Going beyond variation of sets

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The BV sets, functions, and beyond

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- Main questions: If we assume, for example, that just one partial derivative of characteristic function of A is a (signed) Borel measure with finite total variation, can we provide a nice integralgeometric representation of this variation?

Main questions studied

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- This is a delicate question, as the Gauss-Green type theorems of De Giorgi and Federer are not available in this generality.

- We will show that a 'measure-theoretic boundary' plays its role in such representations similarly as for the BV sets.

- Question 2: There is a variety of plausible notions of 'measure-theoretic boundary' and one can address the question to find notions of measure-theoretic boundary that are as fine as possible.

Main results achieved

- The main result concerning Question 1 states that a set A has finite variation in a given direction τ (that is, the distributional derivative of the characteristic function of A in the direction τ is a finite measure) if and only if a suitably defined $(n-1)$-dimensional measure of a suitably defined measure-theoretic boundary is finite, and more precisely the variation of A in the direction τ agrees with the measure of such boundary.

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- Interestingly, our results give also a relatively elementary proof of the classical result of De Giorgi and Federer mentioned above.

- The results show quite clearly that the natural notion of 'area' in this context is not the $(n-1)$-dimensional Hausdorff measure, but the integralgeometric measure (which of course agree in case of rectifiable sets).

Notion of 'directional variation'

- A set $A \subset \mathbb{R}^n$ is said to be a BV set, or a set of finite perimeter if it is Lebesgue measurable and the gradient $D\chi^A$ in the sense of distributions of its characteristic function χ^A is an $\mathbb{R}^n$ valued Borel measure on $\mathbb{R}^n$ with finite total variation. The value of the perimeter of A, denoted by $P(A)$, is then the total variation $\|D\chi^A\|$ of the vector measure $D\chi^A$. Otherwise, let the perimeter of A be equal to $+\infty$.

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- Given a direction $\tau \in S^{n-1}$ a set $A \subset \mathbb{R}^n$ is said to have bounded variation at the direction τ if it is Lebesgue measurable and the directional derivative in the sense of distributions $\partial_\tau \chi^A$ of its characteristic function χ^A is a signed Borel measure with finite total variation on $\mathbb{R}^n$. The value of the variation at direction τ of A, denoted by $P_\tau(A)$, is then the total variation $\|\partial_\tau \chi^A\|$ of the signed measure $\partial_\tau \chi^A$. Otherwise, let $P_\tau(A) = +\infty$.

Representation of 'variation of a set'

- It is well known that, for a Lebesgue measurable set A and $\tau = e_i$ being the standard orthonormal basis direction (and writing briefly P_i instead of P_{e_i}),

$$P_i(A) = \int m_i^A(z)\, dz$$

where $m_i^A(z)$ is the infimum of the variations in x_i of all functions defined on the line $L_i(z)$ (paralel to the x_i axis and meeting z) which are equivalent to $\chi^A|L_i(z)$ and the integration is over the $(n-1)$ space orthogonal to the x_i axis.

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- The perimeter of A (if it is finite) is equal to the $(n-1)$ measure of the set $\mathrm{fr}_r A$ that is called the reduced boundary or equivalently it is equal to $(n-1)$ measure of the essential boundary $\mathrm{fr}_e A$ of A. Specifically, $x \in \mathrm{fr}_r A$ iff there is an $(n-1)$ plane π through x such that the symmetric difference of A and one of the halfspaces determined by π has density zero at x. Further, $x \in \mathrm{fr}_e A$ iff both A and complement of A have positive outer upper density at x.

Representation of 'variation of a set'

- Moreover, if the $(n-1)$ measure of $\mathrm{fr}_e A$ is finite then A is of finite perimeter. Hence $(n-1)$ measure of $\mathrm{fr}_e A$ is equal to the perimeter of A for a general set $A \subset \mathbb{R}^n$ (Our method also offers a simple self-contained proof of this fact for an integralgeometric $(n-1)$ measure.)

