NMMA410 Complex Analysis

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Contents

1 Meromorphic functions

Recall 1.1. Recall the following notions:

- $\Re z$, $\Im z$ the real and imaginary parts of $z \in \mathbb{C}$,
- The Riemannian sphere $\mathbb{S} = \mathbb{C} \cup \{\infty\}$, $\mathbb{S} \simeq S^2$ is compact, the stereographic projection (from the point $(0,0,1)$) $\phi : \mathbb{S} \to S^2$ is given by

$$\phi(x+iy) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right), \phi(\infty) = (0,0,1), \quad \phi^{-1}(a,b,c) = \left(\frac{a}{1-c}, \frac{b}{1-c}\right)$$

- Neighbourhoods of ∞: $P(\infty, \varepsilon) = \{z \in \mathbb{C}\colon\ |z| > \frac{1}{\varepsilon}\}$, $U(\infty,\varepsilon) = P(\infty,\varepsilon) \cup \{\infty\}$,
- $\lim\limits_{z\to\infty} f(z) = \lim\limits_{z\to 0} f\left(\frac{1}{z}\right)$, if at least one of the limits exists; $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$.

Definition 1.2. We say that a function f is holomorphic in a set $F \subseteq \mathbb{C}$ if there is an open set $G, F \subseteq G \subseteq \mathbb{C}$ such that f is holomorphic in G. In particular, it is holomorphic at $z_0 \in \mathbb{C}$ if it is holomorphic on some neighbourhood of z_0.

Definition 1.3. Function f has a removable singularity/pole/essential singularity at ∞ if $f\left(\frac{1}{z}\right)$ has a removable singularity/pole/essential singularity at 0. f is holomorphic at ∞ if $f\left(\frac{1}{z}\right)$ is holomorphic at 0. Let $G \subseteq \mathbb{S}$ be open. Then f is holomorphic on G if f is holomorphic at any and all $z_0 \in G$. Denote $\mathcal{H}(G)$ the set of all functions holomorphic on G.

Example 1.4. $\mathcal{H}(\mathbb{S})$ is the set of all constant functions (continuous bounded on $\mathbb{C}$). So $\mathcal{H}(G)$ is only interesting for $G \subsetneq \mathbb{S}$, WLOG $G \subseteq \mathbb{C}$: for $G \subseteq \mathbb{S} \setminus \{z_0\}, z_0 \in \mathbb{C}$ use the transformation $\phi(z) = \frac{1}{z-z_0}$.

Definition 1.5. Let $G \subseteq \mathbb{S}$ be open. Function f on G is called *meromorphic* if at all points of G the function f is either holomorphic or has a pole. Denote $\mathcal{M}(G)$ the set of all meromorphic functions on G.

Remark 1.6. (i) $\mathcal{H}(G) \subseteq \mathcal{M}(G)$.

(ii) Denote $P_f = \{z \in G\colon f \text{ has a pole at } z\}$. Then P_f has no limit points in G.

(iii) If $f = \infty$ on P_f, then $f : G \to \mathbb{S}$ is continuous. ALWAYS ASSUME: $f = \infty$ on P_f for $f \in \mathcal{M}(G)$.

Example 1.7. $\frac{\pi}{\sin(\pi z)} \in \mathcal{M}(\mathbb{C}), e^{1/z} \notin \mathcal{M}(\mathbb{C})$; the Gamma function $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,\mathrm{dx} \in \mathcal{M}(\mathbb{C}), \Re s > 0$, note that $\Gamma(n+1) = n!, n \in \mathbb{N}_0$; Riemann[1] zeta function $\zeta(z) \in \mathcal{M}(\mathbb{C})$, for $\Re z > 1$ holds $\zeta(z) = \sum_{n=1}^\infty n^{-z} = \Pi_{p\ prime} \frac{1}{1-\frac{1}{p^z}}$.

Example 1.8. $\mathcal{M}(\mathbb{S})$ is the set of rational functions.

Proof. Rational functions are clearly meromorphic. Let $f \in \mathcal{M}(\mathbb{S})$. $\mathbb{S}$ is compact and P_f has no limit points, so P_f must be finite. Assume $P_f \cap \mathbb{C} = \{z_1, \ldots, z_n\}$. There is $h \in \mathcal{H}(\mathbb{C})$ such that $f(z) = h(z) + \sum_{j=1}^n P_j\left(\frac{1}{z-z_j}\right)$ for some polynomials P_j (note that $\frac{1}{z-z_j}$ has a removable singularity at ∞) and $h(z) = \sum_{k=0}^\infty a_k z^k, a_k \in \mathbb{C}$ has essential singularity at infinity if and only if $a_k \neq 0$ for infinitely many k. So h is a polynomial. □

Remark 1.9. $\mathcal{M}(G)$ is interesting for $G \subsetneq \mathbb{S}$. Using the same transformation as in Example 1.4, we can without loss of generality assume $G \subseteq \mathbb{C}$.

Example 1.10. If $G \subseteq \mathbb{C}$ is a domain, $f, g \in \mathcal{H}(G)$, g is not identically zero, then $\frac{f}{g} \in \mathcal{M}(G)$. The inverse is also true (will be shown later, not true for $G = \mathbb{S}$).

Proposition 1.11. *Let $G \subseteq \mathbb{S}$ be open and $M \subseteq G$ have no limit points in G. Then*

(i) $G \setminus M$ is open;

(ii) if $K \subseteq G$ is compact in G, then $K \cap M$ is finite, in particular if $G = \mathbb{S}$, then M is finite;

(iii) M is at most countable, if M is infinite, then $\emptyset \neq \operatorname{der} M \subseteq \partial G$, where $\operatorname{der} M$ is the set of the limit points of M;

(iv) if $G \subseteq \mathbb{C}$ is a domain, then $G \setminus M$ is also a domain.

[1] Georg Friedrich Bernhard Riemann (17 September 1826, Breselenz, Kingdom of Hanover – 20 July 1866, Selasca, Kingdom of Italy)

Proof. (i) Take $z \in G \setminus M$. If for each neighbourhood U of z held $U \cap M \neq \emptyset$, then z would be a limit point of M. Thus there is a neighbourhood U of z which does not intersect M. By passing to a smaller neighbourhood we can get a neighbourhood V such that $V \subset G$ and $V \cap M = \emptyset$.

(ii) If $K \cap M$ was infinite, by compactness we would get that $K \cap M \subseteq M$ has a limit point in K.

(iii) Let $G \subsetneq \mathbb{S}$, WLOG $G \subseteq \mathbb{C}$. There are K_n compact in G such that $G = \bigcup_{n=1}^{\infty} K_n$. Then $M = \bigcup_{n=1}^{\infty} M \cap K_n$, $M \cap K_n$ is finite and so M is countable. If M is infinite, by compactness of $\mathbb{S}$, M has a limit point, but this limit point is not contained in any $K \subset G$ compact.

(iv) Curves are compact, and thus have finite intersection with M. Given a curve connecting two points in G, we can modify it on sufficiently small neighbourhoods of each point of intersection with M so that we get a curve with the same endpoints in $G \setminus M$.

□

Lemma 1.12. *Let $G \subseteq \mathbb{C}$ be open. Then there are compacts $K_n, n \in \mathbb{N}$ such that*

(i) $G = \bigcup_{n=1}^{\infty} K_n$;

(ii) $K_n \subseteq \operatorname{Int} K_{n+1}$;

(iii) for any compact $K \subseteq G$ there is $n_0 \in \mathbb{N}$ such that $K \subseteq K_{n_0}$.

Proof. Set $K_n = \left\{z \in G\colon \operatorname{dist}(z, \mathbb{C} \setminus G) \geq \frac{1}{n}\right\} \cap \overline{U(0,n)}$. □

Theorem 1.13 (Uniqueness of meromorphic functions). *Let $G \subseteq \mathbb{C}$ be a domain, $f \in \mathcal{M}(G)$ not identically zero. Then $N_f = \{z \in G\colon f(z) = 0\}$ has no limit points in G.*

Proof. We know this for holomorphic functions. Set $G_0 = G \setminus P_f$. Then G_0 is also a domain by Proposition 1.11 and $f \in \mathcal{H}(G_0), f \not\equiv 0$ on G_0. Then $N_f \subseteq G_0$ and has no limit points in G_0, nor in P_f. □

Theorem 1.14 (Residue theorem). *Let $G \subseteq \mathbb{C}$ be open, φ be a closed curve in G and $\operatorname{Int}\varphi \subseteq G$ (recall $\operatorname{Int}\varphi = \{z \in \mathbb{C} \setminus \langle\varphi\rangle\colon \operatorname{Ind}_\varphi z \neq 0\}$). Let M be a finite set in $G \setminus \langle\varphi\rangle$ and $f \in \mathcal{H}(G \setminus M)$. Then $\int_\varphi f = 2\pi i \sum_{s \in M} res_s f \cdot \operatorname{Ind}_\varphi s$.*

Remark 1.15. Residue theorem holds true even if instead of finiteness of M we assume that $M \subseteq G \setminus \langle\varphi\rangle$ has no limit points in G. Indeed, we have $M_0 = M \cap \operatorname{Int}\varphi$ is finite, because $\langle\varphi\rangle \cup \operatorname{Int}\varphi$ is compact, and $G_0 :- G \setminus (M \setminus M_0)$ is open, f is holomorphic on $G_0 \setminus M_0 = G \setminus M$ and by Residue theorem for G_0 and M_0 we get that $\int_\varphi f = 2\pi i \sum_{s \in M_0} res_s f \cdot \operatorname{Ind}_\varphi s = 2\pi i \sum_{s \in M} res_s f \cdot \operatorname{Ind}_\varphi s$.

2 Logarithmic integrals

Let $\varphi : [a,b] \to \mathbb{C}$ be a (regular) curve and let f be a nonzero holomorphic function on $\langle\varphi\rangle$. Then

$$I :- \frac{1}{2\pi i}\int_\varphi \frac{f'}{f} = \frac{1}{2\pi i}\int_a^b \frac{f'(\varphi(t))}{f(\varphi(t))}\varphi'(t)\,\mathrm{d}t = \frac{1}{2\pi i}\int_a^b \frac{(f(\varphi(t)))'}{f(\varphi(t))}\,\mathrm{d}t = \frac{1}{2\pi i}\int_{f\circ\varphi}\frac{\mathrm{d}z}{z} = \frac{1}{2\pi i}(\phi(b) - \phi(a)),$$

where ϕ is a branch of logarithm of $f \circ \varphi$. If φ is in addition closed, then $I = \operatorname{Ind}_{f\circ\varphi} 0 = \frac{1}{2\pi}(\theta(b) - \theta(a))$, where θ is branch of argument of $f \circ \varphi$.

Theorem 2.1 (Argument principle). *Let $G \subseteq \mathbb{C}$ be a domain, φ a closed curve in G and $f \in \mathcal{M}(G)$. Let $\operatorname{Int}\varphi \subseteq G$ and $\langle\varphi\rangle \cap N_f = \emptyset = \langle\varphi\rangle \cap P_f$. Then*

$$\frac{1}{2\pi i}\int_\varphi \frac{f'}{f} = \sum_{\substack{s\in\operatorname{Int}\varphi\\ f(s)=0}} n_f(s)\operatorname{Ind}_\varphi s - \sum_{\substack{s\in\operatorname{Int}\varphi\\ f(s)=\infty}} p_f(s)\operatorname{Ind}_\varphi s, \tag{PA}$$

where $n_f(s)$ is the multiplicity of the zero point s of f and $p_f(s)$ is the multiplicity of the pole s of f. Denote the right hand side of the equality (PA) *as $\sum(f,\varphi)$.*

Proof. By Residue theorem (1.14) we have

$$\frac{1}{2\pi i}\int_\varphi \frac{f'}{f} = \sum_{\substack{s\in \operatorname{Int}\varphi \\ s\in N_f\cup P_f}} res_s\left(\frac{f'}{f}\right)\operatorname{Ind}_\varphi s.$$

If $s \in N_f \cup P_f$, then on $P(s) = P(s,r)$ holds

$$\frac{f'(z)}{f(z)} = \frac{pc_p(z-s)^{p-1}+\cdots}{c_p(z-s)^p+\cdots} = \frac{p}{z-s}\cdot\frac{1+HOT}{1+HOT},$$

where by HOT we mean higher order terms. Thus $res_s(f'/f) = p = n_f(s), s \in N_f$ or analogously $res_s(f'/f) = -p_f(s), s \in P_f$. □

Lemma 2.2. *Let $\varphi_1, \varphi_2 : [a,b] \to \mathbb{C}$ be closed curves and $s \in \mathbb{C}\setminus(\langle\varphi_1\rangle\cup\langle\varphi_2\rangle)$. Assume $\forall t \in [a,b]\colon |\varphi_1(t)-\varphi_2(t)| < |\varphi_1(t)-s|$. Then $\operatorname{Ind}_{\varphi_1} s = \operatorname{Ind}_{\varphi_2} s$.*

Proof. For $t \in [a,b]$ we have $|(\varphi_1(t)-s)-(\varphi_2(t)-s)| < |\varphi_1(t)-s| \implies |1-\psi(t)| < 1$ for $\psi(t) = \frac{\varphi_2(t)-s}{\varphi_1(t)-s}$. ψ is a closed curve, $\langle\psi\rangle \subseteq U(1,1)$ and so $0 = \operatorname{Ind}_\psi 0 = \frac{1}{2\pi i}\int_a^b \frac{\psi'}{\psi}$. We calculate $\psi' = \frac{\varphi_2'(\varphi_1-s)-\varphi_1'(\varphi_2-s)}{(\varphi_1-s)^2} = \frac{\varphi_2'}{\varphi_1-s} - \frac{\varphi_1'(\varphi_2-s)}{(\varphi_1-s)^2}$ and $\frac{\psi'}{\psi} = \frac{\varphi_2'}{\varphi_2-s} - \frac{\varphi_1'}{\varphi_1-s}$. Put together we get

$$0 = \frac{1}{2\pi i}\int_a^b \frac{\psi'}{\psi} = \frac{1}{2\pi i}\int_a^b \frac{\varphi_2'}{\varphi_2-s} - \frac{1}{2\pi i}\int_a^b \frac{\varphi_1'}{\varphi_1-s} = \operatorname{Ind}_{\varphi_2} s - \operatorname{Ind}_{\varphi_1} s.$$

□

Theorem 2.3 (Rouché[2])**.** *Let $G \subseteq \mathbb{C}$ be an open domain, $f_1, f_2 \in \mathcal{M}(G)$ and φ be a closed curve in G such that $\operatorname{Int}\varphi \subseteq G$. Assume that $|f_1(z)-f_2(z)| < |f_1(z)| < \infty$ for any $z \in \langle\varphi\rangle$. Then $\sum(f_1,\varphi) = \sum(f_2,\varphi)$.*

Proof. Set $\varphi_j :- f_j\circ\varphi$, $j = 1,2$. Then $\operatorname{Ind}_{\varphi_j} 0 = \frac{1}{2\pi i}\int_\varphi \frac{f_j'}{f_j} \overset{\text{(PA)}}{=} \sum(f_j,\varphi)$. By Lemma 2.2 for $s = 0$, we have $\operatorname{Ind}_{\varphi_1} 0 = \operatorname{Ind}_{\varphi_2} 0$. □

Corollary 2.4. *Let f_1, f_2 be holomorphic on $\overline{U(z_0,r)}$ and $|f_1(z)-f_2(z)| < |f_1(z)|, z \in \partial U(z_0,r)$. Then $\Sigma_1 = \Sigma_2$, where $\Sigma_j = \sum_{\substack{s\in U(z_0,r)\\ f_j(s)=0}} n_{f_j}(s)$.*

Proof. Use the closed curve $\varphi(t) = z_0 + re^{it}, t \in [0,2\pi]$. □

Example 2.5. Let $p(z) = a_n z^n + \cdots + a_0, n \in \mathbb{N}, a_n \neq 0$. Then p has just n roots (including multiplicities) in $\mathbb{C}$.

