Differential equation for $\wp$: We have

$$\text{(DE)} \quad (\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

where $g_2 := 60\, G_4$, $g_3 := 140\, G_6$ and

$$G_r = G_r(L) := \sum_{\omega \in L^*} \frac{1}{\omega^r}$$

For a proof of (DE) we need

$\boxed{\text{LEMMA}}$ We have the Laurent expansions

$$\text{(LE)} \quad \begin{aligned} \wp(u) &= u^{-2} + 3G_4 u^2 + 5G_6 u^4 + \cdots \quad \text{and} \\ \wp'(u) &= -2u^{-3} + 6G_4 u + 20 G_6 u^3 + \cdots \end{aligned}$$

Proof: (i) We have

$$(1-q)^{-1} = \sum_{n=0}^{\infty} q^n, \quad |q|<1$$

$$+ (1-q)^{-2} = \sum_{n=1}^{\infty} n \cdot q^n$$

$$\vdots$$

$$(k-1)!(1-q)^{-k} = \sum_{n=k-1}^{\infty} n \cdot (n-1) \cdots (n-k+2)\, q^{n-k+1}$$

$$(1-q)^{-k} = \sum_{n=k-1}^{+\infty} \binom{n}{k-1} q^{n-k+1} = \sum_{n=0}^{+\infty} \binom{n+k-1}{k-1} q^n, \quad |q|<1.$$

(ii) We have, for $|u| < |\omega|$,

$$(u-\omega)^{-k} = (-\omega)^{-k}\left(1-\frac{u}{\omega}\right)^{-k} = \frac{(-1)^k}{\omega^k}\sum_{n=0}^{\infty}\binom{n+k-1}{k-1}\frac{u^n}{\omega^n},$$ and thus

$$(u-\omega)^{-2} - \omega^{-2} = \sum_{n=1}^{\infty}(n+1)\frac{u^n}{\omega^{n+2}}$$ and

$$\wp(u) = u^{-2} + \sum_{n=1}^{\infty}(n+1)\underbrace{G_{n+2}}_{\substack{= 0 \\ \text{for odd } n}} u^n.$$

Finally, we get

$$\boxed{\wp(u) = \frac{1}{u^2} + \sum_{n=1}^{\infty}(2n+1)\, G_{2n+2}\, u^{2n}}$$ and

$$\boxed{\wp'(u) = -\frac{2}{u^3} + \sum_{n=1}^{\infty}(2n+1)\cdot 2n\, G_{2n+2}\, u^{2n-1}}$$ ∎

Proof of (DE): By (LE), we have

(1) $(\wp'(u))^2 = 4u^{-6} + 2\cdot(-2)\cdot 6\, G_4\, u^{-2} + 2(-2)\,20\, G_6 + \cdots$

(2) $(\wp(u))^3 = u^{-6} + 3\cdot 3\, G_4\, u^{-2} + 3\cdot 5\cdot G_6 + \cdots$

By (1) and (2), the map $(\wp'(u))^2 - 4(\wp(u))^3 + g_2\wp(u) + g_3$ is elliptic, it has no pole and vanishes at 0, thus it vanishes on the whole of $\mathbb{C}$. ∎

<u>Remark</u>: Recall that $e_1 = \wp(\omega_1/2)$, $e_2 = \wp(\omega_2/2)$ and $e_3 = \wp((\omega_1+\omega_2)/2)$. Then e_1, e_2, e_3 are three <u>different</u> <u>simple</u> roots of

$$(*) \quad 4z^3 - g_2 z - g_3 = 0.$$

Indeed, $\wp'(\omega_1/2) = 0 = \wp'(\omega_2/2) = \wp'((\omega_1+\omega_2)/2)$ and use (DE).

$\boxed{\text{Exercise}}$ A cubic equation $(*)$ has three different simple roots iff $g_2^3 \neq 27 g_3^2$.

$\boxed{\Rightarrow}$ Let $p(z) = 4z^3 - g_2 z - g_3 = 0$ and
$p'(z) = 12z^2 - g_2 = 0$.

Then $g_2 = 12z^2$ and $g_3 = -8z^3$. Thus $g_2^3 = 27 g_3^2$.

