding from } M \text{ into } N\}$.

Aharoni (1974)

Let M be a separable metric space. Then $M \underset{L}{\hookrightarrow} c_0$.

More precisely, for any $\varepsilon > 0$, $M \overset{6+\varepsilon}{\hookrightarrow} c_0^+$

Pelant (1994)

Let M be a separable metric space. Then $M \overset{3}{\hookrightarrow} c_0^+$.

Definition

Let (M, d) and (N, δ) be two metric spaces and $f : M \to N$.
We say that f is a *Lipschitz embedding* if there exist $A, B > 0$ such that

$$\forall x, y \in M \quad Ad(x,y) \leq \delta(f(x), f(y)) \leq Bd(x,y).$$

Then the distortion of f is $dist(f) = Lip(f)Lip(f^{-1})$.
If there exists a Lipschitz embedding from M into N, we denote $M \underset{L}{\hookrightarrow} N$.

If this embedding has distortion $D \geq 1$, we denote $M \overset{D}{\hookrightarrow} N$.
Finally $C_M(N) = \inf\{dist(f), \text{ for } f \text{ Lipschitz embedding from } M \text{ into } N\}$.

Aharoni (1974)

Let M be a separable metric space. Then $M \underset{L}{\hookrightarrow} c_0$.

More precisely, for any $\varepsilon > 0$, $M \overset{6+\varepsilon}{\hookrightarrow} c_0^+$

Pelant (1994)

Let M be a separable metric space. Then $M \overset{3}{\hookrightarrow} c_0^+$.

Kalton, L. (2008)

Let M be a separable metric space. Then $M \overset{2}{\hookrightarrow} c_0$.

Kalton, L. (2008)

Let M be a separable metric space. Then $M \overset{2}{\hookrightarrow} c_0$.

Let M be a proper metric space. Then, for any $\varepsilon > 0$, $M \overset{1+\varepsilon}{\hookrightarrow} c_0$

Kalton, L. (2008)

Let M be a separable metric space. Then $M \overset{2}{\hookrightarrow} c_0$.
Let M be a proper metric space. Then, for any $\varepsilon > 0$, $M \overset{1+\varepsilon}{\hookrightarrow} c_0$

This is optimal.

Kalton, L. (2008)

Let M be a separable metric space. Then $M \overset{2}{\hookrightarrow} c_0$.

Let M be a proper metric space. Then, for any $\varepsilon > 0$, $M \overset{1+\varepsilon}{\hookrightarrow} c_0$

This is optimal.

Open questions (around the converse of Aharoni's Theorem).

Kalton, L. (2008)

Let M be a separable metric space. Then $M \overset{2}{\hookrightarrow} c_0$.

Let M be a proper metric space. Then, for any $\varepsilon > 0$, $M \overset{1+\varepsilon}{\hookrightarrow} c_0$

This is optimal.

Open questions (around the converse of Aharoni's Theorem).

Let X be a Banach space

Kalton, L. (2008)

Let M be a separable metric space. Then $M \overset{2}{\hookrightarrow} c_0$.
Let M be a proper metric space. Then, for any $\varepsilon > 0$, $M \overset{1+\varepsilon}{\hookrightarrow} c_0$

This is optimal.

Open questions (around the converse of Aharoni's Theorem).

Let X be a Banach space

1) $c_0 \underset{L}{\hookrightarrow} X \Rightarrow c_0 \simeq Y \subset X\,?$

Kalton, L. (2008)

Let M be a separable metric space. Then $M \overset{2}{\hookrightarrow} c_0$.
Let M be a proper metric space. Then, for any $\varepsilon > 0$, $M \overset{1+\varepsilon}{\hookrightarrow} c_0$

This is optimal.

Open questions (around the converse of Aharoni's Theorem).

Let X be a Banach space

1) $c_0 \underset{L}{\hookrightarrow} X \Rightarrow c_0 \simeq Y \subset X\,?$

2) $\big(\forall\varepsilon > 0 \;\; c_0 \overset{1+\varepsilon}{\hookrightarrow} X\big) \Rightarrow c_0 \simeq Y \subset X\,?$

Kalton, L. (2008)

Let M be a separable metric space. Then $M \overset{2}{\hookrightarrow} c_0$.
Let M be a proper metric space. Then, for any $\varepsilon > 0$, $M \overset{1+\varepsilon}{\hookrightarrow} c_0$

This is optimal.

Open questions (around the converse of Aharoni's Theorem).

Let X be a Banach space

1) $c_0 \underset{L}{\hookrightarrow} X \Rightarrow c_0 \simeq Y \subset X$?

2) $\left(\forall \varepsilon > 0 \;\; c_0 \overset{1+\varepsilon}{\hookrightarrow} X\right) \Rightarrow c_0 \simeq Y \subset X$?

3) $c_0 \underset{L}{\hookrightarrow} X \Rightarrow c_0 \overset{1+\varepsilon}{\hookrightarrow} X$, for all $\varepsilon > 0$?

Kalton, L. (2008)

Let M be a separable metric space. Then $M \overset{2}{\hookrightarrow} c_0$.
Let M be a proper metric space. Then, for any $\varepsilon > 0$, $M \overset{1+\varepsilon}{\hookrightarrow} c_0$

This is optimal.

Open questions (around the converse of Aharoni's Theorem).

Let X be a Banach space

1) $c_0 \underset{L}{\hookrightarrow} X \Rightarrow c_0 \simeq Y \subset X$?

2) $\left(\forall \varepsilon > 0 \;\; c_0 \overset{1+\varepsilon}{\hookrightarrow} X\right) \Rightarrow c_0 \simeq Y \subset X$?

3) $c_0 \underset{L}{\hookrightarrow} X \Rightarrow c_0 \overset{1+\varepsilon}{\hookrightarrow} X$, for all $\varepsilon > 0$?

4) Assume that for any M metric compact (or locally finite, or proper) and any $\varepsilon > 0$, $M \overset{1+\varepsilon}{\hookrightarrow} X$.
Does this imply that $c_0 \simeq Y \subset X$?

Baudier, L. (2008)

Let X be a Banach. Then every locally finite metric space (i.e with finite balls) Lipschitz embeds into X if and only if X contains uniformly the ℓ_∞^n's.

Baudier, L. (2008)

Let X be a Banach. Then every locally finite metric space (i.e with finite balls) Lipschitz embeds into X if and only if X contains uniformly the ℓ_∞^n's.

Sketch of proof.

Baudier, L. (2008)

Let X be a Banach. Then every locally finite metric space (i.e with finite balls) Lipschitz embeds into X if and only if X contains uniformly the ℓ_∞^n's.

Sketch of proof. ($\Leftarrow$ - Schechtman :)

Baudier, L. (2008)

Let X be a Banach. Then every locally finite metric space (i.e with finite balls) Lipschitz embeds into X if and only if X contains uniformly the ℓ_∞^n's.

Sketch of proof. ($\Leftarrow$ - Schechtman :) Note first that if every locally finite metric space Lipschitz embeds into a Banach space X, then every finite metric space embeds into X with uniform distortion.

Baudier, L. (2008)

Let X be a Banach. Then every locally finite metric space (i.e with finite balls) Lipschitz embeds into X if and only if X contains uniformly the ℓ_∞^n's.

Sketch of proof. ($\Leftarrow$ - Schechtman :) Note first that if every locally finite metric space Lipschitz embeds into a Banach space X, then every finite metric space embeds into X with uniform distortion. Let us fix $n \in \mathbb{N}$.

Baudier, L. (2008)

Let X be a Banach. Then every locally finite metric space (i.e with finite balls) Lipschitz embeds into X if and only if X contains uniformly the ℓ_∞^n's.

Sketch of proof. ($\Leftarrow$ - Schechtman :) Note first that if every locally finite metric space Lipschitz embeds into a Banach space X, then every finite metric space embeds into X with uniform distortion. Let us fix $n \in \mathbb{N}$. For $k \in \mathbb{N}$, denote $M_k = (\frac{1}{k}\mathbb{Z}^n \cap B_{\ell_\infty^n}, \| \ \|_\infty)$. Then, there exists $C \geq 1$ such that for all $k \in \mathbb{N}$, there exists $f_k : M_k \to X$ so that $f_k(0) = 0$ and

$$\forall x, y \in M_k \quad \|x - y\|_\infty \leq \|f_k(x) - f_k(y)\|_X \leq C\|x - y\|_\infty.$$

Baudier, L. (2008)

Let X be a Banach. Then every locally finite metric space (i.e with finite balls) Lipschitz embeds into X if and only if X contains uniformly the ℓ_∞^n's.

Sketch of proof. ($\Leftarrow$ - Schechtman :) Note first that if every locally finite metric space Lipschitz embeds into a Banach space X, then every finite metric space embeds into X with uniform distortion. Let us fix $n \in \mathbb{N}$. For $k \in \mathbb{N}$, denote $M_k = (\frac{1}{k}\mathbb{Z}^n \cap B_{\ell_\infty^n}, \|\ \|_\infty)$. Then, there exists $C \geq 1$ such that for all $k \in \mathbb{N}$, there exists $f_k : M_k \to X$ so that $f_k(0) = 0$ and

$$\forall x, y \in M_k \quad \|x - y\|_\infty \leq \|f_k(x) - f_k(y)\|_X \leq C\|x - y\|_\infty.$$

Define now $\lambda_k : B_{\ell_\infty^n} \to M_k$ such that

$$\forall x \in B_{\ell_\infty^n} \quad \|\lambda_k(x) - x\|_\infty = d(x, M_k) \ \text{ and } \ \varphi_k = f_k \circ \lambda_k.$$

Baudier, L. (2008)

Let X be a Banach. Then every locally finite metric space (i.e with finite balls) Lipschitz embeds into X if and only if X contains uniformly the ℓ_∞^n's.

Sketch of proof. ($\Leftarrow$ - Schechtman :) Note first that if every locally finite metric space Lipschitz embeds into a Banach space X, then every finite metric space embeds into X with uniform distortion. Let us fix $n \in \mathbb{N}$. For $k \in \mathbb{N}$, denote $M_k = (\frac{1}{k}\mathbb{Z}^n \cap B_{\ell_\infty^n}, \| \ \|_\infty)$. Then, there exists $C \geq 1$ such that for all $k \in \mathbb{N}$, there exists $f_k : M_k \to X$ so that $f_k(0) = 0$ and

$$\forall x, y \in M_k \ \ \|x - y\|_\infty \leq \|f_k(x) - f_k(y)\|_X \leq C\|x - y\|_\infty.$$

Define now $\lambda_k : B_{\ell_\infty^n} \to M_k$ such that

$$\forall x \in B_{\ell_\infty^n} \ \ \|\lambda_k(x) - x\|_\infty = d(x, M_k) \ \text{ and } \ \varphi_k = f_k \circ \lambda_k.$$

Finally $\varphi(x) = (\varphi_k(x))_{k \in \mathcal{U}}$ is a bilipschitz embedding from $B_{\ell_\infty^n}$ into the ultrapower $X_\mathcal{U} \subset X_\mathcal{U}^{**}$.

Baudier, L. (2008)

Let X be a Banach. Then every locally finite metric space (i.e with finite balls) Lipschitz embeds into X if and only if X contains uniformly the ℓ_∞^n's.

Sketch of proof. ($\Leftarrow$ - Schechtman :) Note first that if every locally finite metric space Lipschitz embeds into a Banach space X, then every finite metric space embeds into X with uniform distortion. Let us fix $n \in \mathbb{N}$. For $k \in \mathbb{N}$, denote $M_k = (\frac{1}{k}\mathbb{Z}^n \cap B_{\ell_\infty^n}, \|\ \|_\infty)$. Then, there exists $C \geq 1$ such that for all $k \in \mathbb{N}$, there exists $f_k : M_k \to X$ so that $f_k(0) = 0$ and

$$\forall x, y \in M_k \ \ \|x - y\|_\infty \leq \|f_k(x) - f_k(y)\|_X \leq C\|x - y\|_\infty.$$

Define now $\lambda_k : B_{\ell_\infty^n} \to M_k$ such that

$$\forall x \in B_{\ell_\infty^n} \ \ \|\lambda_k(x) - x\|_\infty = d(x, M_k) \ \text{ and } \ \varphi_k = f_k \circ \lambda_k.$$

Finally $\varphi(x) = (\varphi_k(x))_{k \in \mathcal{U}}$ is a bilipschitz embedding from $B_{\ell_\infty^n}$ into the ultrapower $X_\mathcal{U} \subset X_\mathcal{U}^{**}$.
So it follows from Heinrich-Mankiewicz's weak*-differentiability theorem, local reflexivity and finite representability of $X_\mathcal{U}$ into X that X uniformly contains the ℓ_∞^n's.

$\underline{(\Rightarrow)}$

(⇒) Let M be a locally finite metric space. Fix $x_0 \in M$ and denote $B_n = B(x_0, 2^{n+1})$.

<u>($\Rightarrow$)</u> Let M be a locally finite metric space. Fix $x_0 \in M$ and denote $B_n = B(x_0, 2^{n+1})$. For $x \in B_n$, set $\phi_n(x) = \big(d(x,y) - d(x_0,y)\big)_{y \in B_n}$, the isometric Fréchet embedding of B_n into $\ell_\infty^{|B_n|}$.

$\underline{(\Rightarrow)}$ Let M be a locally finite metric space. Fix $x_0 \in M$ and denote $B_n = B(x_0, 2^{n+1})$. For $x \in B_n$, set $\phi_n(x) = \big(d(x,y) - d(x_0,y)\big)_{y \in B_n}$, the isometric Fréchet embedding of B_n into $\ell_\infty^{|B_n|}$.

Since $\ell_\infty^n \subseteq X$, there exists a subspace Z of X with a FDD $(Z_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, there exists $T_n : \ell_\infty^{|B_n|} \simeq Z_n$, with $\|T_n\| \leq 2$ and $\|T_n^{-1}\| \leq 1$.

$\underline{(\Rightarrow)}$ Let M be a locally finite metric space. Fix $x_0 \in M$ and denote $B_n = B(x_0, 2^{n+1})$. For $x \in B_n$, set $\phi_n(x) = \big(d(x,y) - d(x_0,y)\big)_{y \in B_n}$, the isometric Fréchet embedding of B_n into $\ell_\infty^{|B_n|}$.

Since $\ell_\infty^n \subseteq X$, there exists a subspace Z of X with a FDD $(Z_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, there exists $T_n : \ell_\infty^{|B_n|} \simeq Z_n$, with $\|T_n\| \leq 2$ and $\|T_n^{-1}\| \leq 1$.

Then set $\psi_n = T_n \circ \phi_n : B_n \to Z_n$.

$\underline{(\Rightarrow)}$ Let M be a locally finite metric space. Fix $x_0 \in M$ and denote $B_n = B(x_0, 2^{n+1})$. For $x \in B_n$, set $\phi_n(x) = \big(d(x,y) - d(x_0,y)\big)_{y \in B_n}$, the isometric Fréchet embedding of B_n into $\ell_\infty^{|B_n|}$.