Proof. By Corollary for $f_1(z) = a_n z^n$ and $f_2(z) = p(z)$ and big enough $U(0,r)$. □

Recall 2.6. Let $z \in \mathbb{C}$, $p \in \mathbb{N}$ and f be a function holomorphic on some neighbourhood of z. We say that z is a *zero point of f of order* p, if $f(z) = f'(z) = \cdots = f^{(p-1)}(z) = 0$ and $f^{(p)}(z) \neq 0$.

Notation 2.7. Let f be a holomorphic function at $z_0 \in \mathbb{C}$. We say that $f(z_0) = w_0$ p-times, $p \in \mathbb{N}$ if z_0 is a zero point of $f - w_0$ of order p. The following statements are equivalent:

(i) $f(z_0) = w_0$ p times,

(ii) $f(z_0) = w_0,\ f'(z_0) = \cdots = f^{(p-1)}(z_0) = 0,\ f^{(p)}(z_0) \neq 0$,

(iii) $f(z) = w_0 + \sum_{k=p}^\infty c_k(z-z_0)^k$ on a neighbourhood of z_0, $c_p \neq 0$.

We say that $f(z_0) = \infty$ p-times, if z_0 is a zero point of $f(\frac{1}{z})$ of order p, i.e. f has a pole of order p at z_0. We say that $f(\infty) = w_0$ p-times if $f(\frac{1}{z})$ attains the value w_0 p-times at 0.

Theorem 2.8 (On multiple values)**.** *Let $z_0, w_0 \in \mathbb{S}$, f be holomorphic on $P(z_0)$ and $f(z_0) = w_0$ p-times for some $p \in \mathbb{N}$. Let $\delta_0 > 0$. Then there are $\varepsilon > 0$ and $\delta \in (0,\delta_0)$ such that $\forall w \in P(w_0,\varepsilon)$ there are exactly p different points $z_1,\ldots,z_p \in P(z_0,\delta)$ with $f(z_j) = w$, $j = 1,\ldots,p$. In addition, $f(z_j) = w$ only once.*

Proof. WLOG $z_0 = w_0 = 0$. Then $z_0 = 0$ is a zero point of f of order p. Choose $\delta \in (0,\delta_0)$ such that $f \neq 0, f' \neq 0$ on $P(0,2\delta)$. Set $\varepsilon = \min_{|z|=\delta}|f| > 0$. Let $w \in P(0,\varepsilon)$, use Rouché's theorem (2.3) for $f_1 = f, f_2 = f - w$ and $\varphi(t) = \delta e^{it}, t \in [0,2\pi]$. We know $|f_1 - f_2| = |w| < \varepsilon \le |f_1|$ on $\langle\varphi\rangle$. Since in $U(0,\delta)$ $f = f_1$ has the only zero point of order p at 0 and $(f-w)' = f' \neq 0$ in $P(0,\delta)$, $f_2 = f - w$ has exactly p simple zero points in $P(0,\delta)$. □

[2] Eugène Rouché (18 August 1832, Sommières, Gard, France – 19 August 1910, Lunel, Hérault)

Corollary 2.9. *Let $G \subseteq \mathbb{S}$ be a domain, $f \in \mathcal{M}(G)$ not constant on G. Then $f : G \to \mathbb{S}$ is an open mapping.*

Proof. Let $\Omega \subseteq G$ be open, $w_0 \in f(\Omega)$. Then there is a $z_0 \in \Omega$ and $p \in \mathbb{N}$ such that $f(z_0) = w_0$ p-times. Choose $\delta_0 > 0$ such that $U(z_0, \delta_0) \subseteq \Omega$. By the Theorem there are $\varepsilon > 0, \delta \in (0, \delta_0)$ such that $P(w_0, \varepsilon) \subseteq f(P(z_0, \delta))$, so $U(w_0, \varepsilon) \subseteq f(U(z_0, \delta)) \subseteq f(\Omega)$. □

Remark 2.10. The Corollary is true for $\mathcal{H}(G)$.

Corollary 2.11. *Let f be holomorphic at $z_0 \in \mathbb{C}$. Then $f'(z_0) \neq 0 \iff$ there is $U(z_0)$ neighbourhood of z_0 such that $f|_{U(z_0)}$ is one-to-one.*

Proof. $\Longrightarrow$: Let $f'(z_0) \neq 0$. Then $f(z_0) = w_0$ just once. Choose $\delta_0 > 0$ such that $f \neq w_0$ on $P(z_0, \delta_0)$. By Theorem 2.8, choose $\varepsilon > 0, \delta \in (0, \delta_0)$, due to continuity choose $\delta_1 < \delta$ such that $f(U(z_0, \delta_1)) \subseteq U(w_0, \varepsilon)$. Then $f|_{U(z_0,\delta_1)}$ is one-to-one.

$\Longleftarrow$: Let $f'(z_0) = 0$ and f be not constant on any neighbourhood of z_0. Then $f(z_0) = w_0$ at least twice. By previous Theorem, f is not one-to-one on any neighbourhood of z_0. □

Example 2.12. $\Longleftarrow$ is not true in $\mathbb{R}$: $f(x) = x^3$ is smooth one-to-one, but $f'(0) = 0$, $f^{-1}(x) = \sqrt[3]{x}$ is not smooth.

Theorem 2.13 (On holomorphic inverse)**.** *Let $G \subseteq \mathbb{C}$ be open and $f : G \to \mathbb{C}$ be a* conformal *(i.e. one-to-one holomorphic) function on G. Then $f' \neq 0$ on G, $\Omega :- f(G)$ is open and $f^{-1} : \Omega \to G$ is onto and holomorphic. In addition, $(f^{-1})' = \frac{1}{f' \circ f^{-1}}$ on Ω.*

Remark 2.14. The statement is not true in $\mathbb{R}$.

Proof. WLOG $G \subseteq \mathbb{C}$ is a domain, otherwise consider connected components of G. By Corollary 2.9, f is an open mapping, so Ω is in fact open and f^{-1} is continuous. Let $z_0 \in G$ and $w_0 = f(z_0)$. By Corollary 2.11, we have $f'(z_0) \neq 0$ and $\frac{1}{f'(z_0)} = \lim\limits_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \lim\limits_{w \to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = (f^{-1})'(w_0)$, where the equality of the limits follows from the Theorem on limit of composition: $f^{-1}(w) \to f^{-1}(w_0)$ for $w \to w_0$ and $f^{-1}(w) \neq f^{-1}(w_0)$ for $w \neq w_0$. □

Theorem 2.15 (Hurwitz[3])**.** *Let $G \subseteq \mathbb{C}$ be a domain, $f_n \in \mathcal{H}(G), f_n \overset{loc}{\rightrightarrows} f$ on G and $f \not\equiv 0$. Let $z_0 \in G$ and $f(z_0) = 0$. Then there are $\{z_n\} \subseteq G$ and $\{f_{k_n}\}$ such that $z_n \to z_0$ and $f_{k_n}(z_n) = 0$.*

Proof. Choose $\delta > 0$ such that $U(z_0, \delta) \subseteq G$ and $f \neq 0$ on $P(z_0, \delta)$. For $n \in \mathbb{N}$ put $\rho_n :- \frac{\delta}{n+1}$ and $\varphi_n(t) = z_0 + \rho_n e^{it}, t \in [0, 2\pi]$. Of course, $\tau_n :- \min_{\langle \varphi_n \rangle} |f| > 0$. For a given n, there is $k_n \in \mathbb{N}$ such that $\forall z \in \langle \varphi_n \rangle$: $|f_{k_n}(z) - f(z)| < \tau_n \leq |f(z)|$. By Rouché's theorem (2.3), there is $z_n \in U(z_0, \rho_n)$ such that $f_{k_n}(z_n) = 0$. We can choose $\{k_n\}$ to be increasing. □

Remark 2.16. The statement does not hold in $\mathbb{R}$. The assumption $f \not\equiv 0$ is necessary. Indeed, take $f_n(z) = \frac{z}{n} \overset{loc}{\rightrightarrows} 0$ on $\mathbb{C}$.

Corollary 2.17. *Let $G \subseteq \mathbb{C}$ be a domain, f_n be conformal functions on G and $f_n \overset{loc}{\rightrightarrows} f$ on G. Then f is either conformal or constant on G.*

Proof. For contradiction assume there is $w_0 \in \mathbb{C}$ such that $f(z') = w_0 = f(z'')$ for some $z', z'' \in \mathbb{C}$, but $f \not\equiv w_0$. WLOG $w_0 = 0$. Choose $\delta > 0$ such that $U(z', \delta) \cap U(z'', \delta) = \emptyset$. By the Hurwitz theorem, there are $\{z'_n\} \subseteq U(z', \delta)$ and $\{f_{k'_n}\} \subseteq \{f_n\}$ such that $z'_n \to z'$ and $f_{k'_n}(z'_n) = 0$. By the Hurwitz theorem again, there are $\{z''_n\} \subseteq U(z'', \delta)$ and $\{f_{k''_n}\} \subseteq \{f_{k'_n}\}$ such that $z''_n \to z''$ and $f_{k''_n}(z''_n) = 0$. Every $f_{k''_n}$ has at least two zero points, which is a contradiction. □

3 The Mittag-Leffler theorem

Recall 3.1. Let $f \in \mathcal{M}(\mathbb{C})$. We know:

(i) P_f has no limit points in $\mathbb{C}$. Hence P_f is either finite or $P_f = \{s_j : j \in \mathbb{N}\}$ and $s_j \to \infty$.

(ii) Let $s \in P_f$. Then the principal part of Laurent expansion of f around s is of the form $H_s(z) = \frac{a_{-k}}{(z-s)^k} + \cdots + \frac{a_{-1}}{z-s} = P_s(\frac{1}{z-s})$, where $a_{-k} \neq 0$ and P_s is a polynomial, $P_s(0) = 0$ and $P_s \not\equiv 0$.

[3] Adolf Hurwitz (26 March 1859, Hildesheim, Kingdom of Hanover – 18 November 1919, Zürich, Switzerland)

Question 3.2. Let $P \subseteq \mathbb{C}$ have no limit points in $\mathbb{C}$. Is there $f \in \mathcal{M}(\mathbb{C})$ such that $P_f = P$? Can we prescribe the principal part for f at points of P? YES! YES! For finite P this is obvious.

Theorem 3.3 (Mittag-Leffler[4]). *Let $(s_j) \in \mathbb{C}^{\mathbb{N}}$ be one-to-one, $s_j \to \infty$ and $|s_1| \leq |s_2| \leq \cdots, s_0 = 0$. Let $P_0, P_1, \ldots$ be a sequence of polynomials such that $P_j(0) = 0, j \geq 0$. Then the function*

$$f(z) :- P_0\left(\frac{1}{z}\right) + \sum_{j=1}^{\infty}\left(P_j\left(\frac{1}{z - s_j}\right) - Q_j(z)\right),$$

for some polynomials Q_j satisfies

(i) the series converges locally uniformly on $\mathbb{C}$, i.e. on any compact $K \subseteq \mathbb{C}$ the series converges uniformly if we omit finitely many terms which have poles in K;

(ii) $f \in \mathcal{M}(\mathbb{C})$ and f has poles at $s_0, s_1, \ldots$, while at s_j the function f has its principal part equal to $P_j\left(\frac{1}{z-s_j}\right)$;

(iii) if $g \in \mathcal{M}(\mathbb{C})$ satisfies (ii), then there is $h \in \mathcal{H}(\mathbb{C})$ such that $g = f + h$ on $\mathbb{C}$.

Proof. Let $k \in \mathbb{N}$. Then $H_k(z) :- P_k\left(\frac{1}{z-s_k}\right) \in \mathcal{H}(U(0, |s_k|))$, so $H_k(z) = \sum_{n=0}^{\infty} c_n^k z^n, |z| < |s_k|$. There is $n_k \in \mathbb{N}$ such that $Q_k(z) :- \sum_{n=0}^{n_k} c_n^k z^n$ satisfies

$$|H_k(z) - Q_k(z)| < \frac{1}{2^k},\ |z| \leq \frac{|s_k|}{2}. \qquad (♗)$$

Let $K \subset \mathbb{C}$ be compact. Choose $k_0 \in \mathbb{N}$ such that $K \subseteq \overline{U(0, |s_{k_0}|/2)}$. If $k \geq k_0$, (♗) holds true on K, which implies (i). (ii) is obviously valid. (iii) follows from the fact that $g - f \in \mathcal{M}(\mathbb{C})$ which has all isolated singularities removable. □

Remark 3.4. The Mittag-Leffler theorem is a generalization of the decomposition of rational functions ($\equiv \mathcal{M}(\mathbb{S})$) into simple partial functions.

Example 3.5. $\pi \cot(\pi z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2-k^2}, z \in \mathbb{C} \setminus \mathbb{Z}$. $f(z) :- \pi \frac{\cos(\pi z)}{\sin(\pi z)}$ has simple poles at integers, $res_{\pm k} f = 1, k \in \mathbb{N}$. Then $f(z) = \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{z-k} + \frac{1}{z+k}\right) + h(z)$ for some $h \in \mathcal{H}(\mathbb{C})$. In exercises it will be shown that $h \equiv 0$.

Theorem 3.6. *Let $\Omega \subseteq \mathbb{C}$ be open, $A \subseteq \Omega$ have no limit points in Ω, $m : A \to \mathbb{N}$ and for $a \in A$ $P_a = \sum_{j=1}^{m(a)} \frac{c_{a,j}}{(z-a)^j}$. Then there exists meromorphic function f whose principal part at each point $a \in A$ is equal to P_a. f has no other poles in Ω.*

Proof. The theorem was formulated and proved as part of the exercises. □

4 Zero points of holomorphic functions

Recall 4.1. Let $G \subseteq \mathbb{C}$ be a domain, $f \in \mathcal{H}(G)$ and $f \not\equiv 0$. Then $N_f :- \{z \in \mathbb{C} \colon f(z) = 0\}$ has no limit points in G (uniqueness theorem for holomorphic functions).

Question 4.2. Let $N \subseteq G$ have no limit points in G. Is there $f \in \mathcal{H}(G)$ such that $N_f = N$?

