Matematická analýza pro fyziky I

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5. Cvičení

1. Dokažte, že je-li $x \in \mathbb{R}$, $x \neq (2k+1)\pi$ pro všechna $k \in \mathbb{Z}$ a $u = \tan\frac{x}{2}$, pak platí

$$\sin x = \frac{2u}{1+u^2}, \qquad \cos x = \frac{1-u^2}{1+u^2}.$$

- $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos^2\frac{x}{2} = 2\tan\frac{x}{2}\frac{1}{1+\tan^2\frac{x}{2}} = \frac{2u}{1+u^2}$, da

$$\sin^2 x + \cos^2 x = 1 \iff \frac{\sin^2 x}{\cos^2 x} + 1 = \frac{1}{\cos^2 x} \iff \tan^2 x + 1 = \frac{1}{\cos^2 x} \iff \cos^2 x = \frac{1}{\tan^2 x + 1}$$

- $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} \overset{1}{=} \cos^2\frac{x}{2}\left(1-\tan^2\frac{x}{2}\right) = \frac{1-\tan^2\frac{x}{2}}{\tan^2 x+1} = \frac{1-u^2}{1+u^2}$

2. Najděte primitivní funkci k f pomocí parciální integrace :

a) $f(x) = x^2 \ \sin(2x)$ b) $f(x) = x^3 \ e^{-x^2}$ c) $f(x) = \ln^2\left(x+\sqrt{1+x^2}\right)$ d) $f(x) = \cos(\ln x)$

zu a :

$$\int x^2 \ \sin 2x \ dx = \begin{bmatrix} u = x^2 & u' = 2x \\ v' = \sin 2x & v = -\frac{\cos 2x}{2} \end{bmatrix} = -\frac{x^2\cos 2x}{2} + \int x \ \cos 2x \ dx$$

$$= \begin{bmatrix} u = x & u' = 1 \\ v' = \cos 2x & v = \frac{\sin 2x}{2} \end{bmatrix} = -\frac{x^2\cos 2x}{2} + \frac{x\sin 2x}{2} + \int \frac{\sin 2x}{2} \ dx = -\frac{x^2\cos 2x}{2} + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}$$

zu b :

$$\int x^3 \ e^{-x^2} \ dx = \begin{bmatrix} u = x^2 & u' = 2x \\ v' = xe^{-x^2} & v = -\frac{e^{-x^2}}{2} \end{bmatrix} = \frac{-x^2e^{-x^2}}{2} + \int xe^{-x^2} \ dx = \frac{-x^2e^{-x^2}}{2} - \frac{e^{-x^2}}{2}$$

zu c :

$$\int \ln^2\left(x+\sqrt{1+x^2}\right) \ dx = \begin{bmatrix} u = \ln^2\left(x+\sqrt{1+x^2}\right) & u' = \frac{2\ln\left(x+\sqrt{1+x^2}\right)}{x+\sqrt{1+x^2}}\left(1+\frac{x}{\sqrt{1+x^2}}\right) \\ v' = 1 & v = x \end{bmatrix}$$

$$= x\ln^2\left(x+\sqrt{1+x^2}\right) - \int \frac{2\ln\left(x+\sqrt{1+x^2}\right)}{x+\sqrt{1+x^2}} \cdot \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} \cdot x \ dx$$

$$= x\ln^2\left(x+\sqrt{1+x^2}\right) - 2\int \frac{x}{\sqrt{1+x^2}}\ln\left(x+\sqrt{1+x^2}\right) \ dx = \begin{bmatrix} u = \ln\left(x+\sqrt{1+x^2}\right) & u' = \frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} \\ v' = \frac{x}{\sqrt{1+x^2}} & v = \sqrt{1+x^2} \end{bmatrix}$$

$$= x\ln^2\left(x+\sqrt{1+x^2}\right) - 2\left[\sqrt{1+x^2}\ln\left(x+\sqrt{1+x^2}\right) - \int \underbrace{\frac{\sqrt{1+x^2}\left(1+\frac{x}{\sqrt{1+x^2}}\right)}{x+\sqrt{1+x^2}}}_{=1} \ dx\right]$$

$$= x\ln^2\left(x+\sqrt{1+x^2}\right) - 2\sqrt{1+x^2}\ln\left(x+\sqrt{1+x^2}\right) + 2x$$

zu d :

$$\int \cos(\ln x) \ dx = \begin{bmatrix} u = \cos(\ln x) & u' = -\frac{\sin(\ln x)}{x} \\ v' = 1 & v = x \end{bmatrix} = x\cos(\ln x) + \int \sin(\ln x) \ dx$$