Since $\ell_\infty^n \subseteq X$, there exists a subspace Z of X with a FDD $(Z_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, there exists $T_n : \ell_\infty^{|B_n|} \simeq Z_n$, with $\|T_n\| \leq 2$ and $\|T_n^{-1}\| \leq 1$.

Then set $\psi_n = T_n \circ \phi_n : B_n \to Z_n$.

Finally we define $\phi : M \to Z$ by embedding $B_n \setminus B_{n-1}$ into $Z_n \oplus Z_{n+1}$ as follows :

($\Rightarrow$) Let M be a locally finite metric space. Fix $x_0 \in M$ and denote $B_n = B(x_0, 2^{n+1})$. For $x \in B_n$, set $\phi_n(x) = \big(d(x,y) - d(x_0,y)\big)_{y \in B_n}$, the isometric Fréchet embedding of B_n into $\ell_\infty^{|B_n|}$.

Since $\ell_\infty^n \subseteq X$, there exists a subspace Z of X with a FDD $(Z_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, there exists $T_n : \ell_\infty^{|B_n|} \simeq Z_n$, with $\|T_n\| \leq 2$ and $\|T_n^{-1}\| \leq 1$.

Then set $\psi_n = T_n \circ \phi_n : B_n \to Z_n$.

Finally we define $\phi : M \to Z$ by embedding $B_n \setminus B_{n-1}$ into $Z_n \oplus Z_{n+1}$ as follows :

$$\forall x \in B_n \setminus B_{n-1} \quad \phi(x) = \lambda(x)\psi_n(x) + (1-\lambda(x))\psi_{n+1}(x),$$

where $\lambda(x) = \frac{2^{n+1} - d(x,x_0)}{2^n}$. □

(⇒) Let M be a locally finite metric space. Fix $x_0 \in M$ and denote $B_n = B(x_0, 2^{n+1})$. For $x \in B_n$, set $\phi_n(x) = \big(d(x,y) - d(x_0,y)\big)_{y \in B_n}$, the isometric Fréchet embedding of B_n into $\ell_\infty^{|B_n|}$.

Since $\ell_\infty^n \subseteq X$, there exists a subspace Z of X with a FDD $(Z_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, there exists $T_n : \ell_\infty^{|B_n|} \simeq Z_n$, with $\|T_n\| \leq 2$ and $\|T_n^{-1}\| \leq 1$.

Then set $\psi_n = T_n \circ \phi_n : B_n \to Z_n$.

Finally we define $\phi : M \to Z$ by embedding $B_n \setminus B_{n-1}$ into $Z_n \oplus Z_{n+1}$ as follows :

$$\forall x \in B_n \setminus B_{n-1} \qquad \phi(x) = \lambda(x)\psi_n(x) + (1 - \lambda(x))\psi_{n+1}(x),$$

where $\lambda(x) = \frac{2^{n+1} - d(x,x_0)}{2^n}$. □

M. Ostrovskii (2012)

Let X and Y be two Banach spaces such that Y is finitely crudely representable in X and M be a locally finite subset of Y. Then M admits a bilipschitz embedding into X.

III. COARSE AND UNIFORM EMBEDDINGS.

Definition

Let (M,d) and (N,δ) be two <u>unbounded</u> metric spaces. A map $f : M \to N$ is said to be a *coarse embedding* if there exist two increasing functions $\rho_1, \rho_2 : [0,\infty) \to [0,\infty)$ such that $\lim_\infty \rho_1 = +\infty$ and

$$\forall x, y \in M \quad \rho_1(d(x,y)) \leq \delta(f(x), f(y)) \leq \rho_2(d(x,y)).$$

Definition

Let (M, d) and (N, δ) be two unbounded metric spaces. A map $f : M \to N$ is said to be a *coarse embedding* if there exist two increasing functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ such that $\lim_\infty \rho_1 = +\infty$ and

$$\forall x, y \in M \quad \rho_1(d(x, y)) \leq \delta(f(x), f(y)) \leq \rho_2(d(x, y)).$$

An important question was to know whether a (separable) reflexive Banach space could be universal for separable metric spaces and coarse embeddings.

Definition

Let (M, d) and (N, δ) be two unbounded metric spaces. A map $f : M \to N$ is said to be a *coarse embedding* if there exist two increasing functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ such that $\lim_\infty \rho_1 = +\infty$ and

$$\forall x, y \in M \quad \rho_1(d(x, y)) \leq \delta(f(x), f(y)) \leq \rho_2(d(x, y)).$$

An important question was to know whether a (separable) reflexive Banach space could be universal for separable metric spaces and coarse embeddings.

Kalton (2007)

Assume that c_0 coarsely embeds into a separable Banach space X. Then one of the iterated duals of X has to be non separable.
In particular, X cannot be reflexive.

Definition

Let (M,d) and (N,δ) be two unbounded metric spaces. A map $f: M \to N$ is said to be a *coarse embedding* if there exist two increasing functions $\rho_1, \rho_2 : [0,\infty) \to [0,\infty)$ such that $\lim_\infty \rho_1 = +\infty$ and

$$\forall x, y \in M \quad \rho_1(d(x,y)) \leq \delta(f(x), f(y)) \leq \rho_2(d(x,y)).$$

An important question was to know whether a (separable) reflexive Banach space could be universal for separable metric spaces and coarse embeddings.

Kalton (2007)

Assume that c_0 coarsely embeds into a separable Banach space X. Then one of the iterated duals of X has to be non separable.
In particular, X cannot be reflexive.

Kalton's graphs : Let $\mathbb{M}$ be an infinite subset of $\mathbb{N}$ and $k \in \mathbb{N}$. We denote

$$G_k(\mathbb{N}) = \{\overline{n} = (n_1, .., n_k),\ n_i \in \mathbb{M} \quad n_1 < .. < n_k\}.$$

Definition

Let (M,d) and (N,δ) be two unbounded metric spaces. A map $f : M \to N$ is said to be a *coarse embedding* if there exist two increasing functions $\rho_1, \rho_2 : [0,\infty) \to [0,\infty)$ such that $\lim_\infty \rho_1 = +\infty$ and

$$\forall x,y \in M \quad \rho_1(d(x,y)) \leq \delta(f(x),f(y)) \leq \rho_2(d(x,y)).$$

An important question was to know whether a (separable) reflexive Banach space could be universal for separable metric spaces and coarse embeddings.

Kalton (2007)

Assume that c_0 coarsely embeds into a separable Banach space X. Then one of the iterated duals of X has to be non separable.
In particular, X cannot be reflexive.

Kalton's graphs : Let $\mathbb{M}$ be an infinite subset of $\mathbb{N}$ and $k \in \mathbb{N}$. We denote

$$G_k(\mathbb{N}) = \{\overline{n} = (n_1, .., n_k),\ n_i \in \mathbb{M} \quad n_1 < .. < n_k\}.$$

We say that $\overline{n} \neq \overline{m} \in G_k(\mathbb{M})$ are adjacent (or $d(\overline{n},\overline{m}) = 1$) if
$m_1 \leq n_1 \leq .. \leq m_k \leq n_k \ \text{ or } \ n_1 \leq m_1 \leq .. \leq n_k \leq m_k.$

Proof : Assume that X is reflexive and fix a non principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. For a bounded function $f : G_k(\mathbb{N}) \to X$ we define $\partial f : G_{k-1}(\mathbb{N}) \to X$ by

$$\forall \overline{n} \in G_{k-1}(\mathbb{N}) \quad \partial f(\overline{n}) = weak - \lim_{n_k \in \mathcal{U}} f(n_1, .., n_{k-1}, n_k).$$

Proof : Assume that X is reflexive and fix a non principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. For a bounded function $f : G_k(\mathbb{N}) \to X$ we define $\partial f : G_{k-1}(\mathbb{N}) \to X$ by

$$\forall \overline{n} \in G_{k-1}(\mathbb{N}) \quad \partial f(\overline{n}) = weak - \lim_{n_k \in \mathcal{U}} f(n_1, .., n_{k-1}, n_k).$$

Note that $\partial^k f \in X$.

Proof : Assume that X is reflexive and fix a non principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. For a bounded function $f : G_k(\mathbb{N}) \to X$ we define $\partial f : G_{k-1}(\mathbb{N}) \to X$ by

$$\forall \overline{n} \in G_{k-1}(\mathbb{N}) \quad \partial f(\overline{n}) = weak - \lim_{n_k \in \mathcal{U}} f(n_1, .., n_{k-1}, n_k).$$

Note that $\partial^k f \in X$.

Lemma 1

Let $h : G_k(\mathbb{N}) \to \mathbb{R}$ be a bounded map and $\varepsilon > 0$. Then there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\forall \overline{n} \in G_k(\mathbb{M}) \quad |h(\overline{n}) - \partial^k h| < \varepsilon.$$

Proof : Assume that X is reflexive and fix a non principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. For a bounded function $f : G_k(\mathbb{N}) \to X$ we define $\partial f : G_{k-1}(\mathbb{N}) \to X$ by

$$\forall \overline{n} \in G_{k-1}(\mathbb{N}) \quad \partial f(\overline{n}) = weak - \lim_{n_k \in \mathcal{U}} f(n_1, .., n_{k-1}, n_k).$$

Note that $\partial^k f \in X$.

Lemma 1

Let $h : G_k(\mathbb{N}) \to \mathbb{R}$ be a bounded map and $\varepsilon > 0$. Then there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\forall \overline{n} \in G_k(\mathbb{M}) \quad |h(\overline{n}) - \partial^k h| < \varepsilon.$$

Lemma 2

Let $f : G_k(\mathbb{N}) \to X$ and $g : G_k(\mathbb{M}) \to X^*$ be two bounded maps.

Proof : Assume that X is reflexive and fix a non principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. For a bounded function $f : G_k(\mathbb{N}) \to X$ we define $\partial f : G_{k-1}(\mathbb{N}) \to X$ by

$$\forall \overline{n} \in G_{k-1}(\mathbb{N}) \quad \partial f(\overline{n}) = weak - \lim_{n_k \in \mathcal{U}} f(n_1, .., n_{k-1}, n_k).$$

Note that $\partial^k f \in X$.

Lemma 1

Let $h : G_k(\mathbb{N}) \to \mathbb{R}$ be a bounded map and $\varepsilon > 0$. Then there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\forall \overline{n} \in G_k(\mathbb{M}) \quad |h(\overline{n}) - \partial^k h| < \varepsilon.$$

Lemma 2

Let $f : G_k(\mathbb{N}) \to X$ and $g : G_k(\mathbb{M}) \to X^*$ be two bounded maps. Define $f \otimes g : G_{2k}(\mathbb{N}) \to \mathbb{R}$ by

$$(f \otimes g)(n_1, .., n_{2k}) = \big\langle f(n_2, n_4, .., n_{2k}), g(n_1, .., n_{2k-1}) \big\rangle.$$

Proof : Assume that X is <u>reflexive</u> and fix a non principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. For a bounded function $f : G_k(\mathbb{N}) \to X$ we define $\partial f : G_{k-1}(\mathbb{N}) \to X$ by

$$\forall \overline{n} \in G_{k-1}(\mathbb{N}) \quad \partial f(\overline{n}) = weak - \lim_{n_k \in \mathcal{U}} f(n_1, .., n_{k-1}, n_k).$$

Note that $\partial^k f \in X$.

Lemma 1

Let $h : G_k(\mathbb{N}) \to \mathbb{R}$ be a bounded map and $\varepsilon > 0$. Then there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\forall \overline{n} \in G_k(\mathbb{M}) \quad |h(\overline{n}) - \partial^k h| < \varepsilon.$$

Lemma 2

Let $f : G_k(\mathbb{N}) \to X$ and $g : G_k(\mathbb{M}) \to X^*$ be two bounded maps. Define $f \otimes g : G_{2k}(\mathbb{N}) \to \mathbb{R}$ by

$$(f \otimes g)(n_1, .., n_{2k}) = \Big\langle f(n_2, n_4, .., n_{2k}), g(n_1, .., n_{2k-1}) \Big\rangle.$$

Then $\partial^2 (f \otimes g) = \partial f \otimes \partial g \ldots$ $\quad \partial^{2k}(f \otimes g) = \langle \partial^k f, \partial^k g \rangle \in \mathbb{R}$.

Lemma 3

Let $f : G_k(\mathbb{N}) \to X$ be a bounded map and $\varepsilon > 0$. Then there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\forall \overline{n} \in G_k(\mathbb{M}) \ \ \|f(\overline{n})\| \leq \|\partial^k f\| + \omega_f(1) + \varepsilon,$$

where ω_f is the modulus of continuity of f.

Lemma 3

Let $f : G_k(\mathbb{N}) \to X$ be a bounded map and $\varepsilon > 0$. Then there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\forall \overline{n} \in G_k(\mathbb{M}) \ \ \|f(\overline{n})\| \leq \|\partial^k f\| + \omega_f(1) + \varepsilon,$$

where ω_f is the modulus of continuity of f.

Proof. For all $\overline{n} \in G_k(\mathbb{N})$, there exists $g(\overline{n}) \in S_{X^*}$ such that $\langle f(\overline{n}), g(\overline{n})\rangle = \|f(\overline{n})\|$.

Lemma 3

Let $f : G_k(\mathbb{N}) \to X$ be a bounded map and $\varepsilon > 0$. Then there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\forall \overline{n} \in G_k(\mathbb{M}) \ \ \|f(\overline{n})\| \leq \|\partial^k f\| + \omega_f(1) + \varepsilon,$$

where ω_f is the modulus of continuity of f.

Proof. For all $\overline{n} \in G_k(\mathbb{N})$, there exists $g(\overline{n}) \in S_{X^*}$ such that $\langle f(\overline{n}), g(\overline{n})\rangle = \|f(\overline{n})\|$. Then

$$|\partial^{2k}(f \otimes g)| = |\langle \partial^k f, \partial^k g\rangle| \leq \|\partial^k f\|.$$

Lemma 3

Let $f : G_k(\mathbb{N}) \to X$ be a bounded map and $\varepsilon > 0$. Then there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\forall \overline{n} \in G_k(\mathbb{M}) \;\; \|f(\overline{n})\| \leq \|\partial^k f\| + \omega_f(1) + \varepsilon,$$

where ω_f is the modulus of continuity of f.

Proof. For all $\overline{n} \in G_k(\mathbb{N})$, there exists $g(\overline{n}) \in S_{X^*}$ such that $\langle f(\overline{n}), g(\overline{n})\rangle = \|f(\overline{n})\|$. Then

$$|\partial^{2k}(f \otimes g)| = |\langle \partial^k f, \partial^k g\rangle| \leq \|\partial^k f\|.$$

By Lemma 1, there is an infinite subset $\mathbb{M}_0$ of $\mathbb{N}$ such that

$$\forall \overline{p} \in G_{2k}(\mathbb{M}_0) : \;\; |(f \otimes g)(\overline{p})| \leq \|\partial^k f\| + \varepsilon.$$

Lemma 3

Let $f : G_k(\mathbb{N}) \to X$ be a bounded map and $\varepsilon > 0$. Then there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\forall \overline{n} \in G_k(\mathbb{M}) \;\; \|f(\overline{n})\| \leq \|\partial^k f\| + \omega_f(1) + \varepsilon,$$

where ω_f is the modulus of continuity of f.

