Free Banach Lattices

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$|x| = x \vee -x$

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- Spaces with unconditional basis with coordinatewise order.

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Duality

For a Banach space E

$$E^* = \{x^* : E \longrightarrow \mathbb{R} \text{ bounded operators}\}$$

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- $Hom(L_p[0,1], \mathbb{R}) = \{0\}$

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- When the lattice operations $\wedge$ and $\vee$ are weakly *sequentially* continuous,
- X is order continuous
- $\cdots$

Order continuous: if $\bigwedge_{i\in I} f_i = 0$, then $\bigwedge\{\|f_{i_1} \wedge \cdots \wedge f_{i_n}\|\} = 0$.

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For $X \in \mathscr{C}$, elements $x_1, \ldots, x_n \in X$ are $\mathscr{C}$-independent if the only equations that they satisfy are those that follow from the axioms.

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That is, $Free_{\mathscr{C}}(A)$ contains A as a set of independent generators.

Independence and free generation

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This is just $\ell_1(A)$. Because this is the free vector space generated by A completed with the largest possible norm.

The free Banach lattice generated by a set A

Definition (de Pagter, Wickstead 2015)

We say that $F = FBL(A)$ if there is an inclusion map $A \longrightarrow B_F$ such that every bounded map $A \longrightarrow X$ extends to a unique Banach lattice homomorphism $FBL(A) \longrightarrow X$ of the same norm.

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- Uniqueness is easy, how to construct it?
- Similarly as before, we first construct the free vector lattice $FVL(A)$ generated by A, and later we complete it with the largest possible norm.

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- Hence, the free vector lattice generated by A, is the vector lattice generated by $\{\delta_a : a \in A\}$ inside $\mathbb{R}^{\mathbb{R}^A}$.

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$$FVL(A) = \langle \delta_a : a \in A \rangle_{VL} \subset \mathbb{R}^{\mathbb{R}^A}$$

- All the functions of $FVL(A)$ are positively homogeneous on $\mathbb{R}^A$ and continuous on $[-1, 1]^A$.

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And this happen to be enough...

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- The proof requires some extra work because the homomorphisms onto $\mathbb{R}$ do not give all the information.

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- In the finite case, $FBL(n)$ is n-isomorphic to $C(\mathbb{S}^n)$.

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They also pose a number of problems on *projective Banach lattices*.

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- The uniqueness of $FBL[E]$ is easy.
- For the existence one can take the quotient of $FBL(E)$ by the ideal generated by all linear combinations of E which are zero.

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The difficulty here is, again, that we cannot reduce to homomorphisms onto $\mathbb{R}$ or onto ℓ_1^n.

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We used that each $f \in FVL(A)$ is the difference of suprema of linear combinations in A,

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We used that each $f \in FVL(A)$ is the difference of suprema of linear combinations in A, and the Riesz-Kantorovich formula:

$$y^*\left(\bigvee_{k=1}^{m} u_k \right) = \sup\left\{ \sum_{k=1}^{m} y_k^*(u_k) : y_k^* \geq 0, \sum_{k=1}^{m} y_k^* = y^* \right\}.$$

More explicit description of $FBL[E]$

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Free Banach lattice generated by ℓ_1

Proposition

The free Banach lattice generated by the Banach space $\ell_1(A)$ coincides with the free Banach lattice generated by a set A.

$$FBL[\ell_1(A)] = FBL(A)$$

Free Banach lattice generated by ℓ_2

Proposition (A., Rodríguez, Tradacete)

In $FBL[\ell_2]$, $\{|e_n| : n \in \mathbb{N}\}$ is equivalent to the basis of ℓ_1.

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$$\|T\| \cdot \left\|\sum r_i |e_i|\right\| \geq \left\| \tilde{T}\left(\sum r_i |e_i|\right)\right\| = \left|\sum r_i\right|$$

Solution to Diestel's question

Corollary (A., Rodríguez, Tradacete)

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Corollary

If $\{e_n : n < \omega\}$ is a sequence equivalent to c_0 in any Banach lattice, then $\{|e_n| : n < \omega\}$ is weakly null.

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Theorem (A., Rodríguez, Tradacete)

All intervals of $FBL[E]$ have the same density as E.

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$$\|f - g\| = \|Td_b - Td_c\| \leq \|d_b - d_c\|$$

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In $FBL(A)$, we have elements that play analogous role, the elements

$$\left|\sum_{a\in A} r_a |\delta_a|\right|$$

for $(r_a)_A \in \ell_1(A)$.

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$$f_n = g \wedge \sum_{i=1}^{2^n} \left| \delta_{u_k^n} \right|$$

- g is any positive element with $\|g\| = 2$.
- $u_k^n = 1_{[(k-1)\cdot 2^{-n}, k\cdot 2^{-n}]}$

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For every $N > 0$ and every uncountable family $\mathscr{F} \subset C_{+h}(B_{E^*})_+$ has an uncountable subfamily $\mathscr{F}'$ such that among every N elements there are two with $f \wedge g \neq 0$.

Chain conditions

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Free Banach lattice generated by a lattice

Now, let $\mathbb{L}$ be a lattice.

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Definition

We say that $F = FBL[\mathbb{L}]$ if there is an inclusion map $\mathbb{L} \longrightarrow B_F$ such that every bounded lattice-morphism $\mathbb{L} \longrightarrow X$ extends to a unique Banach lattice homomorphism $FBL[\mathbb{L}] \longrightarrow X$ of the same norm.

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Again, we can always construct this by making a suitable quotient of $FBL(\mathbb{L})$.

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It seems to us that this description may not be valid for an arbitrary lattice $\mathbb{L}$.

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For a linear order $\mathbb{L}$, the following are equivalent:

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2. $\mathbb{L}$ is order-isomorphic to a subset of $\mathbb{R}$.

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