Generic objects in topology and functional analysis

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1. Part 1

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Part 1

Categories

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- a partial associative composition operation $\circ$ defined on arrows, where $f \circ g$ is defined $\iff$ the domain of g coincides with the domain of f.

Furthermore, for each $a \in \mathrm{Obj}(\mathfrak{K})$ there is an *identity* $\mathrm{id}_a \in \mathfrak{K}(a,a)$ satisfying $\mathrm{id}_a \circ g = g$ and $f \circ \mathrm{id}_a = f$ for $f \in \mathfrak{K}(a,x)$, $g \in \mathfrak{K}(y,a)$, $x, y \in \mathrm{Obj}(\mathfrak{K})$.

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Let $\vec{x}$ be a sequence in $\mathfrak{K}$. The colimit of $\vec{x}$ is a pair $\langle X, \{x_n^\infty\}_{n\in\mathbb{N}}\rangle$ with $x_n^\infty \colon x_n \to X$ satisfying:

1. $x_n^\infty = x_m^\infty \circ x_n^m$ for every $n < m$.
2. If $\langle Y, \{y_n^\infty\}_{n\in\mathbb{N}}\rangle$ with $y_n^\infty \colon x_n \to Y$ satisfies $y_n^\infty = y_m^\infty \circ y_n^m$ for every $n < m$ then there is a unique arrow $f \colon X \to Y$ satisfying $f \circ x_n^\infty = y_n^\infty$ for every $n \in \mathbb{N}$.

The Banach-Mazur game

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More generally, after Odd's move finishing with an object a_{2k-1}, Eve chooses $a_{2k} \in \mathrm{Obj}(\mathfrak{K})$ together with a $\mathfrak{K}$-arrow $a_{2k-1}^{2k} \colon a_{2k-1} \to a_{2k}$.

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Next, Odd chooses $a_{2k+1} \in \mathrm{Obj}(\mathfrak{K})$ together with a $\mathfrak{K}$-arrow $a_{2k}^{2k+1} \colon a_{2k} \to a_{2k+1}$. And so on...

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The result of a play is a sequence $\vec{a}$:

$$a_0 \xrightarrow{a_0^1} a_1 \longrightarrow \cdots \longrightarrow a_{2k-1} \xrightarrow{a_{2k-1}^{2k}} a_{2k} \longrightarrow \cdots$$

Generic objects

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Definition

We say that $U \in \mathtt{Obj}(\mathfrak{L})$ is $\mathfrak{K}$-generic if Odd has a strategy in the Banach-Mazur game BM $(\mathfrak{K})$ such that the colimit of the resulting sequence $\vec{a}$ is always isomorphic to U, no matter how Eve plays.

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A $\mathfrak{K}$-generic object, if exists, is unique up to isomorphism.

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Proof.

The rules for Eve and Odd are the same. □

Part 2

Example 1

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Example 2

Let $\mathfrak{M}_{\mathrm{fin}}$ be the category of finite metric spaces with isometric embeddings.
Then the Urysohn space $\mathbb{U}$ is $\mathfrak{M}_{\mathrm{fin}}$-generic.

The Gurarii space

Theorem (Gurarii 1966)

There exists a separable Banach space $\mathbb{G}$ with the following property.

(G) *For every $\varepsilon > 0$, for every finite-dimensional normed spaces $E \subseteq F$, for every linear isometric embedding $e \colon E \to \mathbb{G}$ there exists a linear ε-isometric embedding $f \colon F \to \mathbb{G}$ such that $f \restriction E = e$.*

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Elementary proof: Solecki & K. 2013.

Theorem (K. 2018)

The Gurarii space $\mathbb{G}$ is generic over the category $\mathfrak{B}_{\mathsf{fd}}$ of finite-dimensional normed spaces with linear isometric embeddings.

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Key Lemma (Solecki & K.)

Let X, Y be finite-dimensional normed spaces, let $f\colon X \to Y$ be an ε-isometry with $0 < \varepsilon < 1$. Then there exist a finite-dimensional normed space Z and isometric embeddings $i\colon X \to Z$, $j\colon Y \to Z$ such that

$$\|i - j \circ f\| \leqslant \varepsilon.$$

The amalgamation property

Definition

We say that $\mathfrak{K}$ has amalgamations at $z \in \mathrm{Obj}(\mathfrak{K})$ if for every $\mathfrak{K}$-arrows $f\colon z \to x$, $g\colon z \to y$ there exist $\mathfrak{K}$-arrows $f'\colon x \to w$, $g'\colon y \to w$ such that $f' \circ f = g' \circ g$.