4.1 The case $N = \emptyset$

Proposition 4.3. *Let f be nonzero holomorphic function on a simply connected domain $G \subseteq \mathbb{C}$. Then there is $L \in \mathcal{H}(G)$ such that $f = e^L$ on G.*

Proof. Since G is simply connected and $f'/f \in \mathcal{H}(G)$, by Cauchy's[5] theorem there is $L_0 \in \mathcal{H}(G)$ such that $L_0' = f'/f$. On G we have $(fe^{-L_0})' = e^{-L_0}(f' - L_0' f) = 0$ on G, hence $fe^{-L_0} = e^c$ for some $c \in \mathbb{C}$. Put $L = L_0 + c$. □

Remark 4.4. Nonzero $\iff$ $f \neq 0$ on G, simply connected means G is a domain and $\mathbb{S} \setminus G$ is connected, e.g. $\mathbb{C}$, convex, star-like.

[4] Magnus Gustaf "Gösta" Mittag-Leffler (16 March 1846, Stockholm – 7 July 1927, Djursholm)
[5] Augustin-Louis Cauchy (21 August 1789, Paris – 23 May 1857, Sceaux, France)

4.2 The case $|N| < \omega$

The polynomial $f(z) = \Pi_{j=1}^{n}(z - z_j)$ has zero points just at points $z_j, \dots, z_n$ with corresponding multiplicities. If $g \in \mathcal{H}(\mathbb{C})$ has some roots with the same multiplicities as f, then there is $L \in \mathcal{H}(\mathbb{C})$ such that $g = fe^L$ on $\mathbb{C}$ - use the previous Proposition for g/f.

4.3 The case $|N| = \omega$

Let $(z_j) \in \mathbb{C}^{\mathbb{N}}, z_j \to \infty$. The naïve approach of setting $f(z) = \Pi_{j=1}^{\infty}(z - z_j)$ does not work, simply because were the product convergent, one would have $z - z_j \to 1$, which is nonsense. Better approach would be to consider $f(z) = z^k \Pi_{j=1}^{\infty}\left(1 - \frac{z}{z_j}\right)$ for some $k \in \mathbb{N}$. What can be said about convergence of this infinite product?

Theorem 4.5. *Let M be a set (in $\mathbb{C}$), $u_j : M \to \mathbb{C}, j \in \mathbb{N}$ be bounded functions and suppose that $\sum_{j=1}^{\infty}|u_j|$ converges uniformly on M. Then $P_n = \Pi_{j=1}^{n}(1 + u_j)$ converge uniformly to a function $f : M \to \mathbb{C}$ and it holds true that $f = \Pi_{j=1}^{\infty}(1 + u_{\pi(j)})$ for any permutation π on $\mathbb{N}$. If $z_0 \in M$, then $f(z_0) = 0$ if and only if $\exists j_0 \in \mathbb{N}$: $u_{j_0}(z_0) = -1$.*

Proof. Denote $P_n^* = \Pi_{j=1}^{n}(1 + |u_j|)$. Then (1) $P_n^* \le \exp\left(\sum_{j=1}^{n}|u_j|\right)$, which follows from the fact that $1 + x \le e^x, x \ge 0$, and (2) $|P_n - 1| \le P_n^* - 1$. This we prove by induction on n. The case $n = 1$ is obvious. Assume (2) holds for $n \in \mathbb{N}$. Then $P_{n+1} - 1 = P_n(1 + u_{n+1}) - 1 = (P_n - 1)(1 + u_{n+1}) + u_{n+1}$, and so we conclude by

$$|P_{n+1} - 1| \le (P_n^* - 1)(1 + |u_{n+1}|) + |u_{n+1}| = P_{n+1}^* - 1.$$

Since $\sum_{j=1}^{\infty}|u_j|$ is bounded on M, $\exists n_0 \in \mathbb{N}$ such that $\sum_{j=n_0+1}^{\infty}|u_j| < 1$. By (1) P_n^* is bounded and by (2) P_n is bounded, so there is $C > 0$ such that $|P_n| \le C, n \in \mathbb{N}$.

Let $\varepsilon \in (0, \frac{1}{2})$. Choose $n_0 \in \mathbb{N}$ such that (3) $\sum_{n=n_0}^{\infty}|u_n| < \varepsilon$ on M. Pick $\pi \in S_\infty$ and put $Q_m = \Pi_{j=1}^{m}(1 + u_{\pi(j)}), m \in \mathbb{N}$. Let $n \ge n_0$ and $m \in \mathbb{N}$ be such that $\{\pi(1), \dots, \pi(m)\} \supseteq \{1, \dots, n\}$. Then

$$|Q_m - P_n| = \left|P_n\left(\prod_{\substack{\pi(j)>n \\ j\le m}}\left(1 + u_{\pi(j)}\right) - 1\right)\right| \overset{(2)}{\le} |P_n|\left(\prod_{\substack{\pi(j)>n \\ j\le m}}\left(1 + \left|u_{\pi(j)}\right|\right) - 1\right) \overset{(1),(3)}{\le} C(e^\varepsilon - 1) \le 2C\varepsilon. \quad (4)$$

Consider $\pi = \mathrm{Id}$. Then $P_m = Q_m$. By (4) $P_n \rightrightarrows f$ and for $n \ge n_0$ we have $|P_n - P_{n_0}| \le 2\varepsilon|P_{n_0}|$, and so $|P_n| \ge |P_{n_0}| - |P_n - P_{n_0}| \ge (1 - 2\varepsilon)|P_{n_0}|$. For $n \to \infty$ we get $|f| \ge (1 - 2\varepsilon)|P_{n_0}|$. Hence $f(z_0) = 0$ if and only if $P_{n_0}(z_0) = 0$. From (4) $Q_m \rightrightarrows f$ on M. □

Corollary 4.6. *Let $G \subseteq \mathbb{C}$ be open, $f_n \in \mathcal{H}(G), f_n \not\equiv 0$ on any component of G. Assume (I) $\sum_{n=1}^{\infty}|1 - f_n| \overset{loc}{\rightrightarrows}$ on G. Then $f = \prod_{n=1}^{\infty} f_n \overset{loc}{\rightrightarrows}$ on G, $f \in \mathcal{H}(G)$ and the infinite product does not depend on the order of the functions f_n. Moreover, (N) $n_f(s) = \sum_{k=1}^{\infty} n_{f_k}(s), s \in G$ ($n_f(s) = 0$ if $f(s) \neq 0$).*

Remark 4.7. The series in (N) contains only finitely many non-zero terms for any $s \in G$.

Proof. (N): Let $s \in G$. There is a neighbourhood U of s such that $f_n \overset{loc}{\rightrightarrows} 1$ on U. Choose $n_0 \in \mathbb{N}$ such that $f_n \neq 0$ on U for $n > n_0$. By the theorem, we get $\prod_{n=n_0+1}^{\infty} f_n \neq 0$ on U. Since $f = \prod_{n=1}^{n_0} f_n \cdot \prod_{n=n_0+1}^{\infty} f_n$, we get $n_f(s) = \sum_{n=1}^{n_0} n_{f_n}(s)$. The rest follows easily from the theorem. □

HW: Under the assumptions of the corollary, prove that $\frac{f'}{f} = \sum_{n=1}^{\infty}\frac{f_n'}{f}$ on $G \setminus N_f$.

Example 4.8. $\sin(\pi z) = \pi z \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right)$ (Euler[6])

Back to our problem. Weierstraß[7]: $(1 - z)e^{-\log(1-z)}$ on $|z| < 1$ and $-\log(1 - z) = \sum_{n=1}^{\infty}\frac{z^n}{n}$.

Lemma 4.9 (Weierstraß factors)**.** *Let $\mathcal{E}_0(z) = (1 - z)$, $\mathcal{E}_m(z) = (1 - z)\exp\left(z + \dots + \frac{z^m}{m}\right)$, $z \in \mathbb{C}, m \in \mathbb{N}$. Then $|1 - \mathcal{E}_m(z)| \le |z|^{m+1}$ for $|z| \le 1$.*

[6] Leonhard Euler (15 April 1707, Basel, Swiss Confederacy – 18 September 1783, Saint Petersburg, Russian Empire)

[7] Karl Theodor Wilhelm Weierstraß(31 October 1815, Ennigerloh, Kingdom of Prussia – 19 February 1897, Berlin, Kingdom of Prussia)

Proof.

$$\mathcal{E}_m'(z) = \exp\left(z + \cdots + \frac{z^m}{m}\right)\left(-1 + (1-z)(1+z+\cdots+z^{m-1})\right) = -z^m \exp\left(z+\cdots+\frac{z^m}{m}\right) = -z^m \sum_{k=0}^{\infty} b_k z^k,$$

where $b_0 = 1$ and $b_k \geq 0$ ($\exp(\cdots) = e^z \cdot e^{z^2/2} \cdot \dots \cdot e^{z^m/m}$, all Taylor series have positive coefficients). Hence $1 - \mathcal{E}_m(z) = -\int_{[0,z]} \mathcal{E}_m'(w)\,\mathrm{dw} = \sum_{k=0}^{\infty} c_k z^{k+m+1}$, $c_k = \frac{b_k}{m+k+1} \geq 0$. If $z \in B(0,1) \setminus \{0\}$, then

$$\left|\frac{1-\mathcal{E}_m(z)}{z^{m+1}}\right| = \left|\sum_{k=0}^{\infty} c_k z^k\right| \leq \sum_{k=0}^{\infty} c_k = 1 - \mathcal{E}_m(1) = 1.$$

□

Theorem 4.10 (Weierstraß factorization in $\mathbb{C}$)**.** *Let $k \in \mathbb{N}_0$ and $0 \neq z_j \to \infty$. Then there are $\{m_j\}_{j=1}^{\infty} \in \mathbb{N}_0^{\mathbb{N}}$ such that*

$$f(z) = z^k \prod_{j=1}^{\infty} \mathcal{E}_{m_j}\left(\frac{z}{z_j}\right) \tag{W}$$

converges locally uniformly on $\mathbb{C}$, $f \in \mathcal{H}(\mathbb{C})$ and f has at 0 a zero point of multiplicity k and "non-zero" zero points just at $z_1, z_2, \dots$ and their multiplicities correspond to the number of their occurrences in $\{z_j\}$. We can always take $m_j = j-1, j \in \mathbb{N}$. If $g \in \mathcal{H}(\mathbb{C})$ has the same zero points as f including multiplicities, then $g = fe^L$ for some $L \in \mathcal{H}(\mathbb{C})$.

Proof. By Corollary 4.6, we know (W) $\overset{loc}{\rightrightarrows}$ on $\mathbb{C}$ if $\sum_{j=1}^{\infty} \left|1 - \mathcal{E}_{m_j}(z/z_j)\right| \overset{loc}{\rightrightarrows}$ on $\mathbb{C}$. By Lemma 4.9 that is true if (♕) $\sum_{j=1}^{\infty} \left|\frac{z}{z_j}\right|^{m_j+1} \overset{loc}{\rightrightarrows}$ on $\mathbb{C}$. Let $r > 0$ and $|z| \leq r$. Choose $j_0 \in \mathbb{N}$ such that $\frac{r}{|z_j|} < \frac{1}{2}$, $j \geq j_0$. If $m_j = j-1$, then $\left|\frac{z}{z_j}\right|^j \leq \frac{1}{2^j}$, $j \geq j_0$, so (♕) $\rightrightarrows$ on $\{|z| \leq r\}$. □

Remark 4.11. If $\sum_{j=1}^{\infty} \frac{1}{|z_j|} < \infty$, one can take $m_j = 0$. If $\sum_{j=1}^{\infty} \frac{1}{|z_j|^2} < \infty$, one can take $m_j = 1$, etc.

Theorem 4.12 (Weierstraß factorization in a general open set)**.** *Let $G \subsetneq \mathbb{S}$ be open, $N \subseteq G$ have no limit points in G, $n : N \to \mathbb{N}$. Then there is $f \in \mathcal{H}(G)$ such that $N_f = N$ and $n_f(s) = n(s), s \in N_f$.*

Remark 4.13. $\mathcal{H}(\mathbb{S}) = \{\text{constant functions}\}$.

Proof. Without loss of generality $\infty \in G \setminus N$. Then $K :- \mathbb{S} \setminus G = \mathbb{C} \setminus G$ is compact. For N finite the claim is obvious. Assume $|N| = \omega$. Enumerate the points of N as $\{s_1, s_2, \dots\}$ in such a way that any $s \in N$ occurs in the enumeration $n(s)$ times. For any $n \in \mathbb{N}$ take $t_n \in K$ such that $|s_n - t_n| = \operatorname{dist}(s_n, K)$. Then $|s_n - t_n| \to 0$: Assume there is $\varepsilon > 0$, $\{n_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ such that $|s_{n_k} - t_{n_k}| \geq \varepsilon$, i.e. $\operatorname{dist}(s_{n_k}, K) \geq \varepsilon$. If s_∞ is a limit point of $\{s_{n_k}\}$, then $\operatorname{dist}(s_\infty, K) \geq \varepsilon$ implying $s_\infty \in G$ which is a contradiction.

Put $f(z) = \prod_{n=1}^{\infty} \mathcal{E}_n\left(\frac{s_n - t_n}{z - t_n}\right)$, $z \in G$. The infinite product $\overset{loc}{\rightrightarrows}$ on G. In fact, let L be compact in G. Denote $d :- \operatorname{dist}(L, K) > 0$ and pick $n_0 \in \mathbb{N}$ such that $|s_n - t_n| < d/2$, $n \geq n_0$. For any $z \in L$ we have $|z - t_n| \geq d$ and thus

$$\sum_{n=n_0}^{\infty} \left|\mathcal{E}_n\left(\frac{s_n - t_n}{z - t_n}\right) - 1\right| \overset{\text{L 4.9}}{\leq} \sum_{n=n_0}^{\infty} \left|\frac{s_n - t_n}{z - t_n}\right|^{n+1} \leq \sum_{n=n_0}^{\infty} \left(\frac{d/2}{d}\right)^{n+1} < \infty.$$

Hence the sum $\sum_{n=1}^{\infty} \left|\mathcal{E}_n\left(\frac{s_n - t_n}{z - t_n}\right) - 1\right|$ converges uniformly on L and by Theorem 4.5 the infinite product $f(z)$ converges uniformly on L. □

Theorem 4.14. *If $G \subseteq \mathbb{C}$ is open and $f \in \mathcal{M}(G)$ then there are $g, h \in \mathcal{H}(G)$ such that $f = \frac{g}{h}$ on G.*

Proof. Let P_f be the set of poles of f. By Weierstraß factorization (4.12) we construct $h \in \mathcal{H}(G)$ such that $N_h = P_f$ and $n_h = p_f$ on P_f. Put $g = fh$. Then $g \in \mathcal{H}(G)$, because g has removable singularities at the points of P_f. □

Remark 4.15 (Algebraic)**.** Recall that $\mathbb{Z}$ is an integral domain, $\mathbb{Q}$ is the field of fractions of $\mathbb{Z}$. If $G \subseteq \mathbb{C}$ is open, then $\mathcal{H}(G)$ is an associative commutative ring, if G is a domain, then $\mathcal{H}(G)$ is an integral domain and its field of fractions is $\mathcal{M}(G)$.

Example 4.16. For any nontrivial domain $\emptyset \neq G \subseteq \mathbb{C}$ there is $f \in \mathcal{H}(G)$ which cannot be extended holomorphically to any bigger domain (Exercises).