<u>**Proof.**</u> For all $\overline{n} \in G_k(\mathbb{N})$, there exists $g(\overline{n}) \in S_{X^*}$ such that $\langle f(\overline{n}), g(\overline{n})\rangle = \|f(\overline{n})\|$. Then

$$|\partial^{2k}(f \otimes g)| = |\langle \partial^k f, \partial^k g\rangle| \leq \|\partial^k f\|.$$

By Lemma 1, there is an infinite subset $\mathbb{M}_0$ of $\mathbb{N}$ such that

$$\forall \overline{p} \in G_{2k}(\mathbb{M}_0) : \;\; |(f \otimes g)(\overline{p})| \leq \|\partial^k f\| + \varepsilon.$$

Then write $\mathbb{M}_0 = \{n_1 < m_1 < .. < n_i < m_i < ..\}$ and set $\mathbb{M} = \{n_1 < n_2 < .. < n_i < ..\}$.

Lemma 3

Let $f : G_k(\mathbb{N}) \to X$ be a bounded map and $\varepsilon > 0$. Then there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\forall \overline{n} \in G_k(\mathbb{M}) \ \ \|f(\overline{n})\| \leq \|\partial^k f\| + \omega_f(1) + \varepsilon,$$

where ω_f is the modulus of continuity of f.

Proof. For all $\overline{n} \in G_k(\mathbb{N})$, there exists $g(\overline{n}) \in S_{X^*}$ such that $\langle f(\overline{n}), g(\overline{n})\rangle = \|f(\overline{n})\|$. Then

$$|\partial^{2k}(f \otimes g)| = |\langle \partial^k f, \partial^k g\rangle| \leq \|\partial^k f\|.$$

By Lemma 1, there is an infinite subset $\mathbb{M}_0$ of $\mathbb{N}$ such that

$$\forall \overline{p} \in G_{2k}(\mathbb{M}_0) : \ |(f \otimes g)(\overline{p})| \leq \|\partial^k f\| + \varepsilon.$$

Then write $\mathbb{M}_0 = \{n_1 < m_1 < .. < n_i < m_i < ..\}$ and set $\mathbb{M} = \{n_1 < n_2 < .. < n_i < ..\}$. Thus for all $\overline{n} = (n_{i_1}, .., n_{i_k}) \in G_k(\mathbb{M})$,

$$\|f(\overline{n})\| = \langle f(\overline{n}), g(\overline{n})\rangle \leq |\langle f(n_{i_1+1}, .., n_{i_k+1}), g(n_{i_1}, .., n_{i_k})\rangle| + \omega_f(1)$$
$$\leq \|\partial^k f\| + \varepsilon + \omega_f(1).$$

□

Lemma 4

Let $\varepsilon > 0$, X be a separable reflexive Banach space and I be an uncountable set. Assume that for each $i \in I$, $f_i : G_k(\mathbb{N}) \to X$ is a bounded map. Then there exist $i \neq j \in I$ and an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\forall \overline{n} \in G_k(\mathbb{M}) \quad \|f_i(\overline{n}) - f_j(\overline{n})\| \leq \omega_{f_i}(1) + \omega_{f_j}(1) + \varepsilon.$$

Lemma 4

Let $\varepsilon > 0$, X be a separable reflexive Banach space and I be an uncountable set. Assume that for each $i \in I$, $f_i : G_k(\mathbb{N}) \to X$ is a bounded map. Then there exist $i \neq j \in I$ and an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\forall \overline{n} \in G_k(\mathbb{M}) \ \ \|f_i(\overline{n}) - f_j(\overline{n})\| \leq \omega_{f_i}(1) + \omega_{f_j}(1) + \varepsilon.$$

Proof. Since X is separable and I is uncountable, there exists $i \neq j \in I$, $\|\partial^k f_i - \partial^k f_j\| < \varepsilon/2$.

Lemma 4

Let $\varepsilon > 0$, X be a separable reflexive Banach space and I be an uncountable set. Assume that for each $i \in I$, $f_i : G_k(\mathbb{N}) \to X$ is a bounded map. Then there exist $i \neq j \in I$ and an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$\forall \overline{n} \in G_k(\mathbb{M}) \quad \|f_i(\overline{n}) - f_j(\overline{n})\| \leq \omega_{f_i}(1) + \omega_{f_j}(1) + \varepsilon.$$

Proof. Since X is separable and I is uncountable, there exists $i \neq j \in I$, $\|\partial^k f_i - \partial^k f_j\| < \varepsilon/2$.
Then just apply Lemma 3 to $f_i - f_j$. □

End of proof. Assume X is reflexive and let $h : c_0 \to X$ be a map which is bounded on bounded subsets of c_0. Let $(e_k)_k$ be the canonical basis of c_0.

End of proof. Assume X is reflexive and let $h : c_0 \to X$ be a map which is bounded on bounded subsets of c_0. Let $(e_k)_k$ be the canonical basis of c_0. For an infinite subset A of $\mathbb{N}$ we now define

$$\forall n \in \mathbb{N} \quad s_A(n) = \sum_{k \leq n,\ k \in A} e_k$$

End of proof. Assume X is reflexive and let $h : c_0 \to X$ be a map which is bounded on bounded subsets of c_0. Let $(e_k)_k$ be the canonical basis of c_0. For an infinite subset A of $\mathbb{N}$ we now define

$$\forall n \in \mathbb{N} \quad s_A(n) = \sum_{k \leq n,\ k \in A} e_k$$

and

$$\forall \overline{n} = (n_1, .., n_k) \in G_k(\mathbb{N}) \quad f_A(\overline{n}) = \sum_{i=1}^{k} s_A(n_i).$$

End of proof. Assume X is reflexive and let $h : c_0 \to X$ be a map which is bounded on bounded subsets of c_0. Let $(e_k)_k$ be the canonical basis of c_0. For an infinite subset A of $\mathbb{N}$ we now define

$$\forall n \in \mathbb{N} \quad s_A(n) = \sum_{k \leq n,\ k \in A} e_k$$

and

$$\forall \overline{n} = (n_1, .., n_k) \in G_k(\mathbb{N}) \quad f_A(\overline{n}) = \sum_{i=1}^{k} s_A(n_i).$$

$(h \circ f_A)_A$ is an uncountable family of bounded maps from $G_k(\mathbb{N})$ to X.

End of proof. Assume X is reflexive and let $h : c_0 \to X$ be a map which is bounded on bounded subsets of c_0. Let $(e_k)_k$ be the canonical basis of c_0. For an infinite subset A of $\mathbb{N}$ we now define

$$\forall n \in \mathbb{N} \;\; s_A(n) = \sum_{k \leq n,\; k \in A} e_k$$

and

$$\forall \overline{n} = (n_1, .., n_k) \in G_k(\mathbb{N}) \;\; f_A(\overline{n}) = \sum_{i=1}^{k} s_A(n_i).$$

$(h \circ f_A)_A$ is an uncountable family of bounded maps from $G_k(\mathbb{N})$ to X.
It follows from Lemma 4 that there are two distinct infinite subsets A and B of $\mathbb{N}$ and another infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that for all $\overline{n} \in G_k(\mathbb{M})$:

$$\|h \circ f_A(\overline{n}) - h \circ f_B(\overline{n})\| \leq \omega_{h \circ f_A}(1) + \omega_{h \circ f_B}(1) + 1 \leq 2\omega_h(1) + 1.$$

End of proof. Assume X is reflexive and let $h : c_0 \to X$ be a map which is bounded on bounded subsets of c_0. Let $(e_k)_k$ be the canonical basis of c_0. For an infinite subset A of $\mathbb{N}$ we now define

$$\forall n \in \mathbb{N} \;\; s_A(n) = \sum_{k \le n,\; k \in A} e_k$$

and

$$\forall \overline{n} = (n_1, .., n_k) \in G_k(\mathbb{N}) \;\; f_A(\overline{n}) = \sum_{i=1}^{k} s_A(n_i).$$

$(h \circ f_A)_A$ is an uncountable family of bounded maps from $G_k(\mathbb{N})$ to X. It follows from Lemma 4 that there are two distinct infinite subsets A and B of $\mathbb{N}$ and another infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that for all $\overline{n} \in G_k(\mathbb{M})$:

$$\|h \circ f_A(\overline{n}) - h \circ f_B(\overline{n})\| \le \omega_{h \circ f_A}(1) + \omega_{h \circ f_B}(1) + 1 \le 2\omega_h(1) + 1.$$

Since $A \ne B$, there is $\overline{n} \in G_k(\mathbb{M})$ with $\|f_A(\overline{n}) - f_B(\overline{n})\| = k$.

End of proof. Assume X is reflexive and let $h : c_0 \to X$ be a map which is bounded on bounded subsets of c_0. Let $(e_k)_k$ be the canonical basis of c_0. For an infinite subset A of $\mathbb{N}$ we now define

$$\forall n \in \mathbb{N} \quad s_A(n) = \sum_{k \le n,\ k \in A} e_k$$

and

$$\forall \overline{n} = (n_1, .., n_k) \in G_k(\mathbb{N}) \quad f_A(\overline{n}) = \sum_{i=1}^{k} s_A(n_i).$$

$(h \circ f_A)_A$ is an uncountable family of bounded maps from $G_k(\mathbb{N})$ to X.
It follows from Lemma 4 that there are two distinct infinite subsets A and B of $\mathbb{N}$ and another infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that for all $\overline{n} \in G_k(\mathbb{M})$:

$$\|h \circ f_A(\overline{n}) - h \circ f_B(\overline{n})\| \le \omega_{h \circ f_A}(1) + \omega_{h \circ f_B}(1) + 1 \le 2\omega_h(1) + 1.$$

Since $A \neq B$, there is $\overline{n} \in G_k(\mathbb{M})$ with $\|f_A(\overline{n}) - f_B(\overline{n})\| = k$.
By taking arbitrarily large values of k we deduce that h cannot be a coarse embedding.

End of proof. Assume X is reflexive and let $h : c_0 \to X$ be a map which is bounded on bounded subsets of c_0. Let $(e_k)_k$ be the canonical basis of c_0. For an infinite subset A of $\mathbb{N}$ we now define

$$\forall n \in \mathbb{N} \ \ s_A(n) = \sum_{k \leq n, \ k \in A} e_k$$

and

$$\forall \overline{n} = (n_1, .., n_k) \in G_k(\mathbb{N}) \ \ f_A(\overline{n}) = \sum_{i=1}^{k} s_A(n_i).$$

$(h \circ f_A)_A$ is an uncountable family of bounded maps from $G_k(\mathbb{N})$ to X.
It follows from Lemma 4 that there are two distinct infinite subsets A and B of $\mathbb{N}$ and another infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that for all $\overline{n} \in G_k(\mathbb{M})$:

$$\|h \circ f_A(\overline{n}) - h \circ f_B(\overline{n})\| \leq \omega_{h \circ f_A}(1) + \omega_{h \circ f_B}(1) + 1 \leq 2\omega_h(1) + 1.$$

Since $A \neq B$, there is $\overline{n} \in G_k(\mathbb{M})$ with $\|f_A(\overline{n}) - f_B(\overline{n})\| = k$.
By taking arbitrarily large values of k we deduce that h cannot be a coarse embedding. ρ_1 cannot tend to $+\infty$. □

Remarks.

Remarks.

1) Similarly, one can show that h cannot be a uniform embedding, by composing h with the maps tf_A and letting t tend to zero.

Remarks.

1) Similarly, one can show that h cannot be a uniform embedding, by composing h with the maps tf_A and letting t tend to zero.

2) It is now easy to adapt this proof in order to obtain the stronger result stated in Kalton's theorem.

Remarks.

1) Similarly, one can show that h cannot be a uniform embedding, by composing h with the maps tf_A and letting t tend to zero.

2) It is now easy to adapt this proof in order to obtain the stronger result stated in Kalton's theorem.

3) On the other hand, Kalton proved that c_0 embeds uniformly and coarsely in a Banach space X with the Schur property. Note that such an X does not contain any subspace linearly isomorphic to c_0.

Remarks.

1) Similarly, one can show that h cannot be a uniform embedding, by composing h with the maps tf_A and letting t tend to zero.

2) It is now easy to adapt this proof in order to obtain the stronger result stated in Kalton's theorem.

3) On the other hand, Kalton proved that c_0 embeds uniformly and coarsely in a Banach space X with the Schur property. Note that such an X does not contain any subspace linearly isomorphic to c_0.

4) Kalton used the same graph distance on $G_k(\omega_1)$ in order to show that the unit balls of ℓ_∞/c_0 or $C([0,\omega_1])$ do not uniformly embed into ℓ_∞.

Remarks.

1) Similarly, one can show that h cannot be a uniform embedding, by composing h with the maps tf_A and letting t tend to zero.

2) It is now easy to adapt this proof in order to obtain the stronger result stated in Kalton's theorem.

3) On the other hand, Kalton proved that c_0 embeds uniformly and coarsely in a Banach space X with the Schur property. Note that such an X does not contain any subspace linearly isomorphic to c_0.

4) Kalton used the same graph distance on $G_k(\omega_1)$ in order to show that the unit balls of ℓ_∞/c_0 or $C([0,\omega_1])$ do not uniformly embed into ℓ_∞.

5) Kalton also proved that any stable metric space can be coarsely and uniformly embedded into a reflexive Banach space.

Remarks.

1) Similarly, one can show that h cannot be a uniform embedding, by composing h with the maps tf_A and letting t tend to zero.

2) It is now easy to adapt this proof in order to obtain the stronger result stated in Kalton's theorem.

3) On the other hand, Kalton proved that c_0 embeds uniformly and coarsely in a Banach space X with the Schur property. Note that such an X does not contain any subspace linearly isomorphic to c_0.

4) Kalton used the same graph distance on $G_k(\omega_1)$ in order to show that the unit balls of ℓ_∞/c_0 or $C([0,\omega_1])$ do not uniformly embed into ℓ_∞.

5) Kalton also proved that any stable metric space can be coarsely and uniformly embedded into a reflexive Banach space.

Open question.

Remarks.

1) Similarly, one can show that h cannot be a uniform embedding, by composing h with the maps tf_A and letting t tend to zero.

2) It is now easy to adapt this proof in order to obtain the stronger result stated in Kalton's theorem.

3) On the other hand, Kalton proved that c_0 embeds uniformly and coarsely in a Banach space X with the Schur property. Note that such an X does not contain any subspace linearly isomorphic to c_0.

4) Kalton used the same graph distance on $G_k(\omega_1)$ in order to show that the unit balls of ℓ_∞/c_0 or $C([0,\omega_1])$ do not uniformly embed into ℓ_∞.

5) Kalton also proved that any stable metric space can be coarsely and uniformly embedded into a reflexive Banach space.