- Moreover, if the $(n-1)$ measure of $\mathrm{fr}_e A$ is finite then A is of finite perimeter. Hence $(n-1)$ measure of $\mathrm{fr}_e A$ is equal to the perimeter of A for a general set $A \subset \mathbb{R}^n$ (Our method also offers a simple self-contained proof of this fact for an integralgeometric $(n-1)$ measure.)
- We can show that the directional variation of a general set $A \subset \mathbb{R}^n$ (without any assumptions on regularity of A) is equal to the measure of projection (with multiplicities taken into account) of the 'measure-theoretic boundary'. The essential boundary $\mathrm{fr}_e A$ can play here the role of such a 'measure-theoretic boundary', but one can aim to replace it even with finer notions of 'measure-theoretic boundary'. We show, for example, that one can replace $\mathrm{fr}_e A$ by finer preponderant boundary $\mathrm{fr}_{pr} A$. Specifically, $x \in \mathrm{fr}_{pr} A$ iff both A and complement of A have the outer upper density at x greater than or equal to $\frac{1}{2}$.

Hausdorff measures

- For an integer $k = 0, 1, \ldots, n$ let H_k stand for the k-dimensional Hausdorff outer measure on $\mathbb{R}^n$, which is normalized in such a way that

$$H_k([0,1]^k x\{0\}^{n-k}) = 1.$$

  In particular, H_0 is the counting measure and H_n coincides with the Lebesgue outer measure on $\mathbb{R}^n$.

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In particular, H_0 is the counting measure and H_n coincides with the Lebesgue outer measure on $\mathbb{R}^n$.

- The constant $V(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ means the volume of the unit ball in $\mathbb{R}^n$ (with $V(0) = 1$), and the constant $A(n) = nV(n) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ means the area of S^{n-1}.

Projection measures μ_τ

- For $\tau \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ the result of Caratheodory's construction from the set function

  $$B \longmapsto H_{n-1}[\ p_\tau(B)\ ]$$

  which is defined on the covering family of all Borel sets in $\mathbb{R}^n$ will be called the projection measure at the direction τ and denoted by μ_τ. Then μ_τ is a Borel regular outer measure on $\mathbb{R}^n$ and $\mu_\tau \leq H_{n-1}$.

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- From Fubini theorem it follows that $H_n(C) = 0$ whenever $C \subset \mathbb{R}^n$ is such that $\mu_\tau(C) < \infty$.

Integralgeometric measure $\mathfrak{I}_1^{n-1}$

- The result of Caratheodory's construction from the set function

$$B \longrightarrow \frac{1}{2V(n-1)} \int_{S^{n-1}} H_{n-1}[\, p_\tau(B)\,]\, dH_{n-1}(\tau)$$

which is defined on the covering family of all Borel sets in $\mathbb{R}^n$ is usually termed $(n-1)$ dimensional integral geometric measure with exponent 1 on $\mathbb{R}^n$ and denoted by $\mathfrak{I}_1^{n-1}$.

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- $\mathfrak{I}_1^{n-1}$ is a Borel regular outer measure on $\mathbb{R}^n$ and $2V(n-1)\mathfrak{I}_1^{n-1} \leq A(n)H_{n-1}$.
  Moreover, it is known that $\mathfrak{I}_1^{n-1} \leq H_{n-1}$.

Densities

- For every set $A \subset \mathbb{R}^n$ and each $x \in \mathbb{R}^n$ we define the upper outer density $\overline{d}(x, A)$ and the lower outer density $\underline{d}(x, A)$ of A at x by the formulas

$$\overline{d}(x, A) = \overline{\lim}_{r \to 0+} \frac{H_n[A \cap B(x, r)]}{H_n[B(x, r)]},$$

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- A point x for which $\underline{d}(x, A) = 1$ is termed the outer density point of A.