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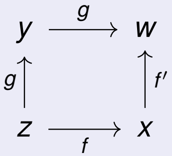

We say that $\mathfrak{K}$ has the amalgamation property (AP) if it has amalgamations at every $z \in \operatorname{Obj}(\mathfrak{K})$.

Theorem (Universality)

Assume $\mathfrak{K}$ has the AP and $X = \lim \vec{x}$, where $\vec{x}$ is a sequence in $\mathfrak{K}$. Assume U is $\mathfrak{K}$-generic.

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Assume $\mathfrak{K}$ has the AP and $X = \lim \vec{x}$, where $\vec{x}$ is a sequence in $\mathfrak{K}$. Assume U is $\mathfrak{K}$-generic. Then there exists an arrow $e \colon X \to U$.

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Part 3

The generic linear operator

The generic linear operator

Theorem (Cabello Sánchez, Garbulińska-Wegrzyn, K. 2014)

There exists a norm-one linear operator $\Omega\colon \mathbb{G} \to \mathbb{G}$ *satisfying the following condition.*

(E) *For every* $\varepsilon > 0$*, for every finite-dimensional Banach spaces* $E \subseteq F$*, for every non-expansive linear operator* $T\colon F \to \mathbb{G}$*, for every linear isometric embedding* $e\colon E \to \mathbb{G}$ *with* $\Omega \circ e = T \restriction E$*, there exists an* ε*-isometric embedding* $f\colon F \to \mathbb{G}$ *such that*

$$f \restriction E = e \qquad \text{and} \qquad \Omega \circ f = T.$$

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Theorem

For every non-expansive linear operator $S \colon X \to \mathbb{G}$ *with* X *separable, there exists a linear isometric embedding* $e \colon X \to \mathbb{G}$ *such that*

$$\Omega \circ e = S.$$

Theorem

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Theorem (Bargetz, Kakol, K. 2017)

There exists a unique graded separable Fréchet space $\mathbb{G}_\infty$ satisfying:

(E) *For every $\varepsilon > 0$, for every finite-dimensional graded Fréchet spaces $E \subseteq F$, for every linear isometric embedding $e\colon E \to \mathbb{G}_\infty$ there exists an ε-isometric embedding $f\colon F \to \mathbb{G}_\infty$ such that $f \restriction E = e$.*

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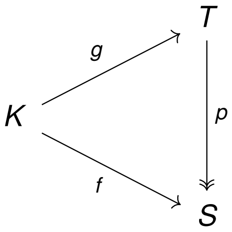

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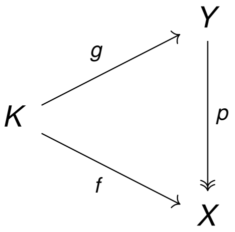

Theorem (Bielas, Walczyńska, K.)

Let 2^ω denote the Cantor set. A continuous mapping $\eta\colon K \to 2^\omega$ is $\mathfrak{K}_K$-generic $\iff$ η is a topological embedding and $\eta[K]$ is nowhere dense in 2^ω.

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Corollary (Knaster & Reichbach 1953)

Let $h\colon A \to B$ be a homeomorphism between closed nowhere dense subsets of 2^ω. Then there exists a homeomorphism $H\colon 2^\omega \to 2^\omega$ such that

$$H \restriction A = h.$$

The pseudo-arc

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The pseudo-arc is $\mathfrak{I}$-generic.

Amalgamations

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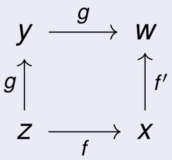

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We say that $\mathfrak{K}$ has the amalgamation property (AP) if it has amalgamations at every $z \in \mathrm{Obj}(\mathfrak{K})$.

Definition

A category $\mathfrak{K}$ is directed if for every $x, y \in \mathrm{Obj}(\mathfrak{K})$ there is $z \in \mathrm{Obj}(\mathfrak{K})$ such that

$$\mathfrak{K}(x, z) \neq \emptyset \qquad \text{and} \qquad \mathfrak{K}(y, z) \neq \emptyset.$$

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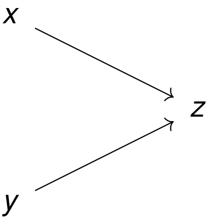

Fraïssé theory

Theorem

Assume $\mathfrak{K}$ is a countable directed category of finitely generated models with embeddings.
If $\mathfrak{K}$ has the AP then there exists a $\mathfrak{K}$-generic (countably generated) model, called the Fraïssé limit of $\mathfrak{K}$.