Open question.

Let X be a separable Banach space such that c_0 embeds coarsely in X.

Remarks.

1) Similarly, one can show that h cannot be a uniform embedding, by composing h with the maps tf_A and letting t tend to zero.

2) It is now easy to adapt this proof in order to obtain the stronger result stated in Kalton's theorem.

3) On the other hand, Kalton proved that c_0 embeds uniformly and coarsely in a Banach space X with the Schur property. Note that such an X does not contain any subspace linearly isomorphic to c_0.

4) Kalton used the same graph distance on $G_k(\omega_1)$ in order to show that the unit balls of ℓ_∞/c_0 or $C([0,\omega_1])$ do not uniformly embed into ℓ_∞.

5) Kalton also proved that any stable metric space can be coarsely and uniformly embedded into a reflexive Banach space.

Open question.

Let X be a separable Banach space such that c_0 embeds coarsely in X. Does it imply that X^{**} is non separable ?

Remarks.

1) Similarly, one can show that h cannot be a uniform embedding, by composing h with the maps tf_A and letting t tend to zero.

2) It is now easy to adapt this proof in order to obtain the stronger result stated in Kalton's theorem.

3) On the other hand, Kalton proved that c_0 embeds uniformly and coarsely in a Banach space X with the Schur property. Note that such an X does not contain any subspace linearly isomorphic to c_0.

4) Kalton used the same graph distance on $G_k(\omega_1)$ in order to show that the unit balls of ℓ_∞/c_0 or $C([0,\omega_1])$ do not uniformly embed into ℓ_∞.

5) Kalton also proved that any stable metric space can be coarsely and uniformly embedded into a reflexive Banach space.

Open question.

Let X be a separable Banach space such that c_0 embeds coarsely in X. Does it imply that X^{**} is non separable?

Problem. Describe the Banach spaces containing uniform bi-Lipschitz copies of the $G_k(\mathbb{N})$'s.

Almost Lipschitz embeddability - with F. Baudier (2015).

Definition

Let (M, d) be a metric space and Y be a Banach space. We say that (M, d) *almost Lipschitz embeds* into Y if there exist $D \geq 1$ such that for any continuous function $\varphi\colon [0, +\infty) \to [0, 1)$ satisfying $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$, there exists a map $f_\varphi\colon M \to Y$ such that

$$\forall x, y \in M \quad \varphi(d_X(x, y))\, d_X(x, y) \leq d_Y(f_\varphi(x), f_\varphi(y)) \leq D d_X(x, y).$$

Almost Lipschitz embeddability - with F. Baudier (2015).

Definition

Let (M, d) be a metric space and Y be a Banach space. We say that (M, d) *almost Lipschitz embeds* into Y if there exist $D \geq 1$ such that for any continuous function $\varphi\colon [0, +\infty) \to [0, 1)$ satisfying $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$, there exists a map $f_\varphi\colon M \to Y$ such that

$$\forall x, y \in M \quad \varphi(d_X(x, y))\, d_X(x, y) \leq d_Y(f_\varphi(x), f_\varphi(y)) \leq D d_X(x, y).$$

A metric space is **proper** if all its closed balls are compact.

Almost Lipschitz embeddability - with F. Baudier (2015).

Definition

Let (M,d) be a metric space and Y be a Banach space. We say that (M,d) *almost Lipschitz embeds* into Y if there exist $D \geq 1$ such that for any continuous function $\varphi\colon [0,+\infty) \to [0,1)$ satisfying $\varphi(0)=0$ and $\varphi(t)>0$ for all $t>0$, there exists a map $f_\varphi\colon M \to Y$ such that

$$\forall x,y \in M \quad \varphi(d_X(x,y))\, d_X(x,y) \leq d_Y(f_\varphi(x), f_\varphi(y)) \leq D d_X(x,y).$$

A metric space is **proper** if all its closed balls are compact.

Theorem

Let $p \in [1,+\infty]$, M a proper subset of L_p, and Y a Banach space containing uniformly the ℓ_p^n's. Then M almost Lipschitz embeds into Y.

Corollary 1

Any proper metric space almost Lipschitz embeds into any Banach space without cotype.

Corollary 1

Any proper metric space almost Lipschitz embeds into any Banach space without cotype.

Corollary 2

Any proper subset of a Hilbert space almost Lipschitz embeds into any infinite dimensional Banach space.

Corollary 1

Any proper metric space almost Lipschitz embeds into any Banach space without cotype.

Corollary 2

Any proper subset of a Hilbert space almost Lipschitz embeds into any infinite dimensional Banach space.

Question. Does ℓ_2 coarsely embed into any infinite dim. Banach space ?

Corollary 1

Any proper metric space almost Lipschitz embeds into any Banach space without cotype.

Corollary 2

Any proper subset of a Hilbert space almost Lipschitz embeds into any infinite dimensional Banach space.

Question. Does ℓ_2 coarsely embed into any infinite dim. Banach space ?

Optimality

Let X be a separable Banach space. Then, there exists a compact subset K of X such that, whenever K almost Lipschitz embeds into a Banach space Y, then X is crudely finitely representable into Y.

Corollary 1

Any proper metric space almost Lipschitz embeds into any Banach space without cotype.

Corollary 2

Any proper subset of a Hilbert space almost Lipschitz embeds into any infinite dimensional Banach space.

Question. Does ℓ_2 coarsely embed into any infinite dim. Banach space ?

Optimality

Let X be a separable Banach space. Then, there exists a compact subset K of X such that, whenever K almost Lipschitz embeds into a Banach space Y, then X is crudely finitely representable into Y.

In particular :

Corollary 1

Any proper metric space almost Lipschitz embeds into any Banach space without cotype.

Corollary 2

Any proper subset of a Hilbert space almost Lipschitz embeds into any infinite dimensional Banach space.

Question. Does ℓ_2 coarsely embed into any infinite dim. Banach space ?

Optimality

Let X be a separable Banach space. Then, there exists a compact subset K of X such that, whenever K almost Lipschitz embeds into a Banach space Y, then X is crudely finitely representable into Y.

In particular :

• For any $p \in [1, \infty)$, there exists a compact subset K_p of L_p such that, K_p almost Lipschitz embeds into Y iff Y uniformly contains the ℓ_p^n's.

Corollary 1

Any proper metric space almost Lipschitz embeds into any Banach space without cotype.

Corollary 2

Any proper subset of a Hilbert space almost Lipschitz embeds into any infinite dimensional Banach space.

Question. Does ℓ_2 coarsely embed into any infinite dim. Banach space ?

Optimality

Let X be a separable Banach space. Then, there exists a compact subset K of X such that, whenever K almost Lipschitz embeds into a Banach space Y, then X is crudely finitely representable into Y.

In particular :

- For any $p \in [1, \infty)$, there exists a compact subset K_p of L_p such that, K_p almost Lipschitz embeds into Y iff Y uniformly contains the ℓ_p^n's.
- There exists a compact subset K_∞ of c_0 such that, K_∞ almost Lipschitz embeds into Y iff Y uniformly contains the ℓ_∞^n's.

Tools for the embedding result.

Let M be a proper subset of L_p and $B_k = \{x \in M,\ \|x\|_p \le 2^{k+1}\}$, $k \in \mathbb{Z}$.

Tools for the embedding result.

Let M be a proper subset of L_p and $B_k = \{x \in M,\ \|x\|_p \leq 2^{k+1}\}$, $k \in \mathbb{Z}$.

• There is a finite rank norm 1 linear ε_n-approximation of the identity φ_n^k on the compact B_k.

Tools for the embedding result.

Let M be a proper subset of L_p and $B_k = \{x \in M,\ \|x\|_p \leq 2^{k+1}\}$, $k \in \mathbb{Z}$.

• There is a finite rank norm 1 linear ε_n-approximation of the identity φ_n^k on the compact B_k.

• We may assume that the image of φ_n^k is included in H_n^k which is 2-isomorphic to some $\ell_p^{d(n,k)}$.

Tools for the embedding result.

Let M be a proper subset of L_p and $B_k = \{x \in M,\ \|x\|_p \leq 2^{k+1}\}$, $k \in \mathbb{Z}$.

- There is a finite rank norm 1 linear ε_n-approximation of the identity φ_n^k on the compact B_k.
- We may assume that the image of φ_n^k is included in H_n^k which is 2-isomorphic to some $\ell_p^{d(n,k)}$.
- We can build a subspace Z of X with an FDD $(G_j)_j$ such that each G_j is 2-isomorphic to exactly one of the H_n^k.

Tools for the embedding result.

Let M be a proper subset of L_p and $B_k = \{x \in M,\ \|x\|_p \leq 2^{k+1}\}$, $k \in \mathbb{Z}$.

• There is a finite rank norm 1 linear ε_n-approximation of the identity φ_n^k on the compact B_k.

• We may assume that the image of φ_n^k is included in H_n^k which is 2-isomorphic to some $\ell_p^{d(n,k)}$.

• We can build a subspace Z of X with an FDD $(G_j)_j$ such that each G_j is 2-isomorphic to exactly one of the H_n^k.

• $f_k = \sum_{n=1}^{\infty} 2^{-n}\varphi_n^k$ embeds B_k into Z.

Tools for the embedding result.

Let M be a proper subset of L_p and $B_k = \{x \in M,\ \|x\|_p \leq 2^{k+1}\}$, $k \in \mathbb{Z}$.

- There is a finite rank norm 1 linear ε_n-approximation of the identity φ_n^k on the compact B_k.
- We may assume that the image of φ_n^k is included in H_n^k which is 2-isomorphic to some $\ell_p^{d(n,k)}$.
- We can build a subspace Z of X with an FDD $(G_j)_j$ such that each G_j is 2-isomorphic to exactly one of the H_n^k.
- $f_k = \sum_{n=1}^{\infty} 2^{-n} \varphi_n^k$ embeds B_k into Z.
- Finally, we use the convex-gluing technique to define, for $x \in B_k \setminus B_{k-1}$:

$$f(x) = \lambda f_k(x) + (1-\lambda) f_{k+1}(x), \quad \text{with} \quad \lambda = \frac{2^{k+1} - \|x\|_p}{2^k}.$$

Steps of the proof of optimality.

• Let $(x_n, x_n^*)_{n=1}^\infty$ biorthogonal in $X \times X^*$ such that $\overline{sp}\{x_n\colon \ n \geq 1\} = X$. Pick a decreasing sequence $(a_n)_{n=1}^\infty$ of positive real numbers such that

$$\sum_{n=1}^\infty a_n \|x_n\| \, \|x_n^*\| \leq 1.$$

Steps of the proof of optimality.

• Let $(x_n, x_n^*)_{n=1}^{\infty}$ biorthogonal in $X \times X^*$ such that $\overline{sp}\{x_n\colon\ n \geq 1\} = X$. Pick a decreasing sequence $(a_n)_{n=1}^{\infty}$ of positive real numbers such that

$$\sum_{n=1}^{\infty} a_n \|x_n\| \, \|x_n^*\| \leq 1.$$

Let $S(x) = \sum_{n=1}^{\infty} a_n x_n^*(x) x_n$ and $K = \overline{S(B_X)}$. K is a compact subset of X.

Steps of the proof of optimality.

• Let $(x_n, x_n^*)_{n=1}^{\infty}$ biorthogonal in $X \times X^*$ such that $\overline{sp}\{x_n\colon\ n \geq 1\} = X$. Pick a decreasing sequence $(a_n)_{n=1}^{\infty}$ of positive real numbers such that

$$\sum_{n=1}^{\infty} a_n \|x_n\| \, \|x_n^*\| \leq 1.$$

Let $S(x) = \sum_{n=1}^{\infty} a_n x_n^*(x) x_n$ and $K = \overline{S(B_X)}$. K is a compact subset of X.

• If $f : K \to Y$ is a bi-Lipschitz embedding, then X linearly embeds into Y^{**} (use the weak*-Gâteaux derivative of $f \circ S$).

Steps of the proof of optimality.

• Let $(x_n, x_n^*)_{n=1}^\infty$ biorthogonal in $X \times X^*$ such that $\overline{sp}\{x_n\colon\ n \geq 1\} = X$. Pick a decreasing sequence $(a_n)_{n=1}^\infty$ of positive real numbers such that

$$\sum_{n=1}^\infty a_n \|x_n\|\,\|x_n^*\| \leq 1.$$

Let $S(x) = \sum_{n=1}^\infty a_n x_n^*(x) x_n$ and $K = \overline{S(B_X)}$. K is a compact subset of X.

• If $f : K \to Y$ is a bi-Lipschitz embedding, then X linearly embeds into Y^{**} (use the weak*-Gâteaux derivative of $f \circ S$).

• If K almost Lipschitz embeds into Y. Then we can build an increasing sequence $(R_n)_n$ of 2^{-n}-nets of K so that the R_n's uniformly bi-Lipschitz embed into Y.

Steps of the proof of optimality.

• Let $(x_n, x_n^*)_{n=1}^\infty$ biorthogonal in $X \times X^*$ such that $\overline{sp}\{x_n : \ n \geq 1\} = X$. Pick a decreasing sequence $(a_n)_{n=1}^\infty$ of positive real numbers such that

$$\sum_{n=1}^\infty a_n \|x_n\| \, \|x_n^*\| \leq 1.$$

Let $S(x) = \sum_{n=1}^\infty a_n x_n^*(x) x_n$ and $K = \overline{S(B_X)}$. K is a compact subset of X.

• If $f : K \to Y$ is a bi-Lipschitz embedding, then X linearly embeds into Y^{**} (use the weak*-Gâteaux derivative of $f \circ S$).

• If K almost Lipschitz embeds into Y. Then we can build an increasing sequence $(R_n)_n$ of 2^{-n}-nets of K so that the R_n's uniformly bi-Lipschitz embed into Y.

• Then, the usual argument shows that K bi-Lipschitz embeds into an ultrapower of Y.

Steps of the proof of optimality.

• Let $(x_n, x_n^*)_{n=1}^\infty$ biorthogonal in $X \times X^*$ such that $\overline{sp}\{x_n : \ n \geq 1\} = X$. Pick a decreasing sequence $(a_n)_{n=1}^\infty$ of positive real numbers such that

$$\sum_{n=1}^\infty a_n \|x_n\| \, \|x_n^*\| \leq 1.$$

Let $S(x) = \sum_{n=1}^\infty a_n x_n^*(x) x_n$ and $K = \overline{S(B_X)}$. K is a compact subset of X.

• If $f : K \to Y$ is a bi-Lipschitz embedding, then X linearly embeds into Y^{**} (use the weak*-Gâteaux derivative of $f \circ S$).

• If K almost Lipschitz embeds into Y. Then we can build an increasing sequence $(R_n)_n$ of 2^{-n}-nets of K so that the R_n's uniformly bi-Lipschitz embed into Y.

• Then, the usual argument shows that K bi-Lipschitz embeds into an ultrapower of Y.

• Therefore X linearly embeds into the bidual of the ultrapower and is therefore finitely crudely representable into Y.