[1] $\cos(x \pm y) = \cos x\cos y \mp \sin x \sin y$

$$= \left[\begin{array}{ll} u=\sin(\ln x) & u'=\frac{\cos(\ln x)}{x} \\ v'=1 & v=x \end{array}\right] = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\ dx$$

$$\Longrightarrow \int \cos(\ln x)\ dx = \frac{x}{2}\left(\cos(\ln x) + \sin(\ln x)\right)$$

3. Najděte primitivní funkce k f :

a) $f(x) = \dfrac{e^x}{e^x + e^{-x}}$ b) $f(x) = \dfrac{\sin x\ \cos^3 x}{1+\cos^2 x}$ $[z = \cos x]$ c) $f(x) = \dfrac{\sqrt{x}}{\left(\sqrt[4]{x^3}+1\right)\sqrt[4]{x^3}}$ $\left[t = \sqrt[4]{x^3}\right]$

d) $f(x) = x(1-x)^{10}$ e) $f(x) = \cos^5 x\sqrt{\sin x}$ $[w = \sin x]$ f) $f(x) = \dfrac{1}{x\ \ln x\ \ln(\ln x)}$ $[u = \ln(\ln x)]$

Návod : Můžete použít navržených substitucí $[\,\cdot\,]$.

zu a :

$$\int \frac{e^x}{e^x+e^{-x}}\ dx = \left[\begin{array}{l} u = e^x \\ du = e^x\ dx = u\ dx \end{array}\right] = \int \frac{u}{u+\frac{1}{u}}\ \frac{du}{u} = \int \frac{1}{u+\frac{1}{u}}\ du = \frac{1}{2}\int \frac{2u}{u^2+1}\ du = \frac{1}{2}\ln\left(e^{2x}+1\right)$$

zu b :

$$\int \frac{\sin x\ \cos^3 x}{1+\cos^2 x}\ dx = \left[\begin{array}{l} u = \cos x \\ du = -\sin x\ dx \end{array}\right] = \int \frac{\sin x\ \cos^3 x}{1+\cos^2 x}\frac{du}{-\sin x} = \int \frac{-u^3}{1+u^2}\ du = \int \frac{-u^3-u}{1+u^2}\ du + \int \frac{u}{1+u^2}\ du$$

$$= -\frac{\cos^3 x}{2} + \frac{1}{2}\ln\left(1+\cos x^2\right)$$

zu c :

$$\int \frac{x^{\frac{1}{2}}}{\left(\sqrt[4]{x^3}+1\right)\sqrt[4]{x^3}}\ dx = \int \frac{x^{\left(\frac{3}{4}-\frac{1}{4}\right)}}{\left(\sqrt[4]{x^3}+1\right)\sqrt[4]{x^3}}\ dx = \int \frac{x^{\frac{3}{4}}x^{-\frac{1}{4}}}{\left(\sqrt[4]{x^3}+1\right)x^{\frac{3}{4}}} = \left[\begin{array}{l} t = x^{\frac{3}{4}} \\ dt = \frac{3}{4}x^{-\frac{1}{4}}\ dx \end{array}\right] = \int \frac{x^{-\frac{1}{4}}}{\left(\sqrt[4]{x^3}+1\right)}\ \frac{dt}{\frac{3}{4}x^{-\frac{1}{4}}}$$

$$= \frac{4}{3}\int \frac{1}{t+1}\ dt = \frac{4}{3}\ln\left(\sqrt[4]{x^3}+1\right)$$

zu d :

$$\int x(1-x)^{10}\ dx = \left[\begin{array}{l} t = 1-x \\ dt = -dx \end{array}\right] = -\int (1-t)t^{10}\ dt = -\frac{(1-x)^{11}}{11} + \frac{(1-x)^{12}}{12}$$

zu e :

$$\int \cos^5 x\sqrt{\sin x}\ dx = \int \cos^{4+1} x\sqrt{\sin x}\ dx = \left[\begin{array}{l} w = \sin x \\ dw = \cos w dx \end{array}\right] = \int \cos^{4+1} x\sqrt{\sin x}\ \frac{dw}{\cos x} = \int \left(1-\sin^2 x\right)^2\sqrt{\sin x}\ dw = \int \left(1-w^2\right)^2 w^{\frac{1}{2}}\ dw = \int \left(1-2w^2+w^4\right) w^{\frac{1}{2}}\ dw = \int \left(w^{\frac{1}{2}} - 2w^{\frac{5}{2}} + w^{\frac{9}{2}}\right)\ dw = \frac{2}{3}(\sin x)^{\frac{3}{2}} + \frac{4}{7}(\sin x)^{\frac{7}{2}} + \frac{2}{11}(\sin x)^{\frac{11}{2}}$$