5 The space $\mathcal{H}(G)$

Recall 5.1. The space $C(K)$: Let K be a compact (topological) space, then $C(K) = \{f : K \to \mathbb{C}; f \text{ continuous}\}$ endowed with the $\|\cdot\|_\infty$ is a Banach[8] space, $f_n \overset{C(K)}{\to} f \iff f_n \rightrightarrows f$ in K.

The Arzelà[9]-Ascoli[10] theorem: Let $\mathcal{F} \subseteq C(K)$ be equibounded ($\exists M > 0\ \forall f \in \mathcal{F}$: $|f| \le M$) and equicontinuous ($\forall \varepsilon > 0\ \exists \delta > 0\ \forall f \in \mathcal{F}\ \forall x, y \in K$: $d(x,y) < \delta \implies |f(x) - f(y)| < \varepsilon$). Then every $(f_n) \in \mathcal{F}^{\mathbb{N}}$ has a uniformly convergent subsequence. Note that $\mathcal{F} \subseteq C(K)$ is equibounded if and only if it is bounded in the sense of Banach spaces. One possible reformulation is that $\mathcal{F}$ is compact in $C(K)$ if and only if it is equibounded and equicontinuous.

5.1 The space $C(G)$

Let $G \subseteq \mathbb{C}$ be open, we consider the space $C(G) = \{f : G \to \mathbb{C}; f \text{ is continuous}\}$ with the topology given by $f_n \overset{C(G)}{\to} f \iff f_n \overset{loc}{\rightrightarrows} f$ on G. For $f \in C(G)$ and $K \subseteq G$ compact we denote $\|f\|_K = \sup_K |f|$. Clearly $\|\cdot\|_K$ is a seminorm on $C(G)$.

Fact 5.2. *Let $f_n, f \in C(G)$ and $K_m \subseteq G$ be compacta such that $\bigcup_{m=1}^\infty K_m = G$ and $K_m \subseteq \operatorname{Int} K_{m+1}$. Then the following conditions are equivalent.*

(i) $f_n \overset{loc}{\rightrightarrows} f$ *on* G.

(ii) $\forall K \subseteq G$ *compact* : $\|f_n - f\|_K \to 0$.

(iii) $\forall m \in \mathbb{N}$: $\|f_n - f\|_{K_m} \overset{n\to\infty}{\to} 0$.

(iv) $\sigma(f_n, f) \to 0$, *where* $\sigma(f_n, f) = \sum_{m=1}^\infty \frac{1}{2^n} \cdot \frac{\|f_n - f\|_{K_m}}{1 + \|f_n - f\|_{K_m}}$ *is a metric on* $C(G)$.

Proof. (i) $\iff$ (ii) $\implies$ (iii) are obvious.

(iii) $\implies$ (ii): Let $K \subseteq G$ be compact. Then $K \subseteq K_{m_0}$ for some $m_0 \in \mathbb{N}$ and $\|f_n - f\|_K \le \|f_n - f\|_{K_{m_0}}$.

(iii) $\iff$ (iv) is left as an exercise to the reader. □

Remark 5.3. $(C(G), \sigma)$ is a complete metric space and $\mathcal{H}(G)$ is a closed subspace. The metric σ is not canonical. σ depends on the choice of K_m's and normalization method. The topology on $C(G)$ is given by the system of seminorms $\|\cdot\|_K$ for all $K \subseteq G$ compact.

Theorem 5.4 (Montel[11]). *Let $G \subseteq \mathbb{C}$ be open and $(f_n) \in \mathcal{H}(G)^{\mathbb{N}}$ be locally equibounded (i.e. $\forall K \subseteq G$ compact $(f_n|_K)$ is equibounded). Then there is (f_{n_k}) which converges locally uniformly on G.*

Proof. Let $z_0 \in G, r > 0, \overline{U(z_0, 2r)} \subseteq G$ and $\varphi(t) = z_0 + 2re^{it}$, $t \in [0, 2\pi]$. Let $z_1, z_2 \in \overline{U(z_0, r)}$. Then by the Cauchy formula we get $f_n(z_j) = \frac{1}{2\pi i} \int_\varphi \frac{f_n(z)}{z - z_j}\,\mathrm{dz}$. There is $M > 0$ such that $\forall n \in \mathbb{N}$: $|f_n| \le M$ on $\langle\varphi\rangle$. We have

$$|f_n(z_1) - f_n(z_2)| = \frac{1}{2\pi} \left| \int_\varphi f_n(z) \left(\frac{1}{z - z_1} - \frac{1}{z - z_2} \right) \mathrm{dz} \right| \le \frac{2\pi 2r}{2\pi} M \frac{|z_1 - z_2|}{r^2} = \frac{2M}{r} |z_1 - z_2|,$$

where in the inequality we have used that

$$\left| \frac{1}{z - z_1} - \frac{1}{z - z_2} \right| = \left| \frac{z_1 - z_2}{(z - z_1)(z - z_2)} \right| \le \frac{|z_1 - z_2|}{r^2}.$$

Hence (f_n) are equicontinuous on $\overline{U(z_0, r)}$. So by the Arzelà-Ascoli theorem there is (f_{n_k}) which is $\rightrightarrows$ on $\overline{U(z_0, r)}$.

Let us cover G by discs $U_j = U(z_j, r_j), j \in \mathbb{N}$ such that $\overline{U(z_j, r_j)} \subseteq G$ (for this we use separability of $\mathbb{C}$). Then we use a diagonal choice argument. First we choose $(f_{n_k^1}) \subseteq (f_n)$ such that $f_{n_k^1} \rightrightarrows$ on $\overline{U_1}$, then we choose $(f_{n_k^2}) \subseteq (f_{n_k^1})$ such that $f_{n_k^r} \rightrightarrows$ on $\overline{U_2}$ and so on. Then $(f_{n_k^k})$ converges uniformly on any $\overline{U_j}$ meaning the sequence converges locally uniformly on G. □

[8] Stefan Banach (30 March 1892, Kraków, – 31 August 1945, Lviv, Ukrainian SSR)
[9] Cesare Arzelà (6 March 1847, Santo Stefano di Magra, La Spezia, Italy – 15 March 1912, Santo Stefano di Magra, Italy)
[10] Giulio Ascoli (20 January 1843, Trieste – 12 July 1896, Milan)
[11] Paul Antoine Aristide Montel (29 April 1876, Nice, France – 22 January 1975, Paris, France)

Definition 5.5. Let E be a (complex) linear space and P be a system of seminorms on E. Then (E,P) is called *a locally convex space* (LCS). In (E,P) we define convergence as $f_n \to f \iff \forall p \in P\colon p(f_n - f) \to 0$ (i.e. the weakest topology such that all $p \in P$ are continuous). $F \subseteq E$ is *bounded* $\iff \forall p \in P\colon p(F)$ is bounded. The dual space E^* is defined in the standard fashion.

Remark 5.6. The space $C(G)$ is Fréchet, so is $\mathcal{H}(G)$ (which is closed in $C(G)$). The topology on $C(G)$ is generated by seminorms $\{\|\cdot\|_K : K \subseteq G \text{ compact}\}$. $U \subseteq C(G)$ is a neighbourhood of $f \iff \exists K \subseteq G$ compact $\exists \varepsilon > 0\colon U \supseteq U_{K,\varepsilon}(f) = \{g \in C(G)\colon \|f-g\|_K < \varepsilon\}$ ($\Longleftarrow$ is obvious, $\Longrightarrow$: $\exists n \in \mathbb{N}, K_1,\dots,K_n$ compacta, $\varepsilon_1,\dots,\varepsilon_n > 0$ such that $U \supseteq \bigcap U_{K_j,\varepsilon_j}$, take $K = \bigcup K_j$, $\varepsilon = \min \varepsilon_i$).

5.2 Compacta

Let $X = \mathbb{R}^n$ ($\mathbb{C}^n$). Then $F \subseteq X$ is compact if and only if F is closed and bounded. Let $X = \mathcal{H}(G)$. Then, in the sense of LCS, $F \subseteq \mathcal{H}(G)$ is bounded if and only if F is locally equibounded on G. By Montel's theorem (which states that $\mathcal{F} \subseteq \mathcal{H}(G)$ is compact iff $\mathcal{F}$ is bounded and closed in the sense of LCS) we get $\overline{F}$ is compact in $\mathcal{H}(G)$. Thus the above characterization holds true even in $\mathcal{H}(G)$. But in $C(G)$ it fails (HW).

Our next aim is to describe the dual space $\mathcal{H}^*(G) = (\mathcal{H}(G))^*$.

5.3 Hahn-Banach theorem

Before we continue, let us quickly mention the Hahn-Banach theorem. In what follows we will consider $E \in \{\mathcal{H}(G), C(G)\colon G \subseteq \mathbb{C} \textit{ open}\}$. For more detailed information on the Hahn-Banach theorem in the setting of general locally convex spaces see for example the lecture Functional Analysis 1.

Lemma 5.7. *Let $L : E \to \mathbb{C}$ be linear. Then L is continuous if and only if there are $K \subseteq G$ compact and $M > 0$ such that $|L(f)| \le M\|f\|_K$, $f \in E$.*

Proof. $\Longleftarrow$: By continuity of $\|\cdot\|_K$.

$\Longrightarrow$: Since $U :- L^{-1}(\{z \in \mathbb{C}\colon |z| < 1\})$ is a neighbourhood of 0 in E, there are $K \subseteq G$ compact and $\varepsilon > 0$ such that $U \supset U_{K,\varepsilon} = \{f \in E\colon \|f\|_K < \varepsilon\}$. Let $f \in E$. First, if $\|f\|_K > 0$ then $\left|L\left(\frac{f}{\|f\|_K}\frac{\varepsilon}{2}\right)\right| < 1$ implying $|L(f)| < \frac{2}{\varepsilon}\|f\|_K$. Put $M = \frac{2}{\varepsilon}$. Second, if $\|f\|_K = 0$, then $\forall n \in \mathbb{N}\colon \|nf\|_K = 0$, so $|L(nf)| < 1 \iff |L(f)| < \frac{1}{n} \to 0$. ☐

Theorem 5.8 (Hahn[12]-Banach)**.** *Let A be a linear subspace of E. Then*

(i) If $L \in A^*$, then there is $\widetilde{L} \in E^*$ such that $\widetilde{L}|_A = L$.

(ii) If A is closed and $b \in E \setminus A$, then there is $L \in E^*$ such that $L(b) = 1$ and $L|_A = 0$.

(iii) $\overline{A} = E$ if and only if $\forall L \in E^*\colon L = 0$ on $A \implies L = 0$ on E.

Proof. (i): Use the lemma and the algebraic version of Hahn-Banach theorem.

(ii) and (iii): can be proved in the same way as for Banach spaces. ☐

5.4 The dual space $\mathcal{H}^*(G)$

In this section we consider $E^* = \{L : E \to \mathbb{C} \text{ continuous and linear}\}$. First we consider the case $G = \mathbb{D}$ where $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$.

Theorem 5.9 (Dual of the unit disc)**.** *Let $L : \mathcal{H}(\mathbb{D}) \to \mathbb{C}$ be linear. Then $L \in \mathcal{H}^*(\mathbb{D})$ if and only if there is a unique $(b_n) \in \mathbb{C}^{\mathbb{N}}$ such that $\limsup_{n\to\infty} \sqrt[n]{|b_n|} < 1$ and*

$$L(f) = \sum_{n=0}^{\infty} a_n b_n \text{ for } f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D}) \qquad (♛)$$

In addition, $b_n = L(z^n), n \in \mathbb{N}$.

Proof. $\Longrightarrow$: Let $L \in \mathcal{H}^*(\mathbb{D})$, $f \in \mathcal{H}(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$ and $\frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} \ge 1$. Then

$$L(f) = L\left(z \mapsto \sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} a_n L(z \mapsto z^n) = \sum_{n=0}^{\infty} a_n b_n,$$

[12]Hans Hahn (27 September 1879, Vienna, Austria-Hungary – 24 July 1934, Vienna, Austria)

where $b_n = L(z \mapsto z^n)$. In the following we will often omit the "$z \mapsto$" part of notation.

We show $r :- \limsup_{n\to\infty} \sqrt[n]{|b_n|} < 1$. For contradiction assume that $r \geq 1$. If $r > 1$, then consider $a_n = 1, n \in \mathbb{N}_0$ for which $\sum_{n=0}^{\infty} a_n b_n$ is divergent. If $r = 1$, there is a subsequence $\{b_{n_k}\}$ such that $0 \neq \sqrt[n_k]{|b_{n_k}|} \to 1$. Then putting $a_j = 1/b_{n_k}$ for $j = n_k$ and $a_j = 0$ otherwise, we get that $\sum_{n=0}^{\infty} a_n b_n$ is divergent (the function f is well defined since $|\sum_{n=0}^{\infty} a_n z^n| \leq C\,|z| + \sum_{n=n_0}^{\infty} |a_n z^n| \leq C\,|z| + 2\sum_{n=n_0}^{\infty} |z|^n$). Thus, a contradiction.

$\Longleftarrow$: Let L satisfy (♕). Pick $\varepsilon > 0$. We want find $\delta > 0$ and $K \subseteq G$ compact satisfying $\forall f \in \mathcal{H}(G)\colon \|f\|_K < \delta \implies |L(f)| < \varepsilon$. Find $R < 1$ and $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0\colon \sqrt[n]{|b_n|} < R$. Pick $R < R' < 1$. Find $C \geq 1$ satisfying $\sqrt[n]{|b_n|} < CR, n \in \mathbb{N}$. Put $K = \overline{U(0, R')}$ and $\delta = \frac{\varepsilon}{C^{n_0}\sum_{n=0}^{\infty}\left(\frac{R}{R'}\right)^n}$. For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D}), \|f\|_K < \delta$ we then use Cauchy's inequality to get

$$|L(f)| = \left|\sum_{n=0}^{\infty} a_n b_n\right| \leq \sum_{n=0}^{\infty} |a_n| \max\{1, C^{n_0}\} R^n \overset{\text{Cauchy}}{\leq} \sum_{n=0}^{\infty} \left((R')^{-n} \sup_{|z|=R'} |f(z)|\right) \max\{1, C^{n_0}\} R^n$$
$$\leq \|f\|_K\, C^{n_0} \sum_{n=0}^{\infty} \left(\frac{R}{R'}\right)^n < \varepsilon.$$

□

Now we will try to find an integral form of L. For this we will use the following notation.

Notation 5.10. Let $A \subseteq \mathbb{S}$. Function f is holomorphic on A if f is holomorphic on an open superset of A. Let f_1, f_2 be holomorphic on A. We say that $f_1 \sim f_2$ if there are open sets $U_1, U_2 \subseteq \mathbb{S}$ such that $A \subseteq U_1 \cap U_2$, f_1 resp. f_2 is holomorphic on U_1 resp. U_2 and $f_1 = f_2$ on $U_1 \cap U_2$. Denote $\mathcal{H}(A) = \{[f] \colon f \textit{ is holomorphic on } A\}$, where $[f]$ is an equivalence class for $\sim$. As usual, we often don't distinguish between $[f]$ and f.