IV. METRIC INVARIANTS.

Definition

Let (M, d) and (N, δ) be two unbounded metric spaces. A map $f : M \to N$ is said to be a *coarse Lipschitz embedding* if there exist $A, B, C, D > 0$ such that

$$\forall x, y \in M \quad Ad(x, y) - B \leq \delta(f(x), f(y)) \leq Cd(x, y) + D.$$

We denote $M \underset{CL}{\hookrightarrow} N$.

Definition

Let (M, d) and (N, δ) be two unbounded metric spaces. A map $f : M \to N$ is said to be a *coarse Lipschitz embedding* if there exist $A, B, C, D > 0$ such that

$$\forall x, y \in M \quad Ad(x, y) - B \leq \delta(f(x), f(y)) \leq Cd(x, y) + D.$$

We denote $M \underset{CL}{\hookrightarrow} N$.

Ribe 1976

Let X and Y be two Banach spaces such that $X \underset{CL}{\hookrightarrow} Y$. Then X is finitely crudely representable into Y : there exists a $C \geq 1$ such that every finite dimensional subspace of X is C-isomorphic to a subspace of Y.

Definition

Let (M, d) and (N, δ) be two unbounded metric spaces. A map $f : M \to N$ is said to be a *coarse Lipschitz embedding* if there exist $A, B, C, D > 0$ such that

$$\forall x, y \in M \quad Ad(x, y) - B \leq \delta(f(x), f(y)) \leq Cd(x, y) + D.$$

We denote $M \underset{CL}{\hookrightarrow} N$.

Ribe 1976

Let X and Y be two Banach spaces such that $X \underset{CL}{\hookrightarrow} Y$. Then X is finitely crudely representable into Y : there exists a $C \geq 1$ such that every finite dimensional subspace of X is C-isomorphic to a subspace of Y.

In other words : the local linear properties of Banach spaces (such as type, cotype, superreflexivity...) are stable under coarse Lipschitz embeddings.

Definition

Let (M, d) and (N, δ) be two unbounded metric spaces. A map $f : M \to N$ is said to be a *coarse Lipschitz embedding* if there exist $A, B, C, D > 0$ such that

$$\forall x, y \in M \quad Ad(x, y) - B \leq \delta(f(x), f(y)) \leq Cd(x, y) + D.$$

We denote $M \underset{CL}{\hookrightarrow} N$.

Ribe 1976

Let X and Y be two Banach spaces such that $X \underset{CL}{\hookrightarrow} Y$. Then X is finitely crudely representable into Y : there exists a $C \geq 1$ such that every finite dimensional subspace of X is C-isomorphic to a subspace of Y.

In other words : the local linear properties of Banach spaces (such as type, cotype, superreflexivity...) are stable under coarse Lipschitz embeddings.

Ribe program (Bourgain-Lindenstrauss). Characterize the local properties of Banach spaces in purely metric terms.

IV.1 Examples of local properties.

linear type et cotype

Let X be a Banach space, $p \in [1,2]$ et $q \in [2,+\infty[$.
We say that X is of type p if there exists $C > 0$ so that

$$\forall x_1,..,x_n \in X \quad 2^{-n} \sum_{\varepsilon_i = \pm 1} \Big\| \sum_{i=1}^{n} \varepsilon_i x_i \Big\| \leq C \Big(\sum_{i=1}^{n} \|x_i\|^p \Big)^{1/p}.$$

IV.1 Examples of local properties.

linear type et cotype

Let X be a Banach space, $p \in [1,2]$ et $q \in [2,+\infty[$.
We say that X is of type p if there exists $C > 0$ so that

$$\forall x_1, .., x_n \in X \quad 2^{-n} \sum_{\varepsilon_i = \pm 1} \Big\| \sum_{i=1}^{n} \varepsilon_i x_i \Big\| \leq C \Big(\sum_{i=1}^{n} \|x_i\|^p \Big)^{1/p}.$$

We say that X is of cotype q if there exists $C > 0$ so that

$$\forall x_1, .., x_n \in X \quad \Big(\sum_{i=1}^{n} \|x_i\|^q \Big)^{1/q} \leq C 2^{-n} \sum_{\varepsilon_i = \pm 1} \Big\| \sum_{i=1}^{n} \varepsilon_i x_i \Big\|.$$

IV.1 Examples of local properties.

linear type et cotype

Let X be a Banach space, $p \in [1,2]$ et $q \in [2,+\infty[$.
We say that X is of <u>type p</u> if there exists $C > 0$ so that

$$\forall x_1, .., x_n \in X \quad 2^{-n} \sum_{\varepsilon_i = \pm 1} \Big\| \sum_{i=1}^{n} \varepsilon_i x_i \Big\| \leq C \Big(\sum_{i=1}^{n} \|x_i\|^p \Big)^{1/p}.$$

We say that X is of <u>cotype q</u> if there exists $C > 0$ so that

$$\forall x_1, .., x_n \in X \quad \Big(\sum_{i=1}^{n} \|x_i\|^q \Big)^{1/q} \leq C 2^{-n} \sum_{\varepsilon_i = \pm 1} \Big\| \sum_{i=1}^{n} \varepsilon_i x_i \Big\|.$$

Maurey-Pisier (1973-1976)

X is of trivial type iff it contains uniformly the ℓ_1^n's (Pisier 73).

IV.1 Examples of local properties.

linear type et cotype

Let X be a Banach space, $p \in [1,2]$ et $q \in [2,+\infty[$.
We say that X is of type p if there exists $C > 0$ so that

$$\forall x_1, .., x_n \in X \quad 2^{-n} \sum_{\varepsilon_i = \pm 1} \Big\| \sum_{i=1}^{n} \varepsilon_i x_i \Big\| \leq C \Big(\sum_{i=1}^{n} \|x_i\|^p \Big)^{1/p}.$$

We say that X is of cotype q if there exists $C > 0$ so that

$$\forall x_1, .., x_n \in X \quad \Big(\sum_{i=1}^{n} \|x_i\|^q \Big)^{1/q} \leq C 2^{-n} \sum_{\varepsilon_i = \pm 1} \Big\| \sum_{i=1}^{n} \varepsilon_i x_i \Big\|.$$

Maurey-Pisier (1973-1976)

X is of trivial type iff it contains uniformly the ℓ_1^n's (Pisier 73).
X is of trivial cotype iff it contains uniformly the ℓ_∞^n's (Maurey-Pisier 76).

Enflo's metric type

Let (M, d) be a metric space and $p \geq 1$. We say that M is of metric type p, if there exists $C > 0$ such that for all $(x_\varepsilon)_{\varepsilon \in \{-1,1\}^n} \subset M$:

$$2^{-n} \sum diagonals \leq C \left(2^{-n} \sum (edges)^p\right)^{1/p},$$

where a diagonal is a pair $(x_\varepsilon, x_{-\varepsilon})$ and an edge is a pair $(x_\varepsilon, x_\delta)$ with ε and δ different on exactly one coordinate.

Enflo's metric type

Let (M, d) be a metric space and $p \geq 1$. We say that M is of metric type p, if there exists $C > 0$ such that for all $(x_\varepsilon)_{\varepsilon \in \{-1,1\}^n} \subset M$:

$$2^{-n} \sum diagonals \leq C \big(2^{-n} \sum (edges)^p\big)^{1/p},$$

where a diagonal is a pair $(x_\varepsilon, x_{-\varepsilon})$ and an edge is a pair $(x_\varepsilon, x_\delta)$ with ε and δ different on exactly one coordinate.

• It is clear that a Banach space of metric type p is of linear type p (consider functions of the sort : $x_\varepsilon = \sum_{i=1}^n \varepsilon_i x_i$).

Enflo's metric type

Let (M, d) be a metric space and $p \geq 1$. We say that M is of metric type p, if there exists $C > 0$ such that for all $(x_\varepsilon)_{\varepsilon \in \{-1,1\}^n} \subset M$:

$$2^{-n} \sum diagonals \leq C \left(2^{-n} \sum (edges)^p\right)^{1/p},$$

where a diagonal is a pair $(x_\varepsilon, x_{-\varepsilon})$ and an edge is a pair $(x_\varepsilon, x_\delta)$ with ε and δ different on exactly one coordinate.

• It is clear that a Banach space of metric type p is of linear type p (consider functions of the sort : $x_\varepsilon = \sum_{i=1}^n \varepsilon_i x_i$).

• An inequality of Pisier (1986) shows that if X is of linear type p, then it is of metric type r, for all $r < p$.

Enflo's metric type

Let (M,d) be a metric space and $p \geq 1$. We say that M is of metric type p, if there exists $C>0$ such that for all $(x_\varepsilon)_{\varepsilon\in\{-1,1\}^n} \subset M$:

$$2^{-n}\sum diagonals \leq C\big(2^{-n}\sum (edges)^p\big)^{1/p},$$

where a diagonal is a pair $(x_\varepsilon, x_{-\varepsilon})$ and an edge is a pair $(x_\varepsilon, x_\delta)$ with ε and δ different on exactly one coordinate.

- It is clear that a Banach space of metric type p is of linear type p (consider functions of the sort : $x_\varepsilon = \sum_{i=1}^n \varepsilon_i x_i$).
- An inequality of Pisier (1986) shows that if X is of linear type p, then it is of metric type r, for all $r<p$.
- It is unknown whether a Banach space of linear type p is always of metric type p.

Enflo's metric type

Let (M,d) be a metric space and $p \geq 1$. We say that M is of <u>metric type p</u>, if there exists $C>0$ such that for all $(x_\varepsilon)_{\varepsilon\in\{-1,1\}^n} \subset M$:

$$2^{-n}\sum diagonals \leq C\big(2^{-n}\sum (edges)^p\big)^{1/p},$$

where a diagonal is a pair $(x_\varepsilon, x_{-\varepsilon})$ and an edge is a pair $(x_\varepsilon, x_\delta)$ with ε and δ different on exactly one coordinate.

- It is clear that a Banach space of metric type p is of linear type p (consider functions of the sort : $x_\varepsilon = \sum_{i=1}^n \varepsilon_i x_i$).
- An inequality of Pisier (1986) shows that if X is of linear type p, then it is of metric type r, for all $r<p$.
- It is unknown whether a Banach space of linear type p is always of metric type p.

Bourgain-Milman-Wolfson (1986)

A metric space is of trivial metric type iff it contains uniformly bi-Lipschitz copies of the Hamming cubes $H_n = (\{-1,1\}^n, \|\ \|_1)$.

Some contributions of Mendel and Naor (2007 and 2008).

Some contributions of Mendel and Naor (2007 and 2008).

• Definition of a better adapted metric type called "Scaled Enflo type", for which they show that a Banach space is of linear type p iff it is of scaled Enflo type p.

Some contributions of Mendel and Naor (2007 and 2008).

• Definition of a better adapted metric type called "Scaled Enflo type", for which they show that a Banach space is of linear type p iff it is of scaled Enflo type p.

• Definition of a good notion of metric cotype for which they showed two major results (Annals of Math. 2008).

Some contributions of Mendel and Naor (2007 and 2008).

• Definition of a better adapted metric type called “Scaled Enflo type”, for which they show that a Banach space is of linear type p iff it is of scaled Enflo type p.

• Definition of a good notion of metric cotype for which they showed two major results (Annals of Math. 2008).

Mendel-Naor (2008)

Some contributions of Mendel and Naor (2007 and 2008).

• Definition of a better adapted metric type called "Scaled Enflo type", for which they show that a Banach space is of linear type p iff it is of scaled Enflo type p.

• Definition of a good notion of metric cotype for which they showed two major results (Annals of Math. 2008).

Mendel-Naor (2008)

• A Banach space is of linear cotype q iff it is of metric cotype q.

Some contributions of Mendel and Naor (2007 and 2008).

• Definition of a better adapted metric type called "Scaled Enflo type", for which they show that a Banach space is of linear type p iff it is of scaled Enflo type p.

• Definition of a good notion of metric cotype for which they showed two major results (Annals of Math. 2008).

Mendel-Naor (2008)

• A Banach space is of linear cotype q iff it is of metric cotype q.

• A metric space is of trivial metric cotype iff it contains uniformly bi-Lipschitz copies of the spaces $C_n^m = (\{1, .., m\}^n, \| \ \|_\infty)$.

Super-reflexivity.

For $N \in \mathbb{N}$, denote $D_N = \{\emptyset\} \cup \cup_{k=1}^{N} \{0,1\}^k$ the dyadic tree of height N, equipped with its geodesic distance ρ.

Super-reflexivity.

For $N \in \mathbb{N}$, denote $D_N = \{\emptyset\} \cup \cup_{k=1}^N \{0,1\}^k$ the dyadic tree of height N, equipped with its geodesic distance ρ.
We denote $D_\infty = \cup_{N\in\mathbb{N}} D_N$, the infinite dyadic tree.

Super-reflexivity.

For $N \in \mathbb{N}$, denote $D_N = \{\emptyset\} \cup \cup_{k=1}^{N} \{0,1\}^k$ the dyadic tree of height N, equipped with its geodesic distance ρ.
We denote $D_\infty = \cup_{N \in \mathbb{N}} D_N$, the infinite dyadic tree.

Bourgain (1986)

A Banach space X is super-reflexive if and only if

$$\lim_{N \to \infty} C_X(D_N) = \infty.$$

Super-reflexivity.

For $N \in \mathbb{N}$, denote $D_N = \{\emptyset\} \cup \cup_{k=1}^N \{0,1\}^k$ the dyadic tree of height N, equipped with its geodesic distance ρ.
We denote $D_\infty = \cup_{N\in\mathbb{N}} D_N$, the infinite dyadic tree.

Bourgain (1986)

A Banach space X is super-reflexive if and only if

$$\lim_{N\to\infty} C_X(D_N) = \infty.$$

Proof : $(\Leftarrow)$: Uses James' criterion.

Super-reflexivity.

For $N \in \mathbb{N}$, denote $D_N = \{\emptyset\} \cup \cup_{k=1}^{N} \{0,1\}^k$ the dyadic tree of height N, equipped with its geodesic distance ρ.
We denote $D_\infty = \cup_{N \in \mathbb{N}} D_N$, the infinite dyadic tree.

Bourgain (1986)

A Banach space X is super-reflexive if and only if

$$\lim_{N \to \infty} C_X(D_N) = \infty.$$

Proof : $(\Leftarrow)$: Uses James' criterion.
$(\Rightarrow)$: Kloeckner's argument (2013).

Super-reflexivity.

For $N \in \mathbb{N}$, denote $D_N = \{\emptyset\} \cup \cup_{k=1}^{N} \{0,1\}^k$ the dyadic tree of height N, equipped with its geodesic distance ρ.
We denote $D_\infty = \cup_{N \in \mathbb{N}} D_N$, the infinite dyadic tree.

Bourgain (1986)

A Banach space X is super-reflexive if and only if

$$\lim_{N \to \infty} C_X(D_N) = \infty.$$

Proof : $(\Leftarrow)$: Uses James' criterion.
$(\Rightarrow)$: Kloeckner's argument (2013).