zu f :

$$\int \frac{1}{x\ \ln x\ \ln(\ln x)}\ dx = \left[\begin{array}{l} u = \ln(\ln x) \\ du = \frac{1}{x\ln x}dx \end{array}\right] = \int \frac{1}{x\ \ln x\ \ln(\ln x)}\ \frac{du}{\frac{1}{x\ln x}} = \int \frac{1}{u}\ du = \ln(\ln(\ln x))$$

4. Najděte primitivní funkci k následujícím funkcím

$$f_1(x) := x\,(1-x)^{10} \qquad f_2(x) := \tfrac{x^2}{(8x^3+27)^{2/3}}$$

$$f_3(x) := \tfrac{x^3}{x^8-2} \qquad f_4(x) := \sin(5x-y) - \sin(5y-x)$$

$$f_5(x) := \tfrac{1}{e^x+e^{-x}} \qquad f_6(x) := \tfrac{1}{16-x^4}$$

$$f_7(x) := \tfrac{1}{x^2+6x+34} \qquad f_8(x) := \tfrac{\cos 2x}{\cos x - \sin x}$$

$$f_9(x) := \tfrac{e^x}{\sqrt{e^{2x}+5}} \qquad f_{10}(x) := \tfrac{x^2}{\sqrt{1+x^2}}$$

f_1 : partielle Integration, $\displaystyle\int f_1(x)\mathrm{d}x = -\frac{(1-x)^{11}(11x+1)}{11\cdot 12}$.

f_2 : $t = 8x^3+27, \mathrm{d}t = 24x^2\mathrm{d}x$, $\displaystyle\int f_2(x)\mathrm{d}x = \frac{(8x^3+27)^{1/3}}{8}$.

f_3 : $y = x^4, \mathrm{d}y = 4x^3\mathrm{d}x$, $\displaystyle\int f_3(x)\mathrm{d}x = \frac{1}{8\sqrt{2}}\ln\left|\frac{x^4-\sqrt{2}}{x^4+\sqrt{2}}\right|$.

f_4 : $\displaystyle\int f_4(x)\mathrm{d}x = -\frac{\cos(5x-y)}{5} - \cos(5y-x)$.

f_5 : $y = e^x, \mathrm{d}y = e^x\mathrm{d}x$, $\displaystyle\int f_5(x)\mathrm{d}x = \operatorname{arctg}(e^x)$.

f_6 : $\displaystyle\int f_6(x)\mathrm{d}x = -\frac{1}{32}\ln\left|\frac{x-2}{x+2}\right| + \frac{1}{16}\operatorname{arctg}(x/2)$.

f_7 : $\displaystyle y = \frac{x+3}{5}, \mathrm{d}y = \frac{\mathrm{d}x}{5}, \int f_7(x)\mathrm{d}x = \int \frac{\mathrm{d}y}{5(y^2+1)} = \frac{1}{5}\operatorname{arctg}\left(\frac{x+3}{5}\right)$.

f_8 : $\displaystyle\int f_8(x)\mathrm{d}x = \int \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x}\mathrm{d}x = \sin x - \cos x$.

f_9 : $\displaystyle z = \frac{e^x}{\sqrt{5}}, \mathrm{d}z = \frac{e^x}{\sqrt{5}}\mathrm{d}x, \int f_9(x)\mathrm{d}x = \operatorname{arcsinh}\left(\frac{e^x}{\sqrt{5}}\right)$.

f_{10} : $\displaystyle\int f_{10}(x)\mathrm{d}x = \int \frac{1+x^2}{\sqrt{1+x^2}}\mathrm{d}x - \int \frac{1}{\sqrt{1+x^2}}\mathrm{d}x = x\sqrt{1+x^2} - \int \frac{x^2}{\sqrt{1+x^2}}\mathrm{d}x - \operatorname{arcsinh} x$. Also $\displaystyle\int f_{10}(x)\mathrm{d}x = \frac{1}{2}x\sqrt{1+x^2} - \frac{1}{2}\operatorname{arcsinh} x$.

5. Najděte primitivní funkci k

$$f(x) := \frac{1}{\sqrt{x^2-3x+2}}\,.$$

Zohledněte, na kterém intervalu (a,b) příslušné výpočty platí.