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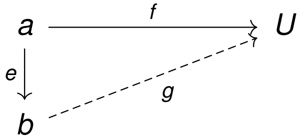
Theorem (Fraïssé 1954)

Let $\mathfrak{K}$ be as above, let $U = \bigcup_{n \in \mathbb{N}} u_n$ with $u_n \in \mathrm{Obj}(\mathfrak{K})$ for every $n \in \mathbb{N}$.
The following conditions are equivalent.

(a) *U is the Fraïssé limit of $\mathfrak{K}$.*

(b) *Every $\mathfrak{K}$-object embeds into U and for every embeddings $e \colon a \to b$, $f \colon a \to U$ with $a, b \in \mathrm{Obj}(\mathfrak{K})$ there exists an embedding $g \colon b \to U$ such that $f = g \circ e$.*


a
f
U
e
g
b

Fact

Finite graphs of vertex degree $\leqslant 2$ fail the amalgamation property.

Weakenings of amalgamation

Definition

We say that $\mathfrak{K}$ has the cofinal amalgamation property (CAP) if for every $z \in \mathrm{Obj}(\mathfrak{K})$ there is a $\mathfrak{K}$-arrow $e \colon z \to z'$ such that $\mathfrak{K}$ has amalgamations at z'.

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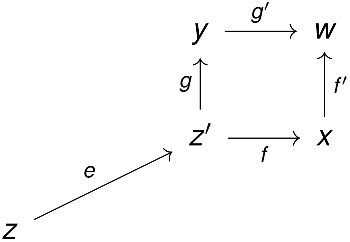
Definition (Ivanov, 1999)

We say that $\mathfrak{K}$ has the weak amalgamation property (WAP) if for every $z \in \operatorname{Obj}(\mathfrak{K})$ there is a $\mathfrak{K}$-arrow $e\colon z \to z'$ such that for every $\mathfrak{K}$-arrows $f\colon z' \to X$, $g\colon z' \to y$ there exist $\mathfrak{K}$-arrows $f'\colon x \to w$, $g'\colon y \to w$ such that $f' \circ f \circ e = g' \circ g \circ e$.

CAP and WAP



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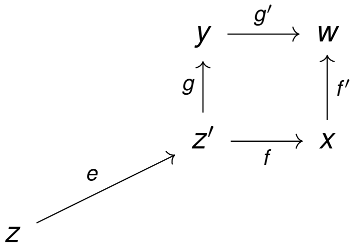

Proposition

Finite graphs of vertex degree $\leqslant 2$ have the CAP.

Theorem (Krawczyk & K. 2016)

Let $\mathfrak{K}$ be a countable directed category of finitely generated models with embeddings.

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Theorem (Krawczyk & K. 2016)

Let $\mathfrak{K}$ be as above and let U be a countably generated model. The following properties are equivalent:

(a) *U is $\mathfrak{K}$-generic.*

(b) *Eve does not have a winning strategy in* $\mathrm{BM}(\mathfrak{K}, U)$.

A more concrete setup

We assume that $\mathfrak{K}$ is a full subcategory of $\mathfrak{L}$ and the following conditions are satisfied.

(L0) All $\mathfrak{L}$-arrows are monic.

(L1) Every $\mathfrak{L}$-object is the co-limit of a sequence in $\mathfrak{K}$.

(L2) Every sequence in $\mathfrak{K}$ has a co-limit in $\mathfrak{L}$.

(L3) Every $\mathfrak{K}$-object is ω-small in $\mathfrak{L}$.

Weak injectivity

Definition

An object $V \in \mathrm{Obj}(\mathfrak{L})$ is weakly $\mathfrak{K}$-injective if

- every $\mathfrak{K}$-object has an $\mathfrak{L}$-arrow into V, and
- for every $\mathfrak{L}$-arrow $e \colon a \to V$ there exists a $\mathfrak{K}$-arrow $i \colon a \to b$ such that for every $\mathfrak{K}$-arrow $f \colon b \to y$ there is an $\mathfrak{L}$-arrow $g \colon y \to V$ satisfying $g \circ f \circ i = e$.

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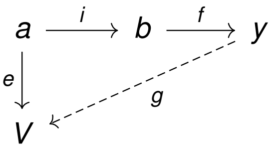

Theorem (K. 2017)

Assume $\mathfrak{K} \subseteq \mathfrak{L}$ *satisfy (L0)–(L3) and* $\mathfrak{K}$ *is locally countable. Given* $V \in \mathrm{Obj}(\mathfrak{L})$*, the following conditions are equivalent.*

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(a) *V is weakly $\mathfrak{K}$-injective.*

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Remark

If there exists a weakly $\mathfrak{K}$-injective object then $\mathfrak{K}$ is directed and has the WAP.

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