$\boxed{\Leftarrow}$ Let $g_2^3 = 27 g_3^2$ and $g_2 \neq 0$. Then
$z := -\frac{3}{2}\frac{g_3}{g_2}$ satisfies $p(z) = 0 = p'(z)$. $\boxtimes$

Geometric interpretation of $\wp$

For a given lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, we have the corresponding Weierstrass function $\wp$ which satisfies

(DE) $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$.

Let us consider the hyperelliptic RS X given by

$$X := \{(z,w) \in \mathbb{C}^2 \mid w^2 = 4z^3 - g_2 z - g_3\}.$$

As we know, by adding one point ∞_0 at infinity to X we can compactify X, i.e., $\overline{X} := X \cup \{\infty_0\}$ is a compact connected RS and the z-projection

$$\pi : \overline{X} \longrightarrow \mathbb{S}$$

$$\pi(z,w) := z, \quad (z,w) \in X;$$

$$\pi(\infty_0) := \infty;$$

is a meromorphic function

Theorem ① The mapping $\phi: \mathbb{C} \to \overline{X}$ defined by $\phi(u) := (\wp(u), \wp'(u))$, $u \in \mathbb{C}$, $u \not\sim 0$;
$:= \infty_0$, $u \sim 0$;
is doubly periodic and holomorphic.

② The corresponding map $\overline{\phi}: \mathbb{C}/L \to \overline{X}$ given by
$$\overline{\phi}([u]) := \phi(u) \text{ if } u \in \mathbb{C} \text{ and } [u] = u + L$$
is a <u>conformal</u> mapping of $\mathbb{C}/L$ onto $\overline{X}$.

In particular, $\mathbb{C}/L \simeq \overline{X}$.

<u>Proof:</u> ① Put $P(z, w) := w^2 - (4z^3 - g_2 z - g_3)$.

(a) Let $u \not\sim 0$ and $\wp'(u) \neq 0$. Then ϕ is holomorphic on a nbh. of u. Indeed, put $(z, w) := (\wp(u), \wp'(u)) \in X$. Then a local map on a nbh. of (z, w) is given by the z-projection π because $\frac{\partial P}{\partial w}(z, w) = 2w \neq 0$. In the local map, we have that $\pi \circ \phi = \wp$ is holomorphic.

(b) Let $u \not\sim 0$ and $\wp'(u) = 0$. Put $(z, w) = (\wp(u), \wp'(u)) \in X$. Since $\frac{\partial P}{\partial z} \neq 0$ at (z, w) a local map on a nbh. of $(z, w) \in X$ is given by the w-projection
$$\rho: X \to \mathbb{C}$$
$$(z, w) \to w$$
and, in the local map, $\rho \circ \phi = \wp'$ is holomorphic.

(c) It is clear that $\lim\limits_{u\to 0} \phi(u) = \infty$.

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(2.) Since $\overline{\phi}$ is a non-constant holomorphic map between compact connected RSs $\overline{\phi}$ is surjective. The map $\overline{\phi}$ is injective. Indeed, $\wp(u) = \infty$ iff $u \sim 0$. Moreover, let $\wp(u) = \wp(v) \in \mathbb{C}$ and $\wp'(u) = \wp'(v)$. Then $v \sim u$.

(a) Let $\wp'(u) \neq 0$. Then $\wp'(-u) = -\wp'(u) \neq \wp'(u)$, so $u \not\sim -u$ and $v \sim \pm u$. By $\wp'(u) = \wp'(v)$, we get $v \sim u$.

(b) Let $\wp'(u) = 0$. Then $\wp$ attains its value at u twice, so $v \sim u$. ▨

$\boxed{\text{Theorem}}$ Let X be a RS. FSAE:

(1.) X is a compact connected RS of genus 1;

(2.) X is conformally equivalent to $\mathbb{C}/L$ for some lattice $L := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ and $\omega_2/\omega_1 \notin \mathbb{R}$;

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(3.) X is conformally equivalent to one point compactification of the hyperelliptic RS given by the equation $w^2 = p(z)$ for some cubic polynomial p having just three simple roots.

Pf: (2.) $\Rightarrow$ (3.) $\Rightarrow$ (1.) We proved these implications in this and previous lectures.

(1.) $\Rightarrow$ (2.) : We do not give a proof of this statement. $\blacksquare$