Auf $(-\infty,-1)$ und $(1,\infty)$ gilt $\displaystyle\int \frac{\mathrm{d}y}{\sqrt{y^2-1}} = \ln|y+\sqrt{y^2-1}|$.

Subst. $\displaystyle y = 2\left(x-\frac{3}{2}\right)$: $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2-3x+2}} = \int \frac{\mathrm{d}y}{\sqrt{y^2-1}}$ falls $|y| > 1$, also für $\displaystyle\left|x-\frac{3}{2}\right| > \frac{1}{2}$, d.h. $x \in (-\infty,1)$ und $x \in (2,\infty)$.

6. Najděte primitivní funkci k

$$f_1(x) := \frac{1}{x^4-1} \qquad f_2(x) := \frac{x+1}{x^4-x}$$

$$f_3(x) := \frac{\log^4 x-1}{x\,(\log^3 x+1)} \qquad f_4(x) := \frac{x^7}{x^4+2}$$

$$f_5(x) := x\ \arctan x \qquad f_6(x) := \frac{1}{1+\sin x+\cos x}$$

$$f_1 : \int \frac{\mathrm{d}x}{x^4-1} = -\frac{1}{2}\int \frac{\mathrm{d}x}{x^2+1} + \frac{1}{4}\int \frac{\mathrm{d}x}{x-1} - \frac{1}{4}\int \frac{\mathrm{d}x}{x+1} = -\frac{1}{2}\operatorname{arctg} x + \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|.$$

$$f_2 : \int f_2(x)\mathrm{d}x = \frac{2}{3}\int \frac{\mathrm{d}x}{x-1} - \int \frac{\mathrm{d}x}{x} + \frac{1}{3}\underbrace{\int \frac{\mathrm{d}x}{x^2+x+1}}_{I} = \frac{2}{3}\ln|x-1| - \ln|x| + \frac{1}{3}I.$$

$$I = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\mathrm{d}x - \frac{3}{2}\underbrace{\int \frac{\mathrm{d}x}{x^2+x+1}}_{II} = \frac{1}{2}\ln(x^2+x+1) - \frac{3}{2}II$$

Subst. $z := \frac{2x+1}{\sqrt{3}}$, $II = \frac{2}{\sqrt{3}}\int \frac{\mathrm{d}z}{z^2+1} = \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}$

f_3 : Subst. $y = \ln x, \mathrm{d}y = \frac{\mathrm{d}x}{x}$. $\int f_3(x)\mathrm{d}x = \int \frac{y^4-1}{y^3+1}\mathrm{d}y = \int y - \frac{1}{y^2-y+1}\mathrm{d}y = \frac{y^2}{2} - \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2y-1}{\sqrt{3}}$

f_4 : Subst. $y = x^4$, $\int f_4(x)\mathrm{d}x = \frac{1}{4}\int \frac{y}{y+2}\mathrm{d}y = \frac{x^4}{4} - \frac{1}{2}\ln(x^4+2)$

$$f_5 : \int x\arctan x\mathrm{d}x = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\mathrm{d}x = \frac{x^2}{2}\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x.$$

f_6 : Subst $y = \tan\frac{x}{2}$

$$\sin\frac{x}{2} = y\cos\frac{x}{2}, \quad \sin^2\frac{x}{2} + \cos^2\frac{x}{2} = 1 \implies \sin^2\frac{x}{2} = \frac{y^2}{1+y^2}, \quad \cos^2\frac{x}{2} = \frac{1}{1+y^2}$$

$$\mathrm{d}y = \frac{1}{2}\frac{\mathrm{d}x}{\cos^2\frac{x}{2}}, \int f_6(x)\mathrm{d}x = \int \frac{\frac{2}{1+y^2}}{1+\frac{2y}{1+y^2}+\frac{1}{1+y^2}-\frac{y^2}{1+y^2}}\mathrm{d}y = \int \frac{1}{1+y}\mathrm{d}y = \ln\left|1+\tan\frac{x}{2}\right|.$$