Baudier (2007)

A Banach space X is not super-reflexive if and only if D_∞ Lipschitz embeds into X.

Super-reflexivity.

For $N \in \mathbb{N}$, denote $D_N = \{\emptyset\} \cup \cup_{k=1}^N \{0,1\}^k$ the dyadic tree of height N, equipped with its geodesic distance ρ.
We denote $D_\infty = \cup_{N\in\mathbb{N}} D_N$, the infinite dyadic tree.

Bourgain (1986)

A Banach space X is super-reflexive if and only if

$$\lim_{N\to\infty} C_X(D_N) = \infty.$$

Proof : $(\Leftarrow)$: Uses James' criterion.
$(\Rightarrow)$: Kloeckner's argument (2013).

Baudier (2007)

A Banach space X is not super-reflexive if and only if D_∞ Lipschitz embeds into X.

Idea of proof : Assume X is not super-reflexive. Combine Bourgain's embedding technique of (D_{2^N}, ρ) into X with the usual convex-gluing technique.

IV.2 Asymptotic properties.

IV.2 Asymptotic properties.

Various asymptotic moduli. Let $(X, \| \ \|)$ be a Banach space.

IV.2 Asymptotic properties.

Various asymptotic moduli. Let $(X, \| \ \|)$ be a Banach space.

For $t > 0$, $x \in S_X$ and Y a closed linear subspace of X, we define

IV.2 Asymptotic properties.

Various asymptotic moduli. Let $(X, \| \ \|)$ be a Banach space.

For $t > 0$, $x \in S_X$ and Y a closed linear subspace of X, we define

$$\overline{\rho}(t, x, Y) = \sup_{y \in S_Y} \|x + ty\| - 1 \quad \text{and} \quad \overline{\delta}(t, x, Y) = \inf_{y \in S_Y} \|x + ty\| - 1.$$

IV.2 Asymptotic properties.

Various asymptotic moduli. Let $(X, \| \ \|)$ be a Banach space.

For $t > 0$, $x \in S_X$ and Y a closed linear subspace of X, we define

$$\overline{\rho}(t,x,Y) = \sup_{y \in S_Y} \|x + ty\| - 1 \quad \text{and} \quad \overline{\delta}(t,x,Y) = \inf_{y \in S_Y} \|x + ty\| - 1.$$

Then

$$\overline{\rho}_X(t) = \sup_{x \in S_X} \inf_{\dim(X/Y)<\infty} \overline{\rho}(t,x,Y); \quad \overline{\delta}_X(t) = \inf_{x \in S_X} \sup_{\dim(X/Y)<\infty} \overline{\delta}(t,x,Y).$$

IV.2 Asymptotic properties.

Various asymptotic moduli. Let $(X, \| \ \|)$ be a Banach space.

For $t > 0$, $x \in S_X$ and Y a closed linear subspace of X, we define

$$\overline{\rho}(t, x, Y) = \sup_{y \in S_Y} \|x + ty\| - 1 \quad \text{and} \quad \overline{\delta}(t, x, Y) = \inf_{y \in S_Y} \|x + ty\| - 1.$$

Then

$$\overline{\rho}_X(t) = \sup_{x \in S_X} \inf_{\dim(X/Y) < \infty} \overline{\rho}(t, x, Y); \quad \overline{\delta}_X(t) = \inf_{x \in S_X} \sup_{\dim(X/Y) < \infty} \overline{\delta}(t, x, Y).$$

- The norm is *asymptotically uniformly smooth* (AUS) if $\lim_{t \to 0} \frac{\overline{\rho}_X(t)}{t} = 0$.

IV.2 Asymptotic properties.

Various asymptotic moduli. Let $(X, \| \ \|)$ be a Banach space.

For $t > 0$, $x \in S_X$ and Y a closed linear subspace of X, we define

$$\overline{\rho}(t, x, Y) = \sup_{y \in S_Y} \|x + ty\| - 1 \quad \text{and} \quad \overline{\delta}(t, x, Y) = \inf_{y \in S_Y} \|x + ty\| - 1.$$

Then

$$\overline{\rho}_X(t) = \sup_{x \in S_X} \inf_{\dim(X/Y) < \infty} \overline{\rho}(t, x, Y); \quad \overline{\delta}_X(t) = \inf_{x \in S_X} \sup_{\dim(X/Y) < \infty} \overline{\delta}(t, x, Y).$$

- The norm is *asymptotically uniformly smooth* (AUS) if $\lim_{t \to 0} \frac{\overline{\rho}_X(t)}{t} = 0$.
- It is *asymptotically uniformly convex* (AUC) if $\overline{\delta}_X(t) > 0$, for all $t > 0$.

IV.2 Asymptotic properties.

Various asymptotic moduli. Let $(X, \| \ \|)$ be a Banach space.

For $t > 0$, $x \in S_X$ and Y a closed linear subspace of X, we define

$$\overline{\rho}(t,x,Y) = \sup_{y \in S_Y} \|x+ty\| - 1 \quad \text{and} \quad \overline{\delta}(t,x,Y) = \inf_{y \in S_Y} \|x+ty\| - 1.$$

Then

$$\overline{\rho}_X(t) = \sup_{x \in S_X} \inf_{\dim(X/Y)<\infty} \overline{\rho}(t,x,Y); \quad \overline{\delta}_X(t) = \inf_{x \in S_X} \sup_{\dim(X/Y)<\infty} \overline{\delta}(t,x,Y).$$

- The norm is *asymptotically uniformly smooth* (AUS) if $\lim_{t \to 0} \frac{\overline{\rho}_X(t)}{t} = 0$.
- It is *asymptotically uniformly convex* (AUC) if $\overline{\delta}_X(t) > 0$, for all $t > 0$.

Similarly, there is in X^* a modulus of weak* asymptotic uniform convexity :

$$\overline{\delta}^*_X(t) = \inf_{x^* \in S_{X^*}} \sup_{E} \inf_{y^* \in S_{E^\perp}} \{\|x^* + ty^*\| - 1\},$$

where E runs through all finite dimensional subspaces of X.

IV.2 Asymptotic properties.

Various asymptotic moduli. Let $(X, \| \ \|)$ be a Banach space.

For $t > 0$, $x \in S_X$ and Y a closed linear subspace of X, we define

$$\overline{\rho}(t,x,Y) = \sup_{y \in S_Y} \|x + ty\| - 1 \quad \text{and} \quad \overline{\delta}(t,x,Y) = \inf_{y \in S_Y} \|x + ty\| - 1.$$

Then

$$\overline{\rho}_X(t) = \sup_{x \in S_X} \inf_{\dim(X/Y)<\infty} \overline{\rho}(t,x,Y); \quad \overline{\delta}_X(t) = \inf_{x \in S_X} \sup_{\dim(X/Y)<\infty} \overline{\delta}(t,x,Y).$$

- The norm is *asymptotically uniformly smooth* (AUS) if $\lim_{t \to 0} \frac{\overline{\rho}_X(t)}{t} = 0$.
- It is *asymptotically uniformly convex* (AUC) if $\overline{\delta}_X(t) > 0$, for all $t > 0$.

Similarly, there is in X^* a modulus of weak* asymptotic uniform convexity :

$$\overline{\delta}^*_X(t) = \inf_{x^* \in S_{X^*}} \sup_{E} \inf_{y^* \in S_{E^\perp}} \{\|x^* + ty^*\| - 1\},$$

where E runs through all finite dimensional subspaces of X.

- The norm of X^* is *weak* uniformly asymptotically convex* (w*-AUC) if $\overline{\delta}^*_X(t) > 0$, for all $t > 0$.

Let $p \in [1, \infty[$.

Let $p \in [1, \infty[$.

- The norm of X is p-AUS if : $\exists C > 0 \;\; \forall t > 0 \;\; \overline{\rho}_X(t) \leq Ct^p$ $(1 < p < \infty)$.

Let $p \in [1, \infty[$.

- The norm of X is p-AUS if : $\exists C > 0 \ \ \forall t > 0 \ \ \overline{\rho}_X(t) \leq Ct^p$ $(1 < p < \infty)$.
- The norm of X is p-AUC if : $\exists C > 0 \ \ \forall t > 0 \ \ \overline{\delta}_X(t) \geq Ct^p$.

Let $p \in [1, \infty[$.

- The norm of X is p-AUS if : $\exists C > 0 \ \ \forall t > 0 \ \ \overline{\rho}_X(t) \leq Ct^p$ $(1 < p < \infty)$.
- The norm of X is p-AUC if : $\exists C > 0 \ \ \forall t > 0 \ \ \overline{\delta}_X(t) \geq Ct^p$.

<u>**Notation :**</u> $X \in \langle P \rangle$ if X admits an equivalent norm with property P.

Let $p \in [1, \infty[$.

- The norm of X is p-AUS if : $\exists C > 0 \;\; \forall t > 0 \;\; \overline{\rho}_X(t) \leq Ct^p$ $(1 < p < \infty)$.
- The norm of X is p-AUC if : $\exists C > 0 \;\; \forall t > 0 \;\; \overline{\delta}_X(t) \geq Ct^p$.

Notation : $X \in \langle P \rangle$ if X admits an equivalent norm with property P.

Property (β) of Rolewicz

We say that X has *property* (β) *of Rolewicz* if for every $t > 0$, there exists $\delta > 0$ such that for any $x \in B_X$ and any t-separated sequence (x_n) in B_X :

$$\exists n \in \mathbb{N} \quad \left\| \frac{x + x_n}{2} \right\| \leq 1 - \delta.$$

Let $p \in [1, \infty[$.

- The norm of X is p-AUS if : $\exists C > 0 \ \ \forall t > 0 \ \ \overline{\rho}_X(t) \leq Ct^p$ $(1 < p < \infty)$.
- The norm of X is p-AUC if : $\exists C > 0 \ \ \forall t > 0 \ \ \overline{\delta}_X(t) \geq Ct^p$.

Notation : $X \in \langle P \rangle$ if X admits an equivalent norm with property P.

Property (β) of Rolewicz

We say that X has *property* (β) *of Rolewicz* if for every $t > 0$, there exists $\delta > 0$ such that for any $x \in B_X$ and any t-separated sequence (x_n) in B_X :

$$\exists n \in \mathbb{N} \quad \left\| \frac{x + x_n}{2} \right\| \leq 1 - \delta.$$

The infimum of such δ's is denoted $\overline{\beta}_X(t)$.

Let $p \in [1, \infty[$.

- The norm of X is p-AUS if : $\exists C > 0 \;\; \forall t > 0 \;\; \overline{\rho}_X(t) \leq Ct^p$ $(1 < p < \infty)$.
- The norm of X is p-AUC if : $\exists C > 0 \;\; \forall t > 0 \;\; \overline{\delta}_X(t) \geq Ct^p$.

Notation : $X \in \langle P \rangle$ if X admits an equivalent norm with property P.

Property (β) of Rolewicz

We say that X has *property* (β) *of Rolewicz* if for every $t > 0$, there exists $\delta > 0$ such that for any $x \in B_X$ and any t-separated sequence (x_n) in B_X :

$$\exists n \in \mathbb{N} \quad \left\| \frac{x + x_n}{2} \right\| \leq 1 - \delta.$$

The infimum of such δ's is denoted $\overline{\beta}_X(t)$.

Examples :

Let $p \in [1, \infty[$.

- The norm of X is p-AUS if : $\exists C > 0 \;\; \forall t > 0 \;\; \overline{\rho}_X(t) \leq Ct^p$ $(1 < p < \infty)$.
- The norm of X is p-AUC if : $\exists C > 0 \;\; \forall t > 0 \;\; \overline{\delta}_X(t) \geq Ct^p$.

Notation : $X \in \langle P \rangle$ if X admits an equivalent norm with property P.

Property (β) of Rolewicz

We say that X has *property* (β) *of Rolewicz* if for every $t > 0$, there exists $\delta > 0$ such that for any $x \in B_X$ and any t-separated sequence (x_n) in B_X :

$$\exists n \in \mathbb{N} \quad \left\| \frac{x + x_n}{2} \right\| \leq 1 - \delta.$$

The infimum of such δ's is denoted $\overline{\beta}_X(t)$.

Examples :

1) Uniformly convex spaces.

Let $p \in [1, \infty[$.

- The norm of X is p-AUS if : $\exists C > 0 \;\; \forall t > 0 \;\; \overline{\rho}_X(t) \leq Ct^p$ $(1 < p < \infty)$.
- The norm of X is p-AUC if : $\exists C > 0 \;\; \forall t > 0 \;\; \overline{\delta}_X(t) \geq Ct^p$.

Notation : $X \in \langle P \rangle$ if X admits an equivalent norm with property P.

Property (β) of Rolewicz

We say that X has *property* (β) *of Rolewicz* if for every $t > 0$, there exists $\delta > 0$ such that for any $x \in B_X$ and any t-separated sequence (x_n) in B_X :

$$\exists n \in \mathbb{N} \quad \left\| \frac{x + x_n}{2} \right\| \leq 1 - \delta.$$

The infimum of such δ's is denoted $\overline{\beta}_X(t)$.

Examples :

1) Uniformly convex spaces.

2) $X = \left(\sum_{n=1}^{\infty} F_n \right)_{\ell_p}$, where $p \in]1, +\infty[$ and the F_n's are finite dimensional.

Let $p \in [1, \infty[$.

- The norm of X is p-AUS if : $\exists C > 0 \;\; \forall t > 0 \;\; \overline{\rho}_X(t) \leq Ct^p$ $(1 < p < \infty)$.
- The norm of X is p-AUC if : $\exists C > 0 \;\; \forall t > 0 \;\; \overline{\delta}_X(t) \geq Ct^p$.

Notation : $X \in \langle P \rangle$ if X admits an equivalent norm with property P.

Property (β) of Rolewicz

We say that X has *property* (β) *of Rolewicz* if for every $t > 0$, there exists $\delta > 0$ such that for any $x \in B_X$ and any t-separated sequence (x_n) in B_X :

$$\exists n \in \mathbb{N} \quad \left\| \frac{x + x_n}{2} \right\| \leq 1 - \delta.$$

The infimum of such δ's is denoted $\overline{\beta}_X(t)$.

Examples :

1) Uniformly convex spaces.

2) $X = \left(\sum_{n=1}^{\infty} F_n \right)_{\ell_p}$, where $p \in]1, +\infty[$ and the F_n's are finite dimensional. If $F_n = \ell_\infty^n$, X is not superreflexive.

Let $p \in [1, \infty[$.

- The norm of X is p-AUS if : $\exists C > 0 \ \ \forall t > 0 \ \ \overline{\rho}_X(t) \leq Ct^p$ $(1 < p < \infty)$.
- The norm of X is p-AUC if : $\exists C > 0 \ \ \forall t > 0 \ \ \overline{\delta}_X(t) \geq Ct^p$.

<u>**Notation :**</u> $X \in \langle P \rangle$ if X admits an equivalent norm with property P.