Theorem 5.11 (Dual of the unit disc, integral form)**.** *$\mathcal{H}^*(\mathbb{D}) \simeq \mathcal{H}_0(\mathbb{S} \setminus \mathbb{D})$. In particular, $L \in \mathcal{H}^*(\mathbb{D})$ if and only if*

$$\exists! \lambda \in \mathcal{H}_0(\mathbb{S} \setminus \mathbb{D})\colon L(f) = \frac{1}{2\pi i}\int_{\varphi} f(z)\lambda(z)\,\mathrm{d}z, f \in \mathcal{H}(\mathbb{D}). \tag{♔}$$

In this case we moreover have

$$L(z \mapsto z^n) = \frac{\lambda^{(n+1)}(\infty)}{(n+1)!}, n \in \mathbb{N}_0 \quad \textit{and} \quad \lambda(w) = L\left(z \mapsto \frac{1}{w-z}\right), |w| \geq 1.$$

Proof. $\Longrightarrow$: Let $(b_n) \in \mathbb{C}^{\mathbb{N}}$ satisfying $r :- \limsup_{n\to\infty} \sqrt[n]{|b_n|} < 1$ be as in the previous theorem. Define $\lambda(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}}$, $|z| > r$. Of course, $\lambda \in \mathcal{H}(\mathbb{S} \setminus \overline{U(0,r)})$, $\lambda(\infty) = 0$ and $b_n = \frac{\lambda^{(n+1)}(\infty)}{(n+1)!} = \frac{(\lambda(1/z))^{(n+1)}(0)}{(n+1)!}, n \in \mathbb{N}_0$. Here $\lambda^{(k)}(\infty)$ is defined as $(\lambda(1/z))^{(k)}(0)$. Let $R \in (r,1), \varphi(t) = Re^{it}, t \in [0, 2\pi]$. Let $f \in \mathcal{H}(\mathbb{D})$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n, z \in \mathbb{D}$. Then

$$\frac{1}{2\pi i}\int_{\varphi} f(z)\lambda(z)\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\varphi}\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty}\frac{b_n}{z^{n+1}}\right)\mathrm{d}z = \frac{1}{2\pi i}\int_{\varphi}\sum_{n,m=0}^{\infty} a_n b_m z^{n-m-1}\,\mathrm{d}z$$
$$= \sum_{n,m=0}^{\infty} a_n b_m \frac{1}{2\pi i}\int_{\varphi} z^{n-m-1}\,\mathrm{d}z = \sum_{n=0}^{\infty} a_n b_n = L(f),$$

where in the second to last equality we use the fact that $\int_{\varphi} z^{n-m-1}\,\mathrm{d}z = 2\pi i \delta_{mn}$ (Kronecker[13] delta).

We have that $\lambda \in \mathcal{H}_0(\mathbb{S} \setminus \mathbb{D}) :- \{\mu \in \mathcal{H}(\mathbb{S} \setminus \mathbb{D})\colon \mu(\infty) = 0\}$. Moreover, (♔) holds and $L(z^n) = \frac{\lambda^{(n+1)}(\infty)}{(n+1)!}$, $n \in \mathbb{N}_0$ and $\lambda(w) = L\left(\frac{1}{w-z}\right)$, $|w| \geq 1$ because

$$\frac{1}{w-z} = \frac{1}{w}\frac{1}{1-\frac{z}{w}} = \sum_{n=0}^{\infty}\frac{z^n}{w^{n+1}},\ z \in \mathbb{D} \implies L\left(\frac{1}{w-z}\right) = L\left(\sum_{n=0}^{\infty}\frac{z^n}{w^{n+1}}\right) = \sum_{n=0}^{\infty}\frac{b_n}{w^{n+1}}.$$

$\Longleftarrow$: Is left to the reader as an exercise. □

The next case we shall discuss is the case when $G = \bigcup_{j=1}^{n} D_j$, where $D_j = U(z_j, r_j)$, $D_j \cap D_k = \emptyset$ for $j \neq k$. Let $L \in \mathcal{H}^*(G)$. For $1 \leq j \leq n$ put $L_j(f) = L(\widetilde{f})$, where $\widetilde{f} = f$ on D_j and $\widetilde{f} = 0$ on $D_k, k \neq j$ for $f \in \mathcal{H}(D_j)$. Then

$$L(f) = \sum_{j=1}^{n} L_j(f|_{D_j}), f \in \mathcal{H}(G). \tag{1}$$

[13] Leopold Kronecker (7 December 1823, Liegnitz, Province of Silesia, Prussia – 29 December 1891, Berlin, German Empire)

By the first case there are $\widetilde{r}_j < r_j$ and $\lambda_j \in \mathcal{H}_0(\mathbb{S} \setminus \overline{U(z_j, \widetilde{r}_j)})$ such that

$$L_j(f) = \frac{1}{2\pi i}\int_{\varphi_j} f(z)\lambda_j(z)\,\mathrm{d}z, \quad f \in \mathcal{H}(D_j), \varphi_j(t) = z_j + R_j e^{it}, t \in [0, 2\pi], R_j \in (\widetilde{r}_j, r_j). \tag{2}$$

In addition we have $L_j(z^n) = \frac{\lambda_j^{(n+1)}(\infty)}{(n+1)!}$, $n \in \mathbb{N}_0$. If $f \in \mathcal{H}(G)$, then $L(f) \overset{(1),\,(2)}{=} \sum_{j=1}^n \frac{1}{2\pi i}\int_{\varphi_j} f(z)\lambda_j(z)\,\mathrm{d}z$. Using the fact that $\int_{\varphi_j} f(z)\lambda_k(z)\,\mathrm{d}z = 0$ for $k \neq j$ ($f \cdot \lambda_k \in \mathcal{H}(D_j)$, Cauchy) we get (3) $L(f) = \frac{1}{2\pi i}\int_\Gamma f(z)\lambda(z)\,\mathrm{d}z$, where $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ and $\lambda = \sum_{j=1}^n \lambda_j$. We have (4) $L(z^n) = \frac{\lambda^{(n+1)}(\infty)}{(n+1)!}, n \in \mathbb{N}_0$. The conclusion is that $\mathcal{H}^*(G) \simeq \mathcal{H}_0(\mathbb{S} \setminus G)$. Indeed, $L \in \mathcal{H}^*(G)$ if and only if there is a unique $\lambda \in \mathcal{H}_0(\mathbb{S} \setminus G)$ such that (3) and (4) hold.

Before continuing to the case of a general open set G, let us mention one application of the previous case.

Theorem 5.12 (Special Runge[14])**.** *Let $G \subseteq \mathbb{C}$ be a finite union of pairwise disjoint open discs. Then for each $f \in \mathcal{H}(G)$ there are polynomials $P_n, n \in \mathbb{N}$ such that $P_n \overset{loc}{\rightrightarrows} f$ on G.*

Proof. Let $\mathcal{P} = \operatorname{span}\{(z \mapsto z^n)\colon n \in \mathbb{N}\}$ be the space of complex polynomials. Then $\mathcal{P} \subseteq \mathcal{H}(G)$. Let $L \in \mathcal{H}^*(G)$ and $L = 0$ on $\mathcal{P}$. We know there is $\lambda \in \mathcal{H}_0(\mathbb{S} \setminus G)$ such that (3) is valid. By (4) $\lambda^{(n)}(\infty) = 0, n \in \mathbb{N}_0$. By the uniqueness theorem we get $\lambda \equiv 0$, so $L = 0$ on $\mathcal{H}(G)$. By the Hahn-Banach theorem then $\overline{\mathcal{P}} = \mathcal{H}(G)$. □

Example 5.13 (Birkhoff)**.** There is a universal entire function, i.e. $f \in \mathcal{H}(\mathbb{C})$ such that $\overline{\{\tau_\gamma f\colon \gamma \in \mathbb{C}\}} = \mathcal{H}(\mathbb{C})$, where $\tau_\gamma f(z) = f(z - \gamma)$.

Example 5.14. There are polynomials $P_n, n \in \mathbb{N}$ such that $P_n \overset{loc}{\rightrightarrows} 0$ in $\{\Re z \leq 0\}$ and $P_n \overset{loc}{\rightrightarrows}$ in $\{\Re z > 1\}$. By Runge, there are polynomials P_n such that $|P_n| < \frac{1}{n}$ on $\overline{U(-n^2, n^2 + \frac{1}{n})}$ and $|P_n - 1| < \frac{1}{n}$ on $\overline{U(n+2, n+2-\frac{2}{n})}$.

Example 5.15. Let $\mathcal{M} = \{f \in \mathcal{H}(\mathbb{D})\colon \lim_{r \to 1^-} f(re^{i\theta})\ doesn't\ exist\ for\ any\ \theta \in \mathbb{R}\}$. If $f \in \mathcal{M}$, then f cannot be extended holomorphically (not even continuously) to a bigger domain. By Runge's Theorem $\mathcal{M} \neq \emptyset$ and it is known that $\mathcal{H}(\mathbb{D}) \setminus \mathcal{M}$ is of first category (HW).

We end this section by giving the description of $\mathcal{H}^*(G)$ for a general $G \subseteq \mathbb{C}$.

Theorem 5.16 (Description of $\mathcal{H}^*(G)$)**.** *Let $G \subseteq \mathbb{C}$ be open. Then $\mathcal{H}^*(G) \simeq \mathcal{H}_0(\mathbb{S} \setminus G)$. Precisely:*
Let $L \in \mathcal{H}^*(G)$. Then there are compact $K \subseteq G$ and $\lambda \in \mathcal{H}_0(\mathbb{S} \setminus K)$ such that

$$L(f) = \frac{1}{2\pi i}\int_\Gamma f(z)\lambda(z)\,\mathrm{d}z, f \in \mathcal{H}(G),$$

where Γ is a cycle in $G \setminus K$ such that $K \subseteq \operatorname{Int}\Gamma \subseteq G$ and $\operatorname{Ind}_\Gamma z_0 = 1$ for $z_0 \in \operatorname{Int}\Gamma$.
In addition, λ is uniquely determined as an element of $\mathcal{H}_0(\mathbb{S} \setminus G)$ by the properties

(a) $\frac{\lambda^{(k+1)}(\infty)}{(k+1)!} = L(z \mapsto z^k), k \in \mathbb{N}_0,$

(b) $\frac{\lambda^{(k)}(z_0)}{k!} = -L(z \mapsto \frac{1}{(z-z_0)^{k+1}})$ *for* $z_0 \in \mathbb{C} \setminus G$, $k \in \mathbb{N}_0$.

Before proving the theorem, we need to do some preparatory work.

Theorem 5.17 (Cauchy's formula for compacta)**.** *Let $G \subseteq \mathbb{C}$ be open and $K \subseteq G$ be compact. Then there is a cycle Γ in G such that $K \subseteq \operatorname{Int}\Gamma$ and $\forall a \in \operatorname{Int}\Gamma\colon \operatorname{Ind}_\Gamma a = 1$. In addition we have that for all $f \in \mathcal{H}(G)$ holds*

$$\text{(1)} \quad \int_\Gamma f = 0 \qquad \text{and} \quad \text{(2)} \quad \forall a \in \operatorname{Int}\Gamma\colon f(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}\,\mathrm{d}z\,.$$

Proof. (1) and (2) follow from the properties of Γ and Residue theorem for cycles, but we will prove them directly.

Let $0 < \delta < \frac{1}{2}\operatorname{dist}(K, \partial G)$ if $G \subsetneq \mathbb{C}$. If $G = \mathbb{C}$, set $\delta = 1$. For $m, n \in \mathbb{N}$ let $Q_{m,n}$ be the closed square with edges parallel to the axes which have the length of δ and $m\delta + in\delta$ is the lower left corner. Denote $Q^* = \{Q_{m,n}\colon Q_{m,n} \cap K \neq \emptyset\}$ and $U = \operatorname{Int}\bigcup Q^*$. Of course, $K \subseteq U \subseteq \bigcup Q^* \subseteq G$. We understand $\partial Q_{m,n}$ as a positively oriented piecewise linear curve. Let Γ be the cycle given by the edges of $\Gamma_1, \ldots, \Gamma_k$ of squares of Q^*, where we omit the edges which appear twice in the opposite direction. Of course, $U = \bigcup Q^* \setminus \langle\Gamma\rangle$.

[14]Carl David Tolmé Runge (30 August 1856, Bremen, German Confederation – 3 January 1927, Göttingen, Weimar Republic)

Let $f \in \mathcal{H}(G)$. Then

$$\int_\Gamma f \stackrel{\text{def}}{=} \sum_{j=1}^{k} \int_{\Gamma_j} f = \sum_{Q_{m,n} \in Q^*} \int_{\partial Q_{m,n}} f = 0 \tag{♘}$$

by Cauchy's theorem. Γ can be viewed as a cycle. In fact, we prove the edges $\Gamma_1, \dots, \Gamma_k$ form finitely many closed simple piecewise linear curves. For $1 \le j \le k$ put $\Gamma_j = [a_j, b_j]$. We'll show that Γ has the property (♙): Every $c \in \mathbb{C}$ is the starting point of some edge of Γ as many times as it is the ending point of some such edge.

Take a polynomial P such that $P(c) = 1$, $P(a) = 0$ if $a \neq c$ and $[a, b] \in \Gamma$, $P(b) = 0$ if $b \neq c$ and $[a, b] \in \Gamma$ (existence as HW). By (♘) we have

$$0 = \int_\Gamma P' = \sum_{j=1}^{k} \int_{\Gamma_j} P' = \sum_{j=1}^{k} P(b_j) - \sum_{j=1}^{k} P(a_j),$$

but $\sum_{j=1}^{k} P(b_j)$ is the number of edges ending at c and $\sum_{j=1}^{k} P(a_j)$ is the number of edges starting at c.

Let L be the longest simple piecewise linear curve consisting of edges of Γ which begin with Γ_1, i.e.

- $L = [c_1, c_2, \dots, c_l] \stackrel{\text{def}}{=} [c_1, c_2] \dot{+} [c_2, c_3] \dot{+} \cdots \dot{+} [c_{l-1}, c_l]$,
- $\Gamma_1 = [c_1, c_2]$, $[c_{j-1}, c_j] \in \Gamma$ for $3 \le j \le l$,
- $c_j \neq c_k$ for $j \neq k$,
- l is maximal.

Since we have (♙), there is $j \in \{1, \dots, l-2\}$ such that $[c_l, c_j] \in \Gamma$. Then $L' = [c_j, c_{j+1}, \dots, c_l, c_j]$ is a closed simple piecewise linear curve. The proper subsystem Γ', which we get from Γ by omitting edges of L' has again the property (♙). We proceed in the same way for Γ' and finish after finitely many steps. Thus we can treat Γ as a cycle.