7. (i) Nechť $R = R(u,v)$ je racionální funkce v u a v. Buďte $n \in \mathbb{N}$, $n \geq 2$, a $a, b \in \mathbb{R}$ s $a \neq 0$. Ukažte, že primitivní funkci k

$$f(x) := R(x, \sqrt[n]{ax+b})$$

lze vždy najít pomocí substituce

$$t := \sqrt[n]{ax+b}$$

(ii) Najděte primitivní funkci k

$$f(x) := x^2\,\sqrt[3]{3x+1}\,.$$

$$t = \sqrt[n]{ax+b}, \quad \mathrm{d}t = \frac{a}{n}t^{1-n}\mathrm{d}x$$

$$\int R(x, \sqrt[n]{ax+b})\mathrm{d}x = \int R\left(\frac{t^n-b}{a}, t\right)\frac{n}{a}t^{n-1}\mathrm{d}t$$

Subst. $t = \sqrt[3]{3x+1}$: $\int x^2\sqrt[3]{3x+1}\mathrm{d}x = \int\left(\frac{t^3-1}{3}\right)^2 \cdot t \cdot t^2\mathrm{d}t = \frac{1}{9}\left(\frac{t^{10}}{10} - \frac{2t^7}{7} + \frac{t^4}{4}\right)$

8. Integrieren Sie die folgenden Funktionen

$$a)\quad f(x) = \frac{1}{x(1+x)(1+x+x^2)} \qquad b)\quad f(x) = \frac{x^4}{x^4+5x^2+4}$$

$$c)\quad f(x) = \frac{\sqrt{x}-\sqrt[3]{x}}{\sqrt[3]{x}-1} \qquad d)\quad f(x) = \frac{x^2}{\sqrt{1-x^2}}$$

$$e)\quad f(x) = \frac{1+\sin x}{\sin x(1+\cos x)} \qquad f)\quad f(x) = \frac{x}{\sqrt[4]{x^3(a-x)}}$$

$$g)\quad f(x) = \frac{x}{x^3+1} \qquad h)\quad f(x) = \frac{1}{(1-x)\sqrt{1-x^2}}$$

$$i)\quad f(x) = \frac{e^x+e^{-x}}{e^{2x}+e^{-2x}}$$

zu a)

Ansatz: $f(x) = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+x+1}$

$$\Rightarrow 1 = A(x+1)(x^2+x+1) + Bx(x^2+x+1) + (Cx+D)x(x+1)$$

Zum Bestimmen der reellen Größen A, B, C und D wird die Gleichung an verschiedenen Stellen x getestet oder es werden die Koeffizienten vor bestimmten Potenzen von x verglichen.

$$\begin{array}{lll} x=0 & \Rightarrow & A=1 \\ x=-1 & \Rightarrow & B=-1 \\ x^3: & \Rightarrow & 0=A+B+C \quad \Rightarrow \quad C=0 \\ x^2: & \Rightarrow & 0=2A+B+C+D \quad \Rightarrow \quad D=-1 \end{array}$$

Jetzt kann das Integral einfacher berechnet werden:

$$\int f(x)dx = \int \left(\frac{1}{x} - \frac{1}{x+1} - \frac{1}{x^2+x+1}\right) dx = \ln|x| - \ln|x+1| - \int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$$
$$= \ln\left|\frac{x}{x+1}\right| - \frac{4}{3}\int \frac{dx}{(\frac{2x+1}{\sqrt{3}})^2+1} = \ln\left|\frac{x}{x+1}\right| - \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

zu b)

$$\frac{x^4}{x^4+5x^2+4} = 1 - \frac{5x^2+4}{x^4+5x^2+4}$$

Zur Partialbruchzerlegung werden die Nullstellen des Nenners ermittelt:

$$x^4+5x^2+4=0 \quad \Rightarrow \quad x^2 = -\frac{5}{2} \pm \sqrt{\frac{25}{4}-4} = -\frac{5}{2} \pm \frac{3}{2} \quad \Rightarrow \quad x_{1,2} = \pm i \quad , \quad x_{3,4} = \pm 2i$$

Die Nenner der Partialbrüche haben bei komplexen Nullstellen z die Form $(x-z)(x-\overline{z}) = x^2 + |z|^2 - 2x\,\Re e(z)$

$$\Rightarrow -\frac{5x^2+4}{x^4+5x^2+4} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} \quad \Rightarrow -5x^2-4 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$$

Die Unbekannten A, B, C und D können analog Aufgabe a) mittels Koeffizientenvergleich ermittelt werden: $A = C = 0 \quad , \quad B = \dfrac{1}{3} \quad , \quad D = -\dfrac{16}{3}$

$$\int \frac{x^4}{x^4+5x^2+4}dx = \int \left(1 + \frac{1}{3(x^2+1)} - \frac{16}{3(x^2+4)}\right) dx = x + \frac{\arctan x}{3} - \frac{16}{3\cdot 4}\int \frac{dx}{\left(\frac{x}{2}\right)^2+1}$$

$$= x + \frac{\arctan x}{3} - \frac{8}{3}\arctan\frac{x}{2} + C$$

zu c)