Essential and preponderant interior and boundary

- We define the essential interior $\mathrm{int}_e A$ and the essential boundary $\mathrm{fr}_e A$ of the set $A \subset \mathbb{R}^n$ by the formulas

$$\mathrm{int}_e A = \{ x \in \mathbb{R}^n : \ \mathrm{d}(x, A^c) = 0 \},$$

$$\mathrm{fr}_e A = \{ x \in \mathbb{R}^n : \ \overline{\mathrm{d}}(x, A) \ > 0 \quad \text{and} \quad \overline{\mathrm{d}}(x, A^c) > 0 \};$$

$\mathrm{int}_e A \cap \mathrm{int}_e (A^c) = \emptyset$, $\mathrm{int}_e A$ is of type $F_{\sigma\delta}$ and $\mathrm{fr}_e A$ is of type $G_{\sigma\delta}$.
It is easy to see that $\mathrm{int}_{pr} A \cap \mathrm{int}_{pr} A^c = \emptyset$,

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  It is easy to see that $\mathrm{int}_{pr} A \cap \mathrm{int}_{pr} A^c = \emptyset$,

- We also define the preponderant interior $\mathrm{int}_{pr} A$ and the preponderant boundary $\mathrm{fr}_{pr} A$ of $A \subset \mathbb{R}^n$ by the formulas

$$\mathrm{int}_{pr} A = \left\{ x \in \mathbb{R}^n : \overline{d}(x, A^c) < \frac{1}{2} \right\},$$

$$\mathrm{fr}_{pr} A = \left\{ x \in \mathbb{R}^n : \overline{d}(x, A) \ \geq \frac{1}{2} \quad \text{and} \quad \overline{d}(x, A^c) \geq \frac{1}{2} \right\};$$

  $\mathrm{int}_{pr} A \cap \mathrm{int}_{pr} A^c = \emptyset$, $\mathrm{int}_{pr} A$ is of type F_σ and $\mathrm{fr}_{pr} A$ is of type G_δ.

BV functions

For a nonempty open set $\Omega \subset \mathbb{R}^n$ and for any $\tau \in \mathbb{R}^n$ we define the space $\mathrm{BV}(\Omega, \tau)$ of all locally (in Ω) H_n summable functions g for which there exists a finite signed Borel measure $\Phi^g_{\Omega,\tau}$ on Ω with the equality

$$\int_\Omega g(x) \cdot \tau \circ \operatorname{grad} \varphi(x)\, dx = -\int_\Omega \varphi(x)\, d\Phi^g_{\Omega,\tau}(x)$$

whenever $\varphi \in C_0^\infty(\Omega)$. $\mathrm{BV}(\Omega)$ is defined as the space of all locally (in Ω) H_n summable functions g such that there exist the finite signed Borel measures $\Phi^g_{\Omega,1}, \Phi^g_{\Omega,2}, \ldots, \Phi^g_{\Omega,n}$ with the equality

$$\int_\Omega g(x) \cdot \operatorname{div} \psi(x)\, dx = -\sum_{i=1}^{n} \int_\Omega \psi_i(x)\, d\Phi^g_{\Omega,i}(x)$$

whenever $\psi = (\psi_1, \psi_2, \ldots, \psi_n) \in C_0^\infty(\Omega, \mathbb{R}^n)$.

Directional variation and perimeter of sets

- For a nonempty open set $\Omega \subset \mathbb{R}^n$ and for any $\tau \in \mathbb{R}^n$ the set functions $P_{\Omega,\tau}$ and P_Ω over the subsets of $\mathbb{R}^n$ are defined for $A \subset \mathbb{R}^n$ by the following:
  If $A \cap \Omega$ is not H_n measurable then we put
  $$P_{\Omega,\tau}(A) = P_\Omega(A) = \infty.$$
  If $A \cap \Omega$ is H_n measurable then we put
  $$P_{\Omega,\tau}(A) = \sup\left\{ \int_\Omega \chi^A(x)\tau \circ D\varphi(x)\,dx : \ \varphi \in C_0^\infty(\Omega) \quad , \quad |\varphi| \le 1 \right\},$$
  $$P_\Omega(A) = \sup\left\{ \int_\Omega \chi^A(x)\,\mathrm{div}\,\psi(x)\,dx : \ \psi \in C_0^\infty(\Omega, \mathbb{R}^n) \quad , \quad |\psi| \le 1 \right\}.$$