Finally let $f \in \mathcal{H}(G)$ and $a \in U = \operatorname{Int}(\bigcup Q^*)$. Let $a \in \operatorname{Int} \widetilde{Q}$ for some $\widetilde{Q} \in Q^*$. Then

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a} \,\mathrm{dz} = \sum_{Q_{m,n} \in Q^*} \frac{1}{2\pi i} \int_{\partial Q_{m,n}} \frac{f(z)}{z-a} \,\mathrm{dz} = \sum_{Q_{m,n} \in Q^*} \begin{cases} f(a), & \textit{if } Q_{m,n} = \widetilde{Q} \ (\textit{by Cauchy}) \\ 0, & Q_{m,n} \neq \widetilde{Q} \end{cases} = f(a).$$

Let $a \in \partial \widetilde{Q}$ for some $\widetilde{Q} \in Q^*$, but $a \notin \langle \Gamma \rangle$. We take points $(a_j) \in (\operatorname{Int} \widetilde{Q})^{\mathbb{N}}$ such that $a_j \to a$. Then

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a} \,\mathrm{dz} \leftarrow \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a_j} \,\mathrm{dz} = f(a_j) \to f(a),$$

where the left convergence follows from the continuity of the integral with respect to a_j. So we have $f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a} \,\mathrm{dz}$. Now we show $U = \operatorname{Int} \Gamma$. Let $a \in \mathbb{C} \setminus (U \cup \langle \Gamma \rangle)$. Then $a \in \mathbb{C} \setminus \bigcup Q^*$. As previously, we show $\operatorname{Ind}_\Gamma a = 0$. If $a \in U$, then by the calculation above we get $\operatorname{Ind}_\Gamma a = 1$. □

Lemma 5.18 ("Fubini[15]"). *Let $K_1, K_2 \subseteq \mathbb{C}$ be compact, $L_j \in C(K_j)^*, j = 1, 2$ and $F \in C(K_1 \times K_2)$. Then*

$$L_1(z \mapsto L_2(w \mapsto F(z, w))) = L_2\left(w \mapsto L_1\left(z \mapsto F(z, w)\right)\right). \tag{♗}$$

Sketch of proof. Obviously (♗) holds for functions of the form $F(z, w) = f(z)g(w)$ for $f \in C(K_1)$, $g \in C(K_2)$. Now we can use the Stone-Weierstraß theorem which implies that

$$\overline{\operatorname{span}}\{(z, w) \mapsto f(z)g(w) \colon f \in C(K_1), g \in C(K_2)\} = C(K_1 \times K_2).$$

□

Lemma 5.19 (Hole filling). *Let $G \subseteq \mathbb{C}$ be open and $K \subseteq G$ compact. There is a compact K_1 such that $K \subseteq K_1 \subseteq G$ and each component of $\mathbb{S} \setminus K_1$ contains some component of $\mathbb{S} \setminus G$.*

Proof. Take $n \in \mathbb{N}$ such that $K_1 = \{z \in G \colon \operatorname{dist}(z, \mathbb{C} \setminus G) \ge \frac{1}{n}\} \cap \overline{U(0, n)} \supset K$. We have $\mathbb{S} \setminus K_1 = \bigcup_{z_0 \in \mathbb{S} \setminus G} U(z_0, \frac{1}{n})$. Let V be a component of $\mathbb{S} \setminus K_1$. There is a point $z_0 \in \mathbb{S} \setminus G$ such that $U(z_0, \frac{1}{n}) \subseteq V$. If W is a component of $\mathbb{S} \setminus G$ containing z_0, then $W \subseteq V$. □

[15] Guido Fubini (19 January 1879, Venice – 6 June 1943, New York)

Proof of theorem 5.16. Let $L \in \mathcal{H}^*(G)$. We show there is a compact $K \subseteq G$ and $L_1 \in C(K)^*$ such that $L(f) = L_1(f|_K), f \in \mathcal{H}(G)$. From Lemma 5.7 we know that there are compact $K \subseteq G$ and $M \in (0,\infty)$ such that $\forall f \in \mathcal{H}(G)\colon |L(f)| \le M \|f\|_K$. By the Hahn-Banach theorem (5.8) we can extend L as $\widetilde{L}_1 \in C(G)$ such that $L = \widetilde{L}_1|_{\mathcal{H}(G)}$ and $\left|\widetilde{L}_1(f)\right| \le M \|f\|_K, f \in C(G)$. For each $f \in C(K)$ we define

$$L_1(f) :- \widetilde{L}_1(\widetilde{f}), \text{ where } \widetilde{f} \in C(G) \text{ and } \widetilde{f}|_K = f.$$

The definition is correct, because by Tietze theorem $\forall f \in C(K)\ \exists \widetilde{f} \in C(G)\colon \widetilde{f}|_K = f$ and if $\widetilde{f_1}, \widetilde{f_2} \in C(G), \widetilde{f_1} = \widetilde{f_2}$ on K, we have $|L_1(f_1) - L_1(f_2)| \le M \|f_1 - f_2\|_K = 0$. Note that by the Riesz representation, for each $L_1 \in C^*(K)$ there is a unique complex Borel measure μ on K such that $L_1(f) = \int_K f d\mu, f \in C(K)$.

By the Cauchy theorem for compacta (5.17), there is a cycle Γ in G such that $K \subseteq \operatorname{Int}\Gamma \subseteq G$, $\forall a \in \operatorname{Int}\Gamma\colon \operatorname{Ind}_\Gamma a = 1$ and for each $f \in \mathcal{H}(G)$ holds $f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\,\mathrm{dw}, z \in K$. Denote

$$L_2(g) = \frac{1}{2\pi i}\int_\Gamma g(w)\,\mathrm{dw}, g \in C(\langle\Gamma\rangle) \quad \text{and} \quad F(z,w) = \frac{f(w)}{w-z} \in C(K \times \langle\Gamma\rangle).$$

Of course, $L_2 \in C(\langle\Gamma\rangle)^*$ and $f(z) = L_2(w \mapsto F(z,w)), z \in K$.

For a given $f \in \mathcal{H}(G)$, $L(f) = L_1(f|_K) = L_1(z \mapsto L_2(F(z,\cdot))) \overset{\text{L 5.18}}{=} L_2(w \mapsto L_1(F(\cdot,w)))$, hence $L(f) = \frac{1}{2\pi i}\int_\Gamma f(w)\lambda(w)\,\mathrm{dw}$ where $\lambda(w) = L_1(z \mapsto \frac{1}{w-z})$, $w \in \mathbb{C}\setminus K$. We want to show that $\lambda \in \mathcal{H}_0(\mathbb{S}\setminus K)$ satisfies (a), (b).

(a): Let $U(\infty,\varepsilon) = \{z \in \mathbb{C}\colon |z| > \frac{1}{\varepsilon}\} \cup \{\infty\} \subseteq \mathbb{S}\setminus K$. in particular, $\forall z \in K\colon |z| \le \frac{1}{\varepsilon}$. For $u \in P(0,\varepsilon)$ we have

$$\lambda\left(\frac{1}{u}\right) = L_1\left(z \mapsto \frac{u}{1-uz}\right) = L_1\left(z \mapsto \sum_{k=0}^{\infty} z^k u^{k+1}\right) \overset{\rightrightarrows}{=} \sum_{k=0}^{\infty} u^{k+1} L_1(z \mapsto z^k),$$

hence $\lambda(\infty) = 0$, $\forall k \in \mathbb{N}_0\colon \frac{\lambda^{(k+1)}(\infty)}{(k+1)!} = L_1(z \mapsto z^k)$. The sum on the right hand side converges for small enough ε since there is $R > 0$ such that $K \subseteq U(0,R)$ and

$$\left|L_1(z \mapsto z^k)\right| \le M \left\|z \mapsto z^k\right\|_K = M\left(\max_{z\in K}|z|^k\right) = MR^k.$$

(b): Let $U(w_0,\varepsilon) \subseteq \mathbb{C}\setminus K$. Then $\forall w \in U(w_0,\varepsilon)\colon \lambda(w) = L_1(\frac{1}{w-\cdot})$.

$$\forall z \in K\colon \frac{1}{w-z} = \frac{1}{(w-w_0)-(z_1-w_0)} = -\frac{1}{z-w_0}\frac{1}{1-\frac{w-w_0}{z-w_0}} = -\sum_{k=0}^{\infty}\frac{(w-w_0)^k}{(z-w_0)^{k+1}},$$

which implies

$$\lambda(z_2) = -L_1\left(z_1 \mapsto \sum_{k=0}^{\infty}\frac{(w-w_0)^k}{(z-w_0)^{k+1}}\right) \overset{\rightrightarrows}{=} -\sum_{k=0}^{\infty}(w-w_0)^k L_1\left(z \mapsto \frac{1}{(z-w_0)^{k+1}}\right).$$

Hence $\frac{\lambda^{(k)}(w_0)}{k!} = -L_1\left(z \mapsto \frac{1}{(z-w_0)^{k+1}}\right), k \in \mathbb{N}_0$.

Let $\lambda_1, \lambda_2 \in \mathcal{H}_0(\mathbb{S}\setminus G)$ satisfy (a), (b). Then there is a compact $K \subseteq G$ such that $\lambda_1, \lambda_2 \in \mathcal{H}_0(\mathbb{S}\setminus K)$. By Lemma 5.19, without loss of generality assume that each component V of $\mathbb{S}\setminus K$ intersects $\mathbb{S}\setminus G$. We will show $\lambda_1 = \lambda_2$ on $\mathbb{S}\setminus K$. Let V be any component of $\mathbb{S}\setminus K$. Then there is $z_0 \in V \cap (\mathbb{S}\setminus G) \neq \emptyset$. By (a), (b) we have $\lambda_1^{(k)}(z_0) = \lambda_2^{(k)}(z_0), k \in \mathbb{N}_0$. By the uniqueness theorem $\lambda_1 = \lambda_2$ on the domain V. So $\lambda_1 = \lambda_2$ on $\mathbb{S}\setminus K$. □

6 Runge's theorem

Notation 6.1. Let $E \subseteq \mathbb{S}$ and $m : E \to \mathbb{N}\cup\{\infty\}$. We call $m(e)$ the multiplicity of $e \in E$. We say that (E,m) has a limit point $e \in \mathbb{S}$ if e is a limit point of E or ($e \in E$ and $m(e) = \infty$).

Denote by $\mathcal{F}(E,m)$ the smallest system of functions which contains:

(i) $z \mapsto \frac{1}{z-e}$ if $e \in E \cap \mathbb{C}, m(e) < \infty$,

(ii) $z \mapsto \frac{1}{(z-e)^k}, k \in \mathbb{N}$ if $e \in E \cap \mathbb{C}, m(e) = \infty$,

(iii) $z \mapsto z^k, k \in \mathbb{N}_0$ if $\infty \in E, m(\infty) = \infty$.

Theorem 6.2 (Runge, modern formulation)**.** *Let $G \subseteq \mathbb{C}$ be open, $E \subseteq \mathbb{S} \setminus G$ and $m : E \to \mathbb{N} \cup \{\infty\}$. If (E, m) has a limit point in every component of $\mathbb{S} \setminus G$, then $\overline{\operatorname{span}}\ \mathcal{F}(E, m) = \mathcal{H}(G)$.*

Proof. We will use the Hahn-Banach theorem (5.8). Let $L \in \mathcal{H}^*(G)$ and $L = 0$ on $\mathcal{F}(E, m)$. We need to show $L = 0$ on $\mathcal{H}(G)$. Let $\lambda \in \mathcal{H}_0(\mathbb{S} \setminus G)$ represent L in the sense of previous chapter. If $e \in E \cap \mathbb{C}, m(e) < \infty$, then $\lambda(e) = -L(\frac{1}{z-e}) = 0$. If $e \in E \cap \mathbb{C}, m(e) = \infty$, then $\frac{\lambda^{(k)(e)}}{k!} = -L\left(\frac{1}{(z-e)^k}\right) = 0, k \in \mathbb{N}_0$. If $\infty \in E$ and $m(\infty) = \infty$, then $\frac{\lambda^{(k+1)}(\infty)}{(k+1)!} = L(z^k) = 0, k \in \mathbb{N}_0$. We show $\lambda = 0$ in $\mathcal{H}_0(\mathbb{S} \setminus G)$. There is a compact set $K \subseteq G$ such that (i) $\lambda \in \mathcal{H}_0(\mathbb{S} \setminus K)$ and (ii) every component of $\mathbb{S} \setminus K$ contains some component of $\mathbb{S} \setminus G$. Let V be any component of $\mathbb{S} \setminus K$. Then V is a domain and by (ii) V contains a limit point e of (E, m). By the uniqueness theorem we get $\lambda = 0$ on V (either zero points of λ have a limit point or at some point λ has all coefficients in its sum representation equal zero) and so on $\mathbb{S} \setminus K$. □

Theorem 6.3 (Runge, classical formulation)**.** *Let $G \subseteq \mathbb{C}$ be open, and let $f \in \mathcal{H}(G)$.*

(a) Then there are rational functions R_n, $n \in \mathbb{N}$ with poles outside G such that $R_n \overset{loc}{\rightrightarrows} f$ on G.

(b) If, in addition, $\mathbb{S} \setminus G$ is connected, there are polynomials $P_n, n \in \mathbb{N}$ such that $P_n \overset{loc}{\rightrightarrows} f$ on G.

Proof. (a): Let $E \subseteq \mathbb{S} \setminus G$ contain at least one point of every component of $\mathbb{S} \setminus G$. Put $m \equiv \infty$ on E. Then $\operatorname{span} \mathcal{F}(E, m)$ is a dense subset of rational functions with poles outside of G.

(b): Let $E = \{\infty\}$, put $m(\infty) = \infty$. Then $\mathcal{F}(E, m) = \{z \mapsto z^k : k \in \mathbb{N}_0\}$. Hence polynomials are dense. □

Theorem 6.4 (Cauchy, for simply connected domains)**.** *Let $G \subseteq \mathbb{C}$ be open and $\mathbb{S} \setminus G$ be connected. If $f \in \mathcal{H}(G)$ and φ is a closed regular curve in G, then $\int_\varphi f = 0$.*

Proof. By Runge's theorem (6.3), there are polynomials P_n such that $P_n \overset{loc}{\rightrightarrows} f$ on G. Then $\int_\varphi P_n \to \int_\varphi f$, but the former integrand has a primitive function and thus the integral is zero. □

Theorem 6.5 (Cauchy, for cycles)**.** *Let $G \subseteq \mathbb{C}$ be open and Γ be a cycle in G (i.e. $\langle \Gamma \rangle \subseteq G$). Then $\int_\Gamma f = 0$ for each $f \in \mathcal{H}(G)$ if and only if $\operatorname{Int} \Gamma \subseteq G$.*

Proof. $\Longrightarrow$: If $z_0 \in \mathbb{C} \setminus G$, then $f(z) = \frac{1}{z-z_0} \in \mathcal{H}(G)$ and the condition implies that $\operatorname{Ind}_\Gamma z_0 = \frac{1}{2\pi i} \int_\Gamma f = 0$.

$\Longleftarrow$: Let $f \in \mathcal{H}(G)$. Then by Runge's theorem (6.3), there are rational functions R_n with poles outside of G such that $R_n \overset{loc}{\rightrightarrows} f$. Then $\int_\Gamma R_n \to \int_\Gamma f$. We prove the former integrals are zero.