Eine Substitution, welche zum Verschwinden aller auftretenden Wurzeln führt, ist z.B.:

$$x = t^6 \;, \quad x \in [0,\infty), \quad t \in [0,\infty), \quad dx = 6t^5\;dt \qquad (\#)$$

$$\int \frac{\sqrt{x}-\sqrt[3]{x}}{\sqrt[3]{x}-1}\;dx \overset{(\#)}{=} \int \frac{t^3-t^2}{t^2-1}\cdot 6t^5\;dt = 6\int t^7 \frac{t-1}{(t-1)(t+1)}\;dt$$

$$= 6\int \left(t^6 - t^5 + t^4 - t^3 + t^2 - t + 1 - \frac{1}{t+1}\right) dt = \frac{6}{7}t^7 - t^6 + \frac{6}{5}t^5 - \frac{3}{2}t^4 + 2t^3 - 3t^2 + 6t - 6\ln(t+1) + C$$

$$\overset{(\#)}{=} 6\sqrt[6]{x}\left(\frac{x}{7} + \frac{(\sqrt[3]{x})^2}{5} + 1\right) + \sqrt[3]{x}\left(\frac{3\sqrt[3]{x}}{2} - 3\right) + 2\sqrt{x} - x - 6\ln(\sqrt[6]{x}+1) + C$$

zu d)

Die Funktion $f(x)$ besitzt den natürlichen Definitionsbereich $x \in (-1,1)$.
Bei Ausdrücken der Form $\sqrt{1-x^2}$ bietet sich eine **trigonometrische Substitution** an.
Hier: $\quad x = \sin t \;, \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \;, \quad dx = \cos t\;dt \qquad (*)$

Man beachte, dass in dem angegebenen Intervall $\frac{dx}{dt} = \cos t > 0$ gilt und damit die Zuordnung eineindeutig ist.

$$\int \frac{x^2}{\sqrt{1-x^2}}\;dx \overset{(*)}{=} \int \frac{\sin^2 t}{|\cos t|}\cos t\;dt \overset{(\cos t>0)}{=} \int \sin^2 t\;dt = \int \frac{1}{2}(1-\cos 2t)\,dt = \frac{t}{2} - \frac{\sin 2t}{4} + C$$

$$= \frac{t}{2} - \frac{\sin t \cos t}{2} + C \overset{(*)}{=} \frac{\arcsin x}{2} - \frac{x\sqrt{1-x^2}}{2} + C$$

zu e)

Bei gebrochenrationalen Funktionen in sin und cos führt folgende Substitution zum Ziel:

$$\tan\frac{x}{2} = t \;, \quad x \in (-\pi,\pi) \;, \quad t \in (-\infty,\infty) \;, \quad dx = \frac{2}{1+t^2}dt$$

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2}$$

$$\int \frac{1+\sin x}{\sin x(1+\cos x)}\;dx = \int \frac{1+\frac{2t}{1+t^2}}{\frac{2t}{1+t^2}\cdot\left(1+\frac{1-t^2}{1+t^2}\right)}\cdot\frac{2}{1+t^2}\;dt = \int \frac{t^2+2t+1}{2t}\;dt = \frac{1}{2}\int\left(t+2+\frac{1}{t}\right)dt$$

$$= \frac{1}{4}\tan^2\frac{x}{2} + \tan\frac{x}{2} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C$$

zu f)

$$\int \frac{x\;dx}{\sqrt[4]{x^3(a-x)}} = \int \sqrt[4]{\frac{x}{a-x}}\;dx$$

Substitution: $\quad x \in \begin{cases} [0,a)\;, & a>0 \\ (a,0]\;, & a<0 \end{cases} \;, \quad \sqrt[4]{\frac{x}{a-x}} = t \;, \quad t \geq 0 \;, \quad x = \frac{at^4}{1+t^4} \;, \quad dx = \frac{4at^3}{(1+t^4)^2}\;dt$

$$\int \sqrt[4]{\frac{x}{a-x}}\;dx = a\int t\cdot\frac{4t^3}{(1+t^4)^2}\;dt = -\frac{at}{1+t^4} + \int \frac{a\;dt}{1+t^4}$$

