

# A FEW QUESTIONS ON NON LINEAR EMBEDDINGS INTO BANACH SPACES

GILLES LANCIEN

The fundamental problem in non linear Banach space theory is to describe to what extent the linear structure of a Banach space is determined by its metric structure. In other words, we try to exhibit the linear properties of Banach spaces that are stable under some particular non linear maps. These non linear maps can be of very different nature: Lipschitz isomorphisms or embeddings, uniform homeomorphisms, uniform or coarse embeddings. Then, the next goal is to characterize these linear properties in purely metric terms. Usually, these characterizations are given by the (non)embeddability of special metric spaces, which are very often fundamental metric trees or graphs. In this series of lectures, we shall illustrate this subject with a few recent results and open questions on non linear embeddings into Banach spaces. These lectures are related to the series that I gave in 2012 in Klenčí, but I will try to keep the overlap to a minimum.

We will first spend some time on isometric embeddings. After recalling the fundamental results by Mazur-Ulam, Figiel and Godefroy-Kalton, we will turn to more recent results, for instance on Banach spaces that are isometrically universal for compact metric spaces. This will lead us to discuss some variants of Figiel's theorem. We will also mention the impact of descriptive set theory through the results of Godefroy-Kalton, improved by Kurka, on Banach spaces that are isometrically universal for strictly convex or reflexive Fréchet smooth separable Banach spaces.

Next, we will turn to bi-Lipschitz embeddings. After recalling Aharoni's result on the universality of  $c_0$  and related questions, we will characterize the Banach spaces that are bi-Lipschitz universal for locally finite metric spaces, as well as very recent results obtained with similar methods.

Then we will explain the following remarkable theorem of Kalton: a Banach space which is universal for separable metric spaces and coarse embeddings cannot be reflexive. Important open questions related to this result will be advertised.

Finally, if time allows, we will describe a few metric invariants characterizing some local properties of Banach spaces, which are part of the so-called Ribe program. We will also exhibit some metric invariants characterizing some asymptotic uniform properties of Banach spaces. I believe that it would be natural to call this last line of research the Kalton program.