Let $\Gamma = \{\varphi_1, \ldots, \varphi_k\}$, where φ_j are closed regular curves in G. Then

$$\int_\Gamma R_n = \sum_{j=1}^{k} \int_{\varphi_j} R_n \overset{\text{Residue thm}}{=} \sum_{j=1}^{m} 2\pi i \sum_{R_n(s)=\infty} res_s R_n \cdot \operatorname{Ind}_{\varphi_j} s = 2\pi i \sum_{R_n(s)=\infty} res_s R_n \cdot \operatorname{Ind}_\Gamma s = 0.$$

In the computation we have used the Residue theorem for star-like domains in $\mathbb{C}$ and the fact that $\operatorname{Ind}_\Gamma s = 0$ for $s \notin G$. □

Theorem 6.6 (Runge, for compacta)**.** *Let K be a compact in $\mathbb{C}$ and let $S \subseteq \mathbb{S} \setminus K$ contain at least one point of any component of $\mathbb{S} \setminus K$. Let $f \in \mathcal{H}(K)$. Then there are rational functions R_n with poles in S such that $R_n \rightrightarrows f$ on K.*

For the proof we will use the technique known as pushing poles.

Recall 6.7. Each rational function R can be uniquely expressed in the following form:

$$R(z) = \sum_{k=1}^{n} \sum_{j=1}^{n_k} \frac{A_j^k}{(z - z_k)^j} + C_0 + C_1 z + \cdots + C_m z^m,$$

where the first summand is the principal part of Laurent expansion around the pole z_k and the rest is the principal part of Laurent expansion around the pole ∞, $n, m \in \mathbb{N}_0, z_k \in \mathbb{C}$ and $A_{n_k}^k \neq 0, C_m \neq 0$. Then z_k is a pole of the function R of multiplicity of n_k and ∞ is a pole of R of multiplicity m.

A rational function is a polynomial if and only if R has a pole at most at ∞.

Notation 6.8. Let K be a compact in $\mathbb{C}$, $U \subseteq \mathbb{S}$ and $U \cap K = \emptyset$. Define the function space $B(K, U) = \overline{\{R|_K : R \text{ rational function with poles in } U\}}$. Here, the closure is taken with respect to the topology of $C(K)$.

Theorem 6.9 (Pushing poles)**.** *Let K be a compact in $\mathbb{C}$, $U \subseteq \mathbb{S}$ be a domain, $K \cap U = \emptyset$ and $z_0 \in U$. If R is a rational function with poles in U, then $R \in B(K, \{z_0\})$.*

Remark 6.10. By the theorem we have $B(K,U) = B(K,z_0)$ $(= B(K,\{z_0\}))$. This follows from the fact that any rational function with poles in U can be approximated by rational functions with a pole at just z_0.

Proof. Put $V = \{\xi \in U\colon (z \mapsto \frac{1}{z-\xi}) \in B(K,z_0) \text{ if } \xi \in \mathbb{C}; (z \mapsto z) \in B(K,z_0) \text{ if } \xi = \infty\}$. $B(K,U)$ is a closed subalgebra of $C(K)$ (HW).

First we show that $B(K,z_0) = B(K,V)$. Indeed, if $\xi \in V$, then $\frac{1}{z-\xi} \in B(K,z_0)$ for $\xi \in \mathbb{C}$. Since $B(K,z_0)$ is an algebra, $\frac{1}{(z-\xi)^k} \in B(K,z_0)$ for $\xi \in \mathbb{C}$, $k \in \mathbb{N}$. Also $z^k \in B(K,z_0)$ for $\xi = \infty$. Then each rational R with poles in V is contained in $B(K,z_0)$. Hence $B(K,V) \subseteq B(K,z_0)$. Since $z_0 \in V$, we have $B(K,z_0) \subseteq B(K,V)$.

Now we will prove that V is closed in U. Let $\xi_n \in V, n \in \mathbb{N}$, $\xi_n \to \xi_0 \in U$. We need to show that $\xi_0 \in V$. Without loss of generality assume that $\xi_n \in \mathbb{C}, n \in \mathbb{N}$. First, let $\xi_0 \in \mathbb{C}$. Then put $\delta = \operatorname{dist}(\xi_0, K) > 0$. Choose $n_0 \in \mathbb{N}$ such that $\operatorname{dist}(\xi_n, K) \geq \delta/2$ for $n > n_0$. Then $\frac{1}{z-\xi_n} \rightrightarrows \frac{1}{z-\xi_0}$ for $z \in K$, because

$$\left| \frac{1}{z-\xi_n} - \frac{1}{z-\xi_0} \right| = \frac{|\xi_n - \xi_0|}{|z-\xi_n|\,|z-\xi_0|} \leq \frac{4}{\delta^2} |\xi_n - \xi_0| \to 0$$

if $n > n_0$ and $z \in K$. Hence $\frac{1}{z-\xi_0} \in B(K,z_0)$, so $\xi_0 \in V$.

Second, let $\xi_0 = \infty$. Then $\frac{\xi_n z}{\xi_n - z} = -\xi_n \left(\frac{\xi_n}{z-\xi_n} + 1 \right) \in B(K,z_0)$ since $B(K,z_0)$ is an algebra. Take $C > 0$ with $|z| \leq C$ for any $z \in K$. Take $n_0 \in \mathbb{N}$ such that $|\xi_n| > C$ for any $n > n_0$. Then $\frac{\xi_n z}{\xi_n - z} \rightrightarrows z$ for $z \in K$, because

$$\left| \frac{\xi_n z}{\xi_n - z} - z \right| = \frac{|z|^2}{|\xi_n - z|} \leq \frac{C^2}{|\xi_n| - C} \to 0$$

for $n > n_0$ and $z \in K$. Hence $z \in B(K,z_0)$ and thus $\infty \in V$.

The next step is to show that V is open. Let $\xi_0 \in V$. Again, first we assume $\xi_0 \in \mathbb{C}$. Put $\delta = \operatorname{dist}(\xi_0, K) > 0$. Let $\xi \in U(\xi_0, \delta/2)$. Then

$$\frac{1}{z-\xi} = \frac{1}{(z-\xi_0) - (\xi - \xi_0)} = \frac{1}{z-\xi_0} \frac{1}{1 - \frac{\xi-\xi_0}{z-\xi_0}} = \sum_{k=0}^{\infty} \frac{(\xi-\xi_0)^k}{(z-\xi_0)^{k+1}}$$

for $\left| \frac{\xi-\xi_0}{z-\xi_0} \right| < 1$. The sum converges uniformly because

$$\left| \frac{(\xi-\xi_0)^k}{(z-\xi_0)^{k+1}} \right| \leq \frac{(\delta/2)^k}{\delta^{k+1}} = \frac{1}{\delta 2^k}$$

for $z \in K$ and $k \in \mathbb{N}$. Hence $\frac{1}{z-\xi} \in B(K,\xi_0) \subseteq B(K,V) = B(K,z_0)$ and so $U(\xi_0, \delta/2) \subseteq V$.

Second, we let $\xi_0 = \infty$. Take $C > 0$ such that $|z| \leq C$ for any $z \in K$. Let $\xi \in \mathbb{C}$ with $|\xi| > 2C$. Then $\frac{1}{z-\xi} = -\frac{1}{\xi} \frac{1}{1-z/\xi} = -\sum_{k=0}^{\infty} \frac{z^k}{\xi^{k+1}}$ converges uniformly for $z \in K$ because

$$\left| \frac{z^k}{\xi^{k+1}} \right| \leq \frac{C^k}{(2C)^{k+1}} = \frac{1}{C 2^{k+1}}$$

for $z \in K$ and $k \in \mathbb{N}_0$. Hence $\frac{1}{z-\xi} \in B(K,\infty) \subseteq B(K,z_0)$, so $U\left(\xi_0, \frac{1}{2C}\right) \subseteq V$.

To finish the proof, we realize that $V = U$, because V is a non-empty clopen subset of the domain U. ☐

Proof of Runge's theorem for compacta. Let f be a holomorphic function on an open set $G \supset K$. Using Runge's theorem for open sets (6.3) there are rational functions $\widetilde{R}_n, n \in \mathbb{N}$ with poles outside G such that $\widetilde{R}_n \rightrightarrows f$ on K. It is enough to show that $\widetilde{R}_n \in B(K,S)$, because then we have $f \in B(K,S)$.

Fix $n \in \mathbb{N}$. All poles of $\widetilde{R}_n$ are contained in finitely many components $C_1, \ldots, C_k$ of $\mathbb{S} \setminus K$. Express $\widetilde{R}_n = \widetilde{Q}_1 + \cdots + \widetilde{Q}_k$ where $\widetilde{Q}_j$ is a rational function with poles in the domain C_j. For each $j \in \{1, \ldots, k\}$ take $s_j \in S \cap C_j$. By pushing poles we have that $\widetilde{Q}_j \in B(K, s_j)$. For a given $\varepsilon > 0$ and $j \in \{1, \ldots, k\}$ there is a rational function Q_j with a pole at s_j such that $\left| Q_j - \widetilde{Q}_j \right| \leq \varepsilon/k$ on K. Put $R_n = Q_1 + \cdots + Q_k \in B(K,S)$. Then $\left| R_n - \widetilde{R}_n \right| \leq \varepsilon$ on K. Hence $\widetilde{R}_n \in B(K,S)$. ☐

7 Characterization of simple connectedness

Recall 7.1. A domain $G \subseteq \mathbb{C}$ is called *simply connected* if $\mathbb{S} \setminus \mathbb{C}$ is connected.

We will start by introducing the notion of homotopic loops and proving some of their basic properties. This will allow us to later give "the right" topological definition of simple connectedness. Without loss of generality we can assume that all curves are defined on the interval $[0,1]$ (otherwise we can take a linear reparametrisation).

Definition 7.2. Let $G \subseteq \mathbb{C}$ be open. A continuous closed curve $\varphi : [0,1] \to G$ is called a *loop* in G. We say that two loops φ, ψ are *homotopic* (in G) if there is a continuous map $H : [0,1]^2 \to G$ such that for $\varphi_s(t) :- H(s,t)$ holds $\varphi_0 = \varphi$, $\varphi_1 = \psi$ and $\varphi_s(0) = \varphi_s(1), s \in [0,1]$.

Remark 7.3. Here φ_s are "continuous deformations" of φ onto ψ.

Example 7.4. If $G \subseteq \mathbb{C}$ is a star-like domain, then every loop in G is homotopic to a constant loop.

Proof. Indeed, let $z_0 \in G$ be such that for each $z \in G$ the line segment $[z, z_0] \subseteq G$. Let φ be a loop in G. Then φ is homotopic to the constant loop $\psi(t) = z_0, t \in [0,1]$ because we can take $H(s,t) = sz_0 + (1-s)\varphi(t)$. □

Fact 7.5. *Let $\Omega \subseteq \mathbb{C}$ be open and let every loop in Ω be homotopic to a constant loop. If $G \subseteq \mathbb{C}$ is homeomorphic to Ω, then G has this property as well.*

Proof. Let h be a homeomorphism of G onto Ω. Let φ be a loop in G. Then $\widetilde{\varphi} :- h \circ \varphi$ is homotopic in Ω with a constant loop (with $\widetilde{H}$), and so is φ (with $H :- h^{-1} \circ \widetilde{H}$). □

Let $\varphi : [0,1] \to \mathbb{C}$ be a loop and $z_0 \in \mathbb{C} \setminus \langle\varphi\rangle$. There are regular closed curves $\varphi_n : [0,1] \to \mathbb{C}$ such that $\varphi_n \rightrightarrows \varphi$. Indeed, using the uniform continuity of φ, φ can be uniformly approximated by piecewise linear closed curves with vertices on φ given by sufficiently fine partition of $[0,1]$. Define $\operatorname{Ind}_\varphi z_0 = \lim_{n\to\infty} \operatorname{Ind}_{\varphi_n} z_0$. By Lemma 2.2, the definition is correct because there is $n_0 \in \mathbb{N}$ such that $\operatorname{Ind}_{\varphi_n} z_0$ are constant for $n \geq n_0$ and $\operatorname{Ind}_\varphi z_0$ does not depend on the choice of $\{\varphi_n\}$. Alternatively one could use a continuous branch of the argument of φ.

Theorem 7.6. *Let φ, ψ be two loops homotopic in an open set $G \subseteq \mathbb{C}$. Then $\operatorname{Ind}_\varphi z_0 = \operatorname{Ind}_\psi z_0$ for any $z_0 \in \mathbb{C} \setminus G$.*

Proof. First we show that Lemma 2.2 holds for loops as well. Indeed, let loops φ_1, φ_2 satisfy the condition from the lemma. Then there are $\widetilde{\varphi}_1, \widetilde{\varphi}_2$ which are regular, satisfy the assumptions of the Lemma and $\operatorname{Ind}_{\varphi_j} z_0 = \operatorname{Ind}_{\widetilde{\varphi}_j} z_0$, $j = 1,2$.

Let $H : [0,1]^2 \to G$ be continuous mapping such that $\varphi_0 = \varphi, \varphi_1 = \psi$ and $\varphi_s(0) = \varphi_s(1), s \in [0,1]$, where $\varphi_s(t) = H(s,t)$. Put $\varepsilon = \operatorname{dist}(z_0, H([0,1]^2)) > 0$ ($H([0,1]^2)$ is a compact set). Since H is uniformly continuous, there is $n \in \mathbb{N}$ such that for each $k = 0, \ldots, n-1$ and $t \in [0,1]$ we have

$$\left|\varphi_{\frac{k}{n}}(t) - \varphi_{\frac{k+1}{n}}(t)\right| = \left|H\left(\frac{k}{n}, t\right) - H\left(\frac{k+1}{n}, t\right)\right| < \varepsilon.$$

In particular, $\varphi_{\frac{k}{n}}$ and $\varphi_{\frac{k+1}{n}}$ satisfy the assumptions of Lemma 2.2. Hence

$$\operatorname{Ind}_{\varphi_0} z_0 = \operatorname{Ind}_{\varphi_{\frac{1}{n}}} z_0 = \operatorname{Ind}_{\varphi_{\frac{2}{n}}} z_0 = \cdots = \operatorname{Ind}_{\varphi_1} z_0.$$

□

Theorem 7.7. *Let $G \subseteq \mathbb{C}$ be open. Then the following statements are equivalent:*

(SC1) *If φ is a closed (regular) curve in G, then $\operatorname{Int}\varphi \subseteq G$.*

(SC2) *$\mathbb{S} \setminus \mathbb{C}$ is connected.*

(SC3) *$\forall f \in \mathcal{H}(G)$ $\exists$ polynomials $P_n : P_n \overset{loc}{\rightrightarrows} f$ on G.*

(SC4) *$\forall f \in \mathcal{H}(G)$: $\int_\varphi f = 0$, where φ is an arbitrary closed regular curve.*

(SC5) *$\forall f \in \mathcal{H}(G)$ $\exists F \in \mathcal{H}(G)$: $F' = f$ on G.*

(SC6) *$\forall f \in \mathcal{H}(G), f \neq 0$ on G $\exists g \in \mathcal{H}(G)$: $f = e^g$ on G.*

(SC7) *$\forall f \in \mathcal{H}(G), f \neq 0$ on G $\exists h \in \mathcal{H}(G)$: $f = h^2$ on G.*

(SC8) *Every loop φ in G is homotopic in G to a constant loop. ("Every loop in G can be shrinked inside G into a point").*

Proof. (SC1) $\implies$ (SC2): Assume that $\mathbb{S}\setminus G$ is not connected. Then there are disjoint closed sets $\emptyset \neq K, L \subseteq \mathbb{S}$ such that $\mathbb{S}\setminus G = K \cup L$. Without loss of generality $\infty \notin K$. Then K is a compact in the complex plane, $G_0 :- G \cup K$ is open in $\mathbb{C}$ ($\mathbb{C}\setminus G_0 = L$) and, by Theorem 5.17 there is a cycle Γ in G_0 such that $K \subseteq \operatorname{Int}\Gamma \subseteq G_0$. Let $z_0 \in K$. Since $\operatorname{Ind}_\Gamma z_0 \neq 0$, there is $\varphi \in \Gamma$ with $\operatorname{Ind}_\varphi z_0 \neq 0$. Of course, $z_0 \in (\mathbb{C}\setminus G) \cap \operatorname{Int}\varphi$.