Der verbleibende Integrant wird in Partialbrüche zerlegt:

$$1+t^4=0 \quad\Leftrightarrow\quad t_{12}=\frac{\sqrt{2}}{2}\pm i\frac{\sqrt{2}}{2}\;,\quad t_{34}=-\frac{\sqrt{2}}{2}\pm i\frac{\sqrt{2}}{2}$$

$$\Rightarrow\quad \text{(vgl. Aufg. b)}\qquad \frac{1}{1+t^4}=\frac{At+B}{t^2+\sqrt{2}t+1}+\frac{Ct+D}{t^2-\sqrt{2}t+1}$$

$$\text{Lösung nach bekannter Methode:}\qquad A=\frac{1}{2\sqrt{2}}\quad C=-\frac{1}{2\sqrt{2}}\quad B=D=\frac{1}{2}$$

$$\int\frac{a\ dt}{1+t^4}=\frac{a}{4\sqrt{2}}\int\left(\frac{2t+2\sqrt{2}}{t^2+\sqrt{2}t+1}-\frac{2t-2\sqrt{2}}{t^2-\sqrt{2}t+1}\right)dt$$

$$=\frac{a}{4\sqrt{2}}\int\frac{2t+\sqrt{2}}{t^2+\sqrt{2}t+1}\ dt+\frac{a}{4\sqrt{2}}\int\frac{\sqrt{2}\ dt}{t^2+\sqrt{2}t+1}-\frac{a}{4\sqrt{2}}\int\frac{2t-\sqrt{2}}{t^2-\sqrt{2}t+1}\ dt+\frac{a}{4\sqrt{2}}\int\frac{\sqrt{2}\ dt}{t^2-\sqrt{2}t+1}$$

$$=\frac{a}{4\sqrt{2}}\ln\left|\frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1}\right|+\frac{a}{4}\int\frac{dt}{\left(t+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}+\frac{a}{4}\int\frac{dt}{\left(t-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}$$

$$=\frac{a}{4\sqrt{2}}\ln\left|\frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1}\right|+\frac{a}{2}\int\frac{dt}{(\sqrt{2}t+1)^2+1}+\frac{a}{2}\int\frac{dt}{(\sqrt{2}t-1)^2+1}$$

$$=\frac{a}{4\sqrt{2}}\ln\left|\frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1}\right|+\frac{a}{2\sqrt{2}}\cdot\Big(\arctan(\sqrt{2}t+1)+\arctan(\sqrt{2}t-1)\Big)+C$$

Zusammen:

$$\int\frac{x\ dx}{\sqrt[4]{x^3(a-x)}}=\left[-\frac{at}{1+t^4}+\frac{a}{4\sqrt{2}}\ln\left|\frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1}\right|+\frac{a}{2\sqrt{2}}\Big(\arctan(\sqrt{2}t+1)+\arctan(\sqrt{2}t-1)\Big)\right]\Bigg|_{t=\sqrt[4]{\frac{x}{a-x}}}+C$$

zu g)

Eine Nullstelle von x^3+1 errät man: $x_0=-1$.
Durch Polynomdivision erhält man den quadratischen Term $(x^3+1):(x+1)=x^2-x+1$, welcher keine rellen Nullstellen besitzt.

$$\frac{x}{x^3+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$$

$$\Rightarrow\qquad A=-\frac{1}{3}\;,\quad B=C=\frac{1}{3}$$

$$\int\frac{x\ dx}{x^3+1}=-\frac{1}{3}\int\frac{dx}{x+1}+\frac{1}{3}\int\frac{x+1}{x^2-x+1}\ dx=-\frac{1}{3}\ln|x+1|+\frac{1}{6}\int\frac{2x-1}{x^2-x+1}\ dx+\frac{1}{2}\int\frac{dx}{x^2-x+1}$$

$$=-\frac{1}{3}\ln|x+1|+\frac{1}{6}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)+C\qquad\text{(vgl. Aufg. a)}$$

zu h)

Analog Teilaufgabe d) gilt $x\in(-1,1)$ und die Substitution $(*)$ führt zum Erfolg.

$$\int\frac{dx}{(1-x)\sqrt{1-x^2}}\overset{(*)}{=}\int\frac{\cos t\ dt}{(1-\sin t)\cos t}=\int\frac{dt}{1-\sin t}\overset{(\tan\frac{t}{2}=y)}{=}\int\frac{1}{1-\frac{2y}{1+y^2}}\cdot\frac{2\ dy}{1+y^2}$$

$$=\int\frac{2\ dy}{1+y^2-2y}=2\int\frac{dy}{(y-1)^2}=-\frac{2}{y-1}+C=-\frac{2}{\tan\left(\frac{\arcsin x}{2}\right)-1}+C$$