Property (β) of Rolewicz

We say that X has *property* (β) *of Rolewicz* if for every $t > 0$, there exists $\delta > 0$ such that for any $x \in B_X$ and any t-separated sequence (x_n) in B_X :

$$\exists n \in \mathbb{N} \quad \left\| \frac{x + x_n}{2} \right\| \leq 1 - \delta.$$

The infimum of such δ's is denoted $\overline{\beta}_X(t)$.

<u>**Examples :**</u>

1) Uniformly convex spaces.

2) $X = \left(\sum_{n=1}^{\infty} F_n\right)_{\ell_p}$, where $p \in]1, +\infty[$ and the F_n's are finite dimensional. If $F_n = \ell_\infty^n$, X is not superreflexive.

<u>**Remark :**</u> Propery (β) implies reflexivity (easy with James'criterion).

Let $p \in [1, \infty[$.
- The norm of X is p-AUS if : $\exists C > 0 \ \ \forall t > 0 \ \ \overline{\rho}_X(t) \leq Ct^p$ $(1 < p < \infty)$.
- The norm of X is p-AUC if : $\exists C > 0 \ \ \forall t > 0 \ \ \overline{\delta}_X(t) \geq Ct^p$.

Notation : $X \in \langle P \rangle$ if X admits an equivalent norm with property P.

Property (β) of Rolewicz

We say that X has *property* (β) *of Rolewicz* if for every $t > 0$, there exists $\delta > 0$ such that for any $x \in B_X$ and any t-separated sequence (x_n) in B_X :

$$\exists n \in \mathbb{N} \quad \left\| \frac{x + x_n}{2} \right\| \leq 1 - \delta.$$

The infimum of such δ's is denoted $\overline{\beta}_X(t)$.

Examples :

1) Uniformly convex spaces.

2) $X = \left(\sum_{n=1}^{\infty} F_n \right)_{\ell_p}$, where $p \in]1, +\infty[$ and the F_n's are finite dimensional. If $F_n = \ell_\infty^n$, X is not superreflexive.

Remark : Propery (β) implies reflexivity (easy with James'criterion). The converse is false (even up to renorming) :

Let $p \in [1, \infty[$.

- The norm of X is p-AUS if : $\exists C > 0 \;\; \forall t > 0 \;\; \overline{\rho}_X(t) \leq Ct^p$ $(1 < p < \infty)$.
- The norm of X is p-AUC if : $\exists C > 0 \;\; \forall t > 0 \;\; \overline{\delta}_X(t) \geq Ct^p$.

Notation : $X \in \langle P \rangle$ if X admits an equivalent norm with property P.

Property (β) of Rolewicz

We say that X has *property* (β) *of Rolewicz* if for every $t > 0$, there exists $\delta > 0$ such that for any $x \in B_X$ and any t-separated sequence (x_n) in B_X :

$$\exists n \in \mathbb{N} \quad \left\| \frac{x + x_n}{2} \right\| \leq 1 - \delta.$$

The infimum of such δ's is denoted $\overline{\beta}_X(t)$.

Examples :

1) Uniformly convex spaces.

2) $X = \left(\sum_{n=1}^{\infty} F_n \right)_{\ell_p}$, where $p \in]1, +\infty[$ and the F_n's are finite dimensional. If $F_n = \ell_\infty^n$, X is not superreflexive.

Remark : Propery (β) implies reflexivity (easy with James'criterion). The converse is false (even up to renorming) : $X = \left(\sum_{n=1}^{\infty} \ell_{1+2^{-n}} \right)_{\ell_2}$.

The Szlenk index.

The Szlenk index.

Let X be a Banach space, K a weak*-compact subset of X^* and $\varepsilon > 0$.

The Szlenk index.

Let X be a Banach space, K a weak*-compact subset of X^* and $\varepsilon > 0$. Denote $\mathcal{V}$ the set of all relatively weak*-open subsets V of K such that the diameter of V is less than ε. Then, define the derivation

$$s_\varepsilon K = K \setminus \cup\{V : V \in \mathcal{V}\}$$

The Szlenk index.

Let X be a Banach space, K a weak*-compact subset of X^* and $\varepsilon > 0$. Denote $\mathcal{V}$ the set of all relatively weak*-open subsets V of K such that the diameter of V is less than ε. Then, define the derivation

$$s_\varepsilon K = K \setminus \cup\{V : V \in \mathcal{V}\}$$

and inductively

$$s_\varepsilon^{\alpha+1} K = s_\varepsilon(s_\varepsilon^\alpha K) \text{ and } s_\varepsilon^\alpha K = \bigcap_{\beta<\alpha} s_\varepsilon^\beta K \text{ if } \alpha \text{ is a limit ordinal.}$$

The Szlenk index.

Let X be a Banach space, K a weak*-compact subset of X^* and $\varepsilon > 0$. Denote $\mathcal{V}$ the set of all relatively weak*-open subsets V of K such that the diameter of V is less than ε. Then, define the derivation

$$s_\varepsilon K = K \setminus \cup\{V : V \in \mathcal{V}\}$$

and inductively

$$s_\varepsilon^{\alpha+1} K = s_\varepsilon(s_\varepsilon^\alpha K) \text{ and } s_\varepsilon^\alpha K = \bigcap_{\beta<\alpha} s_\varepsilon^\beta K \text{ if } \alpha \text{ is a limit ordinal.}$$

Define $\mathrm{Sz}(X, \varepsilon)$ to be the least ordinal α so that $s_\varepsilon^\alpha B_{X^*} = \emptyset$, if it exists. Otherwise we write $\mathrm{Sz}(X, \varepsilon) = \infty$.

The Szlenk index.

Let X be a Banach space, K a weak*-compact subset of X^* and $\varepsilon > 0$. Denote $\mathcal{V}$ the set of all relatively weak*-open subsets V of K such that the diameter of V is less than ε. Then, define the derivation

$$s_\varepsilon K = K \setminus \cup\{V : V \in \mathcal{V}\}$$

and inductively

$$s_\varepsilon^{\alpha+1} K = s_\varepsilon(s_\varepsilon^\alpha K) \ \text{ and } \ s_\varepsilon^\alpha K = \bigcap_{\beta<\alpha} s_\varepsilon^\beta K \text{ if } \alpha \text{ is a limit ordinal.}$$

Define $\mathrm{Sz}(X, \varepsilon)$ to be the least ordinal α so that $s_\varepsilon^\alpha B_{X^*} = \emptyset$, if it exists. Otherwise we write $\mathrm{Sz}(X, \varepsilon) = \infty$.

Finally, the *Szlenk index* of X is $\mathrm{Sz}(X) = \sup_{\varepsilon>0} \mathrm{Sz}(X, \varepsilon)$.

The Szlenk index.

Let X be a Banach space, K a weak*-compact subset of X^* and $\varepsilon > 0$. Denote $\mathcal{V}$ the set of all relatively weak*-open subsets V of K such that the diameter of V is less than ε. Then, define the derivation

$$s_\varepsilon K = K \setminus \cup\{V : V \in \mathcal{V}\}$$

and inductively

$$s_\varepsilon^{\alpha+1} K = s_\varepsilon(s_\varepsilon^\alpha K) \text{ and } s_\varepsilon^\alpha K = \bigcap_{\beta<\alpha} s_\varepsilon^\beta K \text{ if } \alpha \text{ is a limit ordinal.}$$

Define $\mathrm{Sz}(X,\varepsilon)$ to be the least ordinal α so that $s_\varepsilon^\alpha B_{X^*} = \emptyset$, if it exists. Otherwise we write $\mathrm{Sz}(X,\varepsilon) = \infty$.

Finally, the *Szlenk index* of X is $\mathrm{Sz}(X) = \sup_{\varepsilon>0} \mathrm{Sz}(X,\varepsilon)$.

Asymptotic analogue of Bourgain's theorem ?

The Szlenk index.

Let X be a Banach space, K a weak*-compact subset of X^* and $\varepsilon > 0$. Denote $\mathcal{V}$ the set of all relatively weak*-open subsets V of K such that the diameter of V is less than ε. Then, define the derivation

$$s_\varepsilon K = K \setminus \cup\{V : V \in \mathcal{V}\}$$

and inductively

$$s_\varepsilon^{\alpha+1} K = s_\varepsilon(s_\varepsilon^\alpha K) \ \text{ and } \ s_\varepsilon^\alpha K = \bigcap_{\beta<\alpha} s_\varepsilon^\beta K \text{ if } \alpha \text{ is a limit ordinal.}$$

Define $\mathrm{Sz}(X,\varepsilon)$ to be the least ordinal α so that $s_\varepsilon^\alpha B_{X^*} = \emptyset$, if it exists. Otherwise we write $\mathrm{Sz}(X,\varepsilon) = \infty$.

Finally, the *Szlenk index* of X is $\mathrm{Sz}(X) = \sup_{\varepsilon>0} \mathrm{Sz}(X,\varepsilon)$.

Asymptotic analogue of Bourgain's theorem ?

Try to describe the (uniform) Lipschitz embeddability of the countably branching trees : $T_N = \{\emptyset\} \cup \cup_{k=1}^N \mathbb{N}^k$ or of $T_\infty = \cup_{N\in\mathbb{N}} T_N$ (all equipped with the geodesic distance).

Theorem

Let X be a reflexive Banach space. The following assertions are equivalent.

Theorem

Let X be a reflexive Banach space. The following assertions are equivalent.

(i) $\mathrm{Sz}(X) \leq \omega$ and $\mathrm{Sz}(X^*) \leq \omega$.

Theorem

Let X be a reflexive Banach space. The following assertions are equivalent.

(i) $\mathrm{Sz}(X) \leq \omega$ and $\mathrm{Sz}(X^*) \leq \omega$.

(ii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS} \rangle \cap \langle q\text{-AUC} \rangle$.

Theorem

Let X be a reflexive Banach space. The following assertions are equivalent.

(i) $\mathrm{Sz}(X) \leq \omega$ and $\mathrm{Sz}(X^*) \leq \omega$.

(ii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS}\rangle \cap \langle q\text{-AUC}\rangle$.

(iii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS and } q\text{-AUC}\rangle$.

Theorem

Let X be a reflexive Banach space. The following assertions are equivalent.

(i) $\mathrm{Sz}(X) \leq \omega$ and $\mathrm{Sz}(X^*) \leq \omega$.

(ii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS}\rangle \cap \langle q\text{-AUC}\rangle$.

(iii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS and } q\text{-AUC}\rangle$.

(iv) $X \in \langle \beta \rangle$ (with power type modulus).

Theorem

Let X be a reflexive Banach space. The following assertions are equivalent.

(i) $\mathrm{Sz}(X) \leq \omega$ and $\mathrm{Sz}(X^*) \leq \omega$.

(ii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS}\rangle \cap \langle q\text{-AUC}\rangle$.

(iii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS and } q\text{-AUC}\rangle$.

(iv) $X \in \langle \beta \rangle$ (with power type modulus).

(v) $\lim_{N \to +\infty} C_X(T_N) = +\infty$.

Theorem

Let X be a reflexive Banach space. The following assertions are equivalent.

(i) $\mathrm{Sz}(X) \leq \omega$ and $\mathrm{Sz}(X^*) \leq \omega$.

(ii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS}\rangle \cap \langle q\text{-AUC}\rangle$.

(iii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS and } q\text{-AUC}\rangle$.

(iv) $X \in \langle \beta \rangle$ (with power type modulus).

(v) $\lim_{N \to +\infty} C_X(T_N) = +\infty$.

(vi) T_∞ does not bi-Lipschitz embed into X.

Theorem

Let X be a reflexive Banach space. The following assertions are equivalent.

(i) $\mathrm{Sz}(X) \leq \omega$ and $\mathrm{Sz}(X^*) \leq \omega$.

(ii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS}\rangle \cap \langle q\text{-AUC}\rangle$.

(iii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS and } q\text{-AUC}\rangle$.

(iv) $X \in \langle \beta \rangle$ (with power type modulus).

(v) $\lim_{N \to +\infty} C_X(T_N) = +\infty$.

(vi) T_∞ does not bi-Lipschitz embed into X.

Remark : There is a characterization of super-reflexivity similar to (i) in terms of dentability indices.

Theorem

Let X be a reflexive Banach space. The following assertions are equivalent.

(i) $\mathrm{Sz}(X) \leq \omega$ and $\mathrm{Sz}(X^*) \leq \omega$.

(ii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS}\rangle \cap \langle q\text{-AUC}\rangle$.

(iii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS and } q\text{-AUC}\rangle$.

(iv) $X \in \langle \beta \rangle$ (with power type modulus).

(v) $\lim_{N \to +\infty} C_X(T_N) = +\infty$.

(vi) T_∞ does not bi-Lipschitz embed into X.

Remark : There is a characterization of super-reflexivity similar to (i) in terms of dentability indices.

"Proof".

Theorem

Let X be a reflexive Banach space. The following assertions are equivalent.

(i) $\mathrm{Sz}(X) \leq \omega$ and $\mathrm{Sz}(X^*) \leq \omega$.

(ii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS}\rangle \cap \langle q\text{-AUC}\rangle$.

(iii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS and } q\text{-AUC}\rangle$.

(iv) $X \in \langle \beta \rangle$ (with power type modulus).

(v) $\lim_{N\to+\infty} C_X(T_N) = +\infty$.

(vi) T_∞ does not bi-Lipschitz embed into X.

Remark : There is a characterization of super-reflexivity similar to (i) in terms of dentability indices.

"Proof".

(iv)⇒(v). Reproduce Kloeckner's argument (Baudier-Zhang, 2016).

Theorem

Let X be a reflexive Banach space. The following assertions are equivalent.

(i) $\mathrm{Sz}(X) \leq \omega$ and $\mathrm{Sz}(X^*) \leq \omega$.

(ii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS}\rangle \cap \langle q\text{-AUC}\rangle$.

(iii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS and } q\text{-AUC}\rangle$.

(iv) $X \in \langle \beta \rangle$ (with power type modulus).

(v) $\lim_{N\to+\infty} C_X(T_N) = +\infty$.

(vi) T_∞ does not bi-Lipschitz embed into X.

Remark : There is a characterization of super-reflexivity similar to (i) in terms of dentability indices.

"Proof".

(iv)⇒(v). Reproduce Kloeckner's argument (Baudier-Zhang, 2016).

(v)⇒(vi). Clear.

Theorem

Let X be a reflexive Banach space. The following assertions are equivalent.

(i) $\mathrm{Sz}(X) \leq \omega$ and $\mathrm{Sz}(X^*) \leq \omega$.

(ii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS} \rangle \cap \langle q\text{-AUC} \rangle$.

(iii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS and } q\text{-AUC} \rangle$.

(iv) $X \in \langle \beta \rangle$ (with power type modulus).

(v) $\lim_{N \to +\infty} C_X(T_N) = +\infty$.

(vi) T_∞ does not bi-Lipschitz embed into X.

Remark : There is a characterization of super-reflexivity similar to (i) in terms of dentability indices.

"Proof".

(iv)⇒(v). Reproduce Kloeckner's argument (Baudier-Zhang, 2016).