Integralgeometric characterization of variations

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Theorem

Let $\Omega \subset \mathbb{R}^n$ *be nonempty open,* $A \subset \mathbb{R}^n$ *be arbitrary and* $\tau \in S^{n-1}$. *Then*

$$P_{\Omega,\tau}(A) = \mu_\tau(\Omega \cap \mathrm{fr}_e A) = \mu_\tau(\Omega \cap \mathrm{fr}_{pr} A).$$

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Corollary: Let $\Omega \subset \mathbb{R}^n$ be nonempty open and $A \subset \mathbb{R}^n$ be arbitrary. Then the following are equivalent :

(i) $P_\Omega(A) < \infty$.

(ii) There exist linearly independent vectors $\tau_1, \tau_2, \ldots, \tau_n \in \mathbb{R}^n$ such that $\mu_{\tau_i}(\Omega \cap \mathrm{fr}_{pr} A) < \infty$ for $i = 1, 2, \ldots, n$.

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Let $\Omega \subset \mathbb{R}^n$ *be nonempty open and* $A \subset \mathbb{R}^n$ *be arbitrary. Then*

$$P_\Omega(A) = \frac{1}{2V(n-1)} \int_{S^{n-1}} P_{\Omega,\tau}(A)\, dH_{n-1}(\tau).$$

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Theorem

Let $\Omega \subset \mathbb{R}^n$ *be nonempty open and* $A \subset \mathbb{R}^n$ *be arbitrary. Then the following equalities hold:*

$$P_\Omega(A) = \mathfrak{I}_1^{n-1}(\Omega \cap \mathrm{fr}_e A) = \mathfrak{I}_1^{n-1}(\Omega \cap \mathrm{fr}_{pr} A).$$

Some open questions

- We have seen that there is a variety of notions of ‘measure theoretic boundary’ that play an important role in integralgeometric representations of various notions of variation of a general set $A \subset \mathbb{R}^n$. We demonstrated this here using the essential boundary, and the slightly finer preponderant boundary.

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- We have seen that there is a variety of notions of ‘measure theoretic boundary’ that play an important role in integralgeometric representations of various notions of variation of a general set $A \subset \mathbb{R}^n$. We demonstrated this here using the essential boundary, and the slightly finer preponderant boundary.

- While for the sets of bounded variation there is plenty of such notions of boundary that can be used, much less is known about which notions of 'boundary’ can be used for integral representations of variations of more general sets.

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- While for the sets of bounded variation there is plenty of such notions of boundary that can be used, much less is known about which notions of 'boundary' can be used for integral representations of variations of more general sets.

- Even for the usual notion of the perimeter $P(A)$ of a set $A \subset \mathbb{R}^n$ we aim to understand for which notions of 'fine boundary', $\mathrm{fr}_{fine}(A)$, we can say that $P(A)$ is equal to $(n-1)$-dimensional measure of $\mathrm{fr}_{fine}(A)$ for general sets $A \subset \mathbb{R}^n$.

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- One of natural choices for such finer notions of 'boundary' that need to be understand for general sets is the following 'strong boundary',

$$\mathrm{fr}_s(A) = \{\, x \in \mathbb{R}^n : \ \underline{d}(x, A) > 0 \quad \text{and} \quad \underline{d}(x, A^c) > 0 \,\}.$$

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- Or one can suggest its finer version, $\mathrm{fr}_{s,\delta}(A)$ for $0 < \delta \leq 0.5$,

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- For BV sets these 'boundaries' can be used to represent their variation, but what can be said about them for general sets? Is their $(n-1)$-dimensional measure always equal to $P(A)$, or can it be 'small' for some set A of infinite variation?

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