Zusätzlich gilt: $\dfrac{\sin\varphi}{\cos\varphi+1}=\dfrac{2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}}{\cos^2\frac{\varphi}{2}-\sin^2\frac{\varphi}{2}+\cos^2\frac{\varphi}{2}+\sin^2\frac{\varphi}{2}}=\dfrac{\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}}=\tan\dfrac{\varphi}{2}$

$$\Rightarrow \tan\left(\frac{\arcsin x}{2}\right) = \frac{x}{\sqrt{1-x^2}+1} \quad \Rightarrow \int \frac{dx}{(1-x)\sqrt{1-x^2}} = -\frac{2\sqrt{1-x^2}+2}{x-\sqrt{1-x^2}-1} + C$$

zu i)

Subst. $t = e^x, \mathrm{d}t = \mathrm{d}x$: $\displaystyle\int \frac{e^x + e^{-x}}{e^{2x}+e^{-2x}}\mathrm{d}x = \int \frac{e^{2x}+1}{e^{4x}+1} e^x \mathrm{d}x = \int \frac{t^2+1}{t^4+1}\mathrm{d}t = \frac{1}{2}\int \frac{1}{t^2+\sqrt{2}t+1} + \frac{1}{t^2-\sqrt{2}t+1}\mathrm{d}t = \frac{\sqrt{2}}{2}\left(\arctan(\sqrt{2}t+1) + \arctan(\sqrt{2}t-1)\right) = \frac{\sqrt{2}}{2}\left(\arctan(\sqrt{2}e^x+1) + \arctan(\sqrt{2}e^x-1)\right)$

9. Man berechne eine Stammfunktion der folgenden Funktion:

$$f(x) := \frac{1+\sqrt{x}}{\sqrt{x}-\sqrt[3]{x}}.$$

Hinweis: Benutzen Sie eine Substitution, welche zum Verschwinden aller auftretenden Wurzeln führt.

Subst. $x = y^6, \mathrm{d}x = 6y^5\mathrm{d}y$.

$\displaystyle\int \frac{1+\sqrt{x}}{\sqrt{x}-\sqrt[3]{x}}\mathrm{d}x = 6\int \frac{(1+y^3)y^5}{(y^3-y^2)}\mathrm{d}y = 6\int y^5+y^4+y^3+2y^2+2y+2+\frac{2}{y-1}\mathrm{d}y = = x + \frac{6x^{5/6}}{5} + \frac{3x^{2/3}}{2} + 4x^{1/2} + 6x^{1/3} + 12x^{1/6} + 2\ln|x^{1/6}-1|.$

10. Man berechne eine Stammfunktion der folgenden Funktion:

$$f(x) := \frac{1}{(x-1)^2}\sqrt[3]{\frac{x+1}{x-1}}.$$

Subst. $t = \sqrt[3]{\frac{x+1}{x-1}}, x = \frac{t^3+1}{t^3-1}, \mathrm{d}x = -\frac{6t^2}{(t^3-1)^2}\mathrm{d}t$

$$\int f(x)\mathrm{d}x = \int \frac{1}{\left(\frac{t^3+1}{t^3-1}-1\right)^2} \cdot t \cdot \frac{-6t^2}{(t^3-1)^2}\mathrm{d}t = -\frac{8}{3}\left(\frac{x+1}{x-1}\right)^{4/3}$$

11. Buď

$$f(x) = \begin{cases} x^2\sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Ukažte, že

(i) f je spojitá,

(ii) f je diferencovatelná,

(iii) f *není* spojitě diferencovatelná.

Tedy je f' nespojitá funkce, která má primitivní funkci.

zu (i):

f ist offensichtlich stetig in allen x mit $x \neq 0$. Stetigkeit in Null folgt aus der Abschätzung

$$|f(x) - f(0)| = |f(x)| \leq x^2.$$

zu (ii):

$f'(x) = 2x\sin(1/x) - \cos(1/x), x \neq 0.$

$f'(0) : \left|\frac{f(h)-f(0)}{h}\right| \leq \left|\frac{h^2}{h}\right| = |h|$, also $f'(0) = 0$.

zu (iii):

f' ist *nicht* Stetig: $f'\left(\dfrac{1}{2k\pi}\right) = -1$ für alle $k \in \mathbb{N}$, aber $f'(0) = 0$.

Folgende Skizze zeigt f und f' bei Null.