(v)⇒(vi). Clear.

(vi)⇒(i). Baudier-Kalton-L. (2010).

Theorem

Let X be a reflexive Banach space. The following assertions are equivalent.

(i) $\mathrm{Sz}(X) \leq \omega$ and $\mathrm{Sz}(X^*) \leq \omega$.

(ii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS}\rangle \cap \langle q\text{-AUC}\rangle$.

(iii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS and } q\text{-AUC}\rangle$.

(iv) $X \in \langle \beta \rangle$ (with power type modulus).

(v) $\lim_{N\to+\infty} C_X(T_N) = +\infty$.

(vi) T_∞ does not bi-Lipschitz embed into X.

Remark : There is a characterization of super-reflexivity similar to (i) in terms of dentability indices.

"Proof".

(iv)⇒(v). Reproduce Kloeckner's argument (Baudier-Zhang, 2016).

(v)⇒(vi). Clear.

(vi)⇒(i). Baudier-Kalton-L. (2010). (details later ?)

Theorem

Let X be a reflexive Banach space. The following assertions are equivalent.

(i) $\mathrm{Sz}(X) \leq \omega$ and $\mathrm{Sz}(X^*) \leq \omega$.

(ii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS}\rangle \cap \langle q\text{-AUC}\rangle$.

(iii) There exist $p, q > 1$ so that $X \in \langle p\text{-AUS and } q\text{-AUC}\rangle$.

(iv) $X \in \langle \beta \rangle$ (with power type modulus).

(v) $\lim_{N \to +\infty} C_X(T_N) = +\infty$.

(vi) T_∞ does not bi-Lipschitz embed into X.

Remark : There is a characterization of super-reflexivity similar to (i) in terms of dentability indices.

"Proof".

(iv)⇒(v). Reproduce Kloeckner's argument (Baudier-Zhang, 2016).

(v)⇒(vi). Clear.

(vi)⇒(i). Baudier-Kalton-L. (2010). (details later ?)

In [B-K-L 2010], there is a direct complicated proof of (i)⇒(v).

Knaust-Odell-Schlumprecht (1999), Raja (2010)

Let X be a Banach space. Then $\mathrm{Sz}(X) \leq \omega$ if and only if there exists $p \in (1, \infty)$ such that $X \in \langle p\text{-AUS} \rangle$.

Knaust-Odell-Schlumprecht (1999), Raja (2010)

Let X be a Banach space. Then $\mathrm{Sz}(X) \leq \omega$ if and only if there exists $p \in (1, \infty)$ such that $X \in \langle p\text{-AUS} \rangle$.

Easy fact. Let $p \in (1, \infty)$ and p' be its conjugate exponent. Then $\| \, \|_X$ is p-AUS if and only if $\| \, \|_{X^*}$ is p'-weak* AUC.

Knaust-Odell-Schlumprecht (1999), Raja (2010)

Let X be a Banach space. Then $\mathrm{Sz}(X) \leq \omega$ if and only if there exists $p \in (1, \infty)$ such that $X \in \langle p\text{-AUS}\rangle$.

Easy fact. Let $p \in (1, \infty)$ and p' be its conjugate exponent. Then $\| \ \|_X$ is p-AUS if and only if $\| \ \|_{X^*}$ is p'-weak* AUC.

(i)⇒(ii). Follows immediately from this remark and the previous theorem.

Knaust-Odell-Schlumprecht (1999), Raja (2010)

Let X be a Banach space. Then $\mathrm{Sz}(X) \leq \omega$ if and only if there exists $p \in (1, \infty)$ such that $X \in \langle p\text{-AUS} \rangle$.

Easy fact. Let $p \in (1, \infty)$ and p' be its conjugate exponent. Then $\| \ \|_X$ is p-AUS if and only if $\| \ \|_{X^*}$ is p'-weak* AUC.

(i)⇒(ii). Follows immediately from this remark and the previous theorem.

(iii)⇒(iv). Kutzarova (1990) and Dilworth-Kutzarova-Randrianarivony-Revalski-Zhivkov (2013).

Knaust-Odell-Schlumprecht (1999), Raja (2010)

Let X be a Banach space. Then $\mathrm{Sz}(X) \leq \omega$ if and only if there exists $p \in (1, \infty)$ such that $X \in \langle p\text{-AUS} \rangle$.

Easy fact. Let $p \in (1, \infty)$ and p' be its conjugate exponent. Then $\| \ \|_X$ is p-AUS if and only if $\| \ \|_{X^*}$ is p'-weak* AUC.

(i)⇒(ii). Follows immediately from this remark and the previous theorem.

(iii)⇒(iv). Kutzarova (1990) and Dilworth-Kutzarova-Randrianarivony-Revalski-Zhivkov (2013).

Focus on (ii)⇒(iii).

Knaust-Odell-Schlumprecht (1999), Raja (2010)

Let X be a Banach space. Then $\mathrm{Sz}(X) \leq \omega$ if and only if there exists $p \in (1, \infty)$ such that $X \in \langle p\text{-AUS}\rangle$.

Easy fact. Let $p \in (1, \infty)$ and p' be its conjugate exponent. Then $\| \ \|_X$ is p-AUS if and only if $\| \ \|_{X^*}$ is p'-weak* AUC.

(i)⇒(ii). Follows immediately from this remark and the previous theorem.

(iii)⇒(iv). Kutzarova (1990) and Dilworth-Kutzarova-Randrianarivony-Revalski-Zhivkov (2013).

Focus on (ii)⇒(iii).

Let $(X, \| \ \|)$ be a Banach space. Denote $\mathcal{S}$ the set of continuous semi-norms on X.

Knaust-Odell-Schlumprecht (1999), Raja (2010)

Let X be a Banach space. Then $\mathrm{Sz}(X) \leq \omega$ if and only if there exists $p \in (1, \infty)$ such that $X \in \langle p\text{-AUS} \rangle$.

Easy fact. Let $p \in (1, \infty)$ and p' be its conjugate exponent. Then $\| \ \|_X$ is p-AUS if and only if $\| \ \|_{X^*}$ is p'-weak* AUC.

(i)⇒(ii). Follows immediately from this remark and the previous theorem.

(iii)⇒(iv). Kutzarova (1990) and Dilworth-Kutzarova-Randrianarivony-Revalski-Zhivkov (2013).

Focus on (ii)⇒(iii).

Let $(X, \| \ \|)$ be a Banach space. Denote $\mathcal{S}$ the set of continuous semi-norms on X. We equip $\mathcal{S}$ with the metric

$$\forall N, M \in \mathcal{S} \quad d(N, M) = \sup_{x \in B_X} |N(x) - M(x)|.$$

Knaust-Odell-Schlumprecht (1999), Raja (2010)

Let X be a Banach space. Then $Sz(X) \leq \omega$ if and only if there exists $p \in (1, \infty)$ such that $X \in \langle p\text{-AUS}\rangle$.

Easy fact. Let $p \in (1, \infty)$ and p' be its conjugate exponent. Then $\| \|_X$ is p-AUS if and only if $\| \|_{X^*}$ is p'-weak* AUC.

(i)$\Rightarrow$(ii). Follows immediately from this remark and the previous theorem.

(iii)$\Rightarrow$(iv). Kutzarova (1990) and Dilworth-Kutzarova-Randrianarivony-Revalski-Zhivkov (2013).

Focus on (ii)$\Rightarrow$(iii).

Let $(X, \| \|)$ be a Banach space. Denote $\mathcal{S}$ the set of continuous semi-norms on X. We equip $\mathcal{S}$ with the metric

$$\forall N, M \in \mathcal{S} \quad d(N, M) = \sup_{x \in B_X} |N(x) - M(x)|.$$

It is clear that $(\mathcal{S}, d)$ is complete and that the set $\mathcal{P}$ of all equivalent norms on X is open in $\mathcal{S}$. So $(\mathcal{P}, d)$ is a Baire space.

Knaust-Odell-Schlumprecht (1999), Raja (2010)

Let X be a Banach space. Then $\mathrm{Sz}(X) \leq \omega$ if and only if there exists $p \in (1, \infty)$ such that $X \in \langle p\text{-AUS}\rangle$.

Easy fact. Let $p \in (1, \infty)$ and p' be its conjugate exponent. Then $\| \ \|_X$ is p-AUS if and only if $\| \ \|_{X^*}$ is p'-weak* AUC.

(i)⇒(ii). Follows immediately from this remark and the previous theorem.

(iii)⇒(iv). Kutzarova (1990) and Dilworth-Kutzarova-Randrianarivony-Revalski-Zhivkov (2013).

Focus on (ii)⇒(iii).

Let $(X, \| \ \|)$ be a Banach space. Denote $\mathcal{S}$ the set of continuous semi-norms on X. We equip $\mathcal{S}$ with the metric

$$\forall N, M \in \mathcal{S} \quad d(N, M) = \sup_{x \in B_X} |N(x) - M(x)|.$$

It is clear that $(\mathcal{S}, d)$ is complete and that the set $\mathcal{P}$ of all equivalent norms on X is open in $\mathcal{S}$. So $(\mathcal{P}, d)$ is a Baire space.

Finish it on the board.

This proof gives a norm which is AUS and AUC, This proof gives a norm which is AUS and AUC, but does not give the result about power types moduli.

This proof gives a norm which is AUS and AUC, but does not give the result about power types moduli.Indeed, the conditions “p-AUS” and “q-AUC” are not $\mathcal{G}_\delta$.

This proof gives a norm which is AUS and AUC, but does not give the result about power types moduli.Indeed, the conditions “p-AUS” and “q-AUC” are not $\mathcal{G}_\delta$.
However it is enough to deduce (iv).

This proof gives a norm which is AUS and AUC, but does not give the result about power types moduli.Indeed, the conditions “p-AUS” and “q-AUC” are not $\mathcal{G}_\delta$.
However it is enough to deduce (iv).

• For the quantitative result, one has to adapt a proof by John and Zizler (1979).

This proof gives a norm which is AUS and AUC, but does not give the result about power types moduli.Indeed, the conditions "p-AUS" and "q-AUC" are not $\mathcal{G}_\delta$.
However it is enough to deduce (iv).

- For the quantitative result, one has to adapt a proof by John and Zizler (1979).

Assume that N and M are equivalent norms on X with

$$N \leq M \leq C\, M \quad \overline{\delta}_N(t) \geq at^q \ \text{ and } \ \overline{\rho}_M(t) \leq bt^p.$$

This proof gives a norm which is AUS and AUC, but does not give the result about power types moduli.Indeed, the conditions "p-AUS" and "q-AUC" are not $\mathcal{G}_\delta$.
However it is enough to deduce (iv).

• For the quantitative result, one has to adapt a proof by John and Zizler (1979).

Assume that N and M are equivalent norms on X with

$$N \leq M \leq C\, M \quad \overline{\delta}_N(t) \geq at^q \text{ and } \overline{\rho}_M(t) \leq bt^p.$$

Set

$$\| \, \|_n^* = N^* + \frac{1}{n} M^* \text{ and } | \, | = \sum_{n=1}^{\infty} \frac{1}{n^3} \| \, \|_n.$$

This proof gives a norm which is AUS and AUC, but does not give the result about power types moduli.Indeed, the conditions "p-AUS" and "q-AUC" are not $\mathcal{G}_\delta$.
However it is enough to deduce (iv).

• For the quantitative result, one has to adapt a proof by John and Zizler (1979).

Assume that N and M are equivalent norms on X with

$$N \leq M \leq C\,M \quad \overline{\delta}_N(t) \geq at^q \ \text{ and } \ \overline{\rho}_M(t) \leq bt^p.$$

Set

$$\|\ \|_n^* = N^* + \frac{1}{n}M^* \ \text{ and } \ |\ | = \sum_{n=1}^{\infty} \frac{1}{n^3}\|\ \|_n.$$

Then $|\ |$ is p-AUS and $4q$-AUC.

This proof gives a norm which is AUS and AUC, but does not give the result about power types moduli.Indeed, the conditions “p-AUS” and “q-AUC” are not $\mathcal{G}_\delta$.
However it is enough to deduce (iv).

• For the quantitative result, one has to adapt a proof by John and Zizler (1979).

Assume that N and M are equivalent norms on X with

$$N \leq M \leq C\,M \quad \overline{\delta}_N(t) \geq at^q \ \text{ and } \ \overline{\rho}_M(t) \leq bt^p.$$

Set

$$\|\ \|_n^* = N^* + \frac{1}{n}M^* \ \text{ and } \ |\ | = \sum_{n=1}^{\infty} \frac{1}{n^3}\|\ \|_n.$$

Then $|\ |$ is p-AUS and $4q$-AUC.

<u>**Question.**</u> We do not know if both exponents can be preserved in (ii)⇒(iii).

This proof gives a norm which is AUS and AUC, but does not give the result about power types moduli.Indeed, the conditions "p-AUS" and "q-AUC" are not $\mathcal{G}_\delta$.
However it is enough to deduce (iv).

• For the quantitative result, one has to adapt a proof by John and Zizler (1979).
Assume that N and M are equivalent norms on X with

$$N \leq M \leq C\, M \quad \overline{\delta}_N(t) \geq at^q \;\text{ and }\; \overline{\rho}_M(t) \leq bt^p.$$

Set

$$\|\ \|_n^* = N^* + \frac{1}{n} M^* \;\text{ and }\; |\ | = \sum_{n=1}^{\infty} \frac{1}{n^3} \|\ \|_n.$$

Then $|\ |$ is p-AUS and $4q$-AUC.

<u>**Question.**</u> We do not know if both exponents can be preserved in (ii)⇒(iii). The similar question for UC and US renormings is a famous open question.

Corollary

Having an equivalent norm with property (β) is stable under coarse Lipschitz embeddings.

Corollary

Having an equivalent norm with property (β) is stable under coarse Lipschitz embeddings.

Indeed a Banach space which coarse Lipschitz embeds into a reflexive AUS space is reflexive (Kalton-Randrianarivony).

Corollary

Having an equivalent norm with property (β) is stable under coarse Lipschitz embeddings.

Indeed a Banach space which coarse Lipschitz embeds into a reflexive AUS space is reflexive (Kalton-Randrianarivony).

Open questions :

Corollary

Having an equivalent norm with property (β) is stable under coarse Lipschitz embeddings.

Indeed a Banach space which coarse Lipschitz embeds into a reflexive AUS space is reflexive (Kalton-Randrianarivony).

Open questions :

1) Describe the non reflexive spaces that contain a bi-Lipschitz copy of T_∞ or uniform bi-Lipschitz copies of the T_N's.

Corollary

Having an equivalent norm with property (β) is stable under coarse Lipschitz embeddings.

Indeed a Banach space which coarse Lipschitz embeds into a reflexive AUS space is reflexive (Kalton-Randrianarivony).

Open questions :

1) Describe the non reflexive spaces that contain a bi-Lipschitz copy of T_∞ or uniform bi-Lipschitz copies of the T_N's.

2) Is ($\langle AUS \rangle$ + reflexive) stable under coarse Lipschitz embeddings ?

FIN.