(SC2) $\implies$ (SC3): Proof is the same as for the classical version of Runge's theorem 6.3.

(SC3) $\implies$ (SC4): See the proof of Cauchy theorem for simply connected domains 6.4.

(SC4) $\iff$ (SC5): We know from Introduction to Complex analysis.

(SC5) $\implies$ (SC6): See the proof of Proposition 4.3.

(SC6) $\implies$ (SC7): Put $h = e^{\frac{1}{2}g}$.

(SC7) $\implies$ (SC8): Let φ be a loop in G. Let G_0 be a component of G containing $\langle\varphi\rangle$. If $G_0 = \mathbb{C}$, then by Example 7.4, φ is homotopic to a constant loop. If $G_0 \subsetneq \mathbb{C}$, then G_0 is a non-trivial proper subdomain of $\mathbb{C}$ which also satisfies the condition (SC7). By Riemann's theorem 8.1, G_0 is homeomorphic to the unit disc, which satisfies the condition (SC8) by Example 7.4 and so we can conclude using Fact 7.5.

(SC8) $\implies$ (SC1): Of course, every constant loop ψ has $\operatorname{Int}\psi = \emptyset$. Hence this implication follows from Theorem 7.6. ☐

8 The Riemann theorem

In this section we will prove the Riemann theorem. This is probably the most important result proved on this lecture.

Theorem 8.1 (Riemann)**.** *Let $\emptyset \neq G \subsetneq \mathbb{C}$ be a domain such that $\forall f \in \mathcal{H}(G), f \neq 0$ on G $\exists h \in \mathcal{H}(G)\colon h^2 = f$ on G. Then there is a holomorphic bijection $f : G \to \mathbb{D}$.*

Remark 8.2.

(i) $f^{-1} : \mathbb{D} \overset{onto}{\to} G$ is conformal.

(ii) The condition we require of G is exactly condition (SC7) from the characterization of simple connectedness (Theorem 7.7). By this, we finish the proof of said theorem.

Theorem 8.3 (Schwartz[16] lemma)**.** *Let $f \in \mathcal{H}(\mathbb{D}), f(\mathbb{D}) \subseteq \mathbb{D}$, and $f(0) = 0$. Then*

(i) $|f(z)| \leq |z|$, $z \in \mathbb{D}$,

(ii) $|f'(0)| \leq 1$.

If equality occurs in (i) for some $z \in \mathbb{D}\setminus\{0\}$ or in (ii), then f is a rotation, i.e. $f(z) = \lambda z, z \in \mathbb{D}$ for some $\lambda \in S_{\mathbb{C}}$.

Proof. Put $g(z) = \frac{f(z)}{z}, z \in \mathbb{D}\setminus\{0\}$, $g(0) = f'(0)$. Note that $g \in \mathcal{H}(\mathbb{D})$. Let $0 < r < 1$. Then $|g(z)| \leq \frac{1}{r}, |z| = r$. By the maximum modulus theorem we get (♙) $|g(z)| \leq \frac{1}{r}, |z| \leq r$. Let $z \in \mathbb{D}$. For $r \in (1-\delta, 1)$ for some δ small enough, we have (♙) and letting $r \to 1^-$ we obtain $|g(z)| \leq 1$. If $|g(z)| = 1$ for some $z \in \mathbb{D}$, then by maximum modulus theorem g is constant on $\mathbb{D}$. ☐

Lemma 8.4. *For $\alpha \in \mathbb{D}$ put $\varphi_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$. Then*

(i) φ_α *is one-to-one and* $\varphi_\alpha^{-1} = \varphi_{-\alpha}$,

(ii) $\varphi_\alpha \in \mathcal{H}(\mathbb{C}\setminus\{\frac{1}{\overline{\alpha}}\})$, $\varphi_\alpha(\mathbb{D}) = \mathbb{D}$, $\varphi_\alpha(\mathbb{T}) = \mathbb{T}$, *where* $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$,

(iii) $\varphi_\alpha(\alpha) = 0, \varphi_\alpha'(\alpha) = \frac{1}{1-|\alpha|^2}$, $\varphi_\alpha(0) = 1 - |\alpha|^2$.

Proof. (i):

$$\begin{aligned} w &= \frac{z-\alpha}{1-\overline{\alpha}z} \\ w - \overline{\alpha}wz &= z - \alpha \\ w + \alpha &= z(1+\overline{\alpha}w) \\ z &= \frac{w+\alpha}{1+\overline{\alpha}w} = \varphi_{-\alpha}(w) \end{aligned}$$

[16]Laurent-Moïse Schwartz (5 March 1915, Paris – 4 July 2002, Paris)

(ii): If $z \in \mathbb{T}$, then using $1 = |z|^2 = z\overline{z}$ we get

$$|\varphi_\alpha| = \left|\frac{z-\alpha}{1-\overline{\alpha}z}\right| = \left|\frac{z-\alpha}{\overline{z}z-\overline{\alpha}z}\right| = \frac{|z-\alpha|}{|\overline{z-\alpha}|\,|z|} = 1.$$

Hence $\varphi_\alpha(\mathbb{T}) \subseteq \mathbb{T}$. The same is true for $\varphi_\alpha^{-1} = \varphi_{-\alpha}$, so $\varphi_\alpha(\mathbb{T}) = \mathbb{T}$. By the fact $\varphi_\alpha(\mathbb{T}) = \mathbb{T}$ and maximum modulus principle, we get $\varphi_{-\alpha}(\mathbb{D}) \subseteq \mathbb{D}$, so $\varphi_\alpha(\mathbb{D}) = \mathbb{D}$.

(iii):

$$\varphi_\alpha'(\alpha) = \lim_{z\to\alpha}\frac{\varphi_\alpha(z)}{z-\alpha} = \frac{1}{1-|\alpha|^2} \quad\text{and}\quad \varphi_\alpha'(0) = \left.\frac{1-\overline{\alpha}z+(z-\alpha)\overline{\alpha}}{(1-\overline{\alpha}z)^2}\right|_{z=0} = 1-|\alpha|^2.$$

□

Theorem 8.5 (Conformal transformations of $\mathbb{D}$)**.** *A fucntion f is conformal map of $\mathbb{D}$ onto $\mathbb{D}$ if and only if there are $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that $f(z) = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z}$, $z \in \mathbb{D}$.*

Proof. $\Longleftarrow$: Follows from the previous lemma.

$\Longrightarrow$: Let $\alpha \in \mathbb{D}$ and $f(\alpha) = 0$. Then $g :- f \circ \varphi_{-\alpha}$ is a conformal map of $\mathbb{D}$ onto $\mathbb{D}$ and $g(0) = 0$. By the Schwartz lemma (8.3), for $z \in \mathbb{D}$ we have $|g(z)| \le |z|$, $\left|g^{-1}(z)\right| \le |z|$, so $|g(z)| = |z|$. By Schwartz, g is a rotation. □

Lemma 8.6 (Schwartz-Pick[17])**.** *Let $F \in \mathcal{H}(\mathbb{D}), F(\mathbb{D}) \subseteq \mathbb{D}$ and $F(\alpha) = \beta$. Then $|F'(\alpha)| \le \frac{1-|\beta|^2}{1-|\alpha|^2}$. If equality occurs, then $F(z) = \varphi_{-\beta}(\lambda\varphi_\alpha(z)), z \in \mathbb{D}$ for some $\lambda \in \mathbb{T}$. In particular, $|F'(0)| < 1$ unless F is a rotation.*

Here φ_α are mappings defined in the previous lemma.

Proof. Use the Schwartz lemma (8.3) for the function $f :- \varphi_\beta \circ F \circ \varphi_{-\alpha}$ to obtain $|f'(0)| \le 1$ and use Lemma 8.4 to calculate

$$f'(0) = \varphi_\beta'(\beta)\cdot F'(\alpha)\cdot\varphi_{-\alpha}'(0) = \frac{1}{1-|\beta|^2}\cdot F'(\alpha)\cdot(1-|\alpha|^2) \implies |F'(\alpha)| = \left|\frac{1-|\alpha|^2}{1-|\beta|^2}f'(0)\right| \le \frac{1-|\alpha|^2}{1-|\beta|^2}.$$

If $\alpha = 0 = \beta$ and F is not a rotation, then $|F'(0)| < 1$. □

Proof of the Riemann theorem. Let $\emptyset \ne G \subsetneq \mathbb{C}$ be a domain with (SC7). Take a point $z_0 \in G$. Denote by Σ the set of all conformal mappings $\psi : G \to \mathbb{D}$. Then we have:

(i) $\Sigma \ne \emptyset$,

(ii) If $\psi \in \Sigma$ and $\psi(G) \ne \mathbb{D}$, then there is $\widetilde{\psi} \in \Sigma$ such that $\left|\widetilde{\psi}'(z_0)\right| > |\psi'(z_0)|$.

We defer the proof of these properties to the end of this proof.

Put $\eta = \sup\{|\psi'(z_0)| : \psi \in \Sigma\}$. Take $\psi \in \Sigma$. Since ψ is one-to-one, we have $\psi'(z_0) \ne 0$ and hence $\eta > 0$. By the definition of η there are $\psi_n \in \Sigma$, $n \in \mathbb{N}$ such that $|\psi_n'(z_0)| \overset{n\to\infty}{\to} \eta$. Since $\psi_n, n \in \mathbb{N}$ are uniformly bounded, by the Montel theorem (5.4) there is a subsequence $\{\psi_{n_k}\}$ such that $\psi_{n_k} \overset{loc}{\rightrightarrows} f$ on G. By the Weierstraß theorem, $f \in \mathcal{H}(G)$ and $|f'(z_0)| = \eta \in (0,\infty)$. Since f is not constant, the Hurwitz theorem (2.17) implies that f is one-to-one. Of course, $f(G) \subseteq \overline{\mathbb{D}}$, but by openness of f we have $f(G) \subseteq \mathbb{D}$. Hence $f \in \Sigma$ and by the second property $f(G) = \mathbb{D}$.

Now we prove the properties (i) and (ii).

(i): Let $w_0 \in \mathbb{C}\setminus G$. Then by (SC7) there is $\varphi \in \mathcal{H}(G)$ such that $z - w_0 = \varphi^2(z), z \in G$. If $\varphi(z_1) = \pm\varphi(z_2)$, then $z_1 = z_2$: Indeed, $z_1 - w_0 = \varphi^2(z_1) = \varphi^2(z_2) = z_2 - w_0$. So φ is one-to-one and ($\times$) $0 \ne w \in \varphi(G) \implies -w \notin \varphi(G)$. Since $\emptyset \ne \varphi(G)$ is open, there is $0 \notin U(a,r) \subseteq \varphi(G)$. By ($\times$), we have $U(-a,r) \cap \varphi(G) = \emptyset$, i.e. $|\varphi(z) + a| \ge r, z \in G$. Put $\psi = \frac{r}{2(\varphi(z)+a)}, z \in G$. Then $|\psi| \le \frac{1}{2}$ on G, so $\psi \in \Sigma$.

(ii): Pick $\psi \in \Sigma$ and $\alpha \in \mathbb{D}\setminus\psi(G)$. Consider the map $\varphi_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}, z \in \mathbb{D}$. Then $\varphi_\alpha \circ \psi \in \Sigma$ and $\varphi_\alpha \circ \psi \ne 0$ on G. By (SC7) there is $g \in \mathcal{H}(G)$ such that (1) $\varphi_\alpha \circ \psi = g^2$ on G. Then g is one-to-one, because $g(z_1) = g(z_2) \implies \varphi_\alpha \circ \psi(z_1) = \varphi_\alpha \circ \psi(z_2) \implies z_1 = z_2$. Hence $g \in \Sigma$.

Denote $\beta :- g(z_0)$ and put (2) $\widetilde{\psi} = \varphi_\beta \circ g$. Of course, $\widetilde{\psi} \in \Sigma$ and $\widetilde{\psi}(z_0) = 0$. Denoting $s(w) :- w^2, w \in \mathbb{C}$, we have by (1) and (2) that (3) $\psi = (\varphi_{-\alpha} \circ s \circ \varphi_{-\beta}) \circ \widetilde{\psi} = F \circ \widetilde{\psi}$, where $F = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta}$. We have $F \in \mathcal{H}(\mathbb{D}), F(\mathbb{D}) \subseteq \mathbb{D}$ and F is not a rotation (because F is not one-to-one). By the Schwartz-Pick lemma (8.6), we have $|F'(0)| < 1$. Since $\psi'(z_0) = F'(0)\widetilde{\psi}'(z_0)$, we have $0 < |\psi'(z_0)| < \left|\widetilde{\psi}'(z_0)\right|$. □

[17]Georg Alexander Pick (10 August 1859, Vienna, Austria-Hungary – 26 July 1942, Theresienstadt concentration camp, Czechoslovakia)

Definition 8.7. Let $G \subseteq \mathbb{S}$ be open. We say that $f : G \to \mathbb{S}$ is a *conformal map* if f is one-to-one meromorphic on G.

Remark 8.8. For meromorphic functions we always assume that $f = \infty$ on the set P_f of its poles.

Definition 8.9. Let $\Omega, G \subseteq \mathbb{S}$ be open. We say that G and Ω are *conformally equivalent* (we write $G \sim \Omega$) if there is a conformal map $f : G \overset{onto}{\to} \Omega$. This relation is an equivalence (note that conformal mapping onto is a meromorphic bijection).

Example 8.10. As HW, show that there are just 4 classes of conformally equivalent simply connected domains in $\mathbb{S}$, namely

(i) $\emptyset$,

(ii) $\mathbb{S}$,

(iii) $[\mathbb{C}] = \{\mathbb{S} \setminus \{z_0\} \colon z_0 \in \mathbb{S}\}$,

(iv) $[\mathbb{D}]$ consists of the rest (Riemann's theorem).

Proof. No two of these sets are conformally equivalent. This is clear for the empty set. $\mathbb{S}$ is compact and conformally equivalent sets are homeomorphic. By Liouville theorem $\mathbb{C} \not\sim \mathbb{D}$. To show $\mathbb{S} \setminus \{z_0\} \in [\mathbb{C}]$ use the transformation $z \mapsto \frac{1}{z - z_0}$. □

Remark 8.11. Let $G, \Omega \subseteq \mathbb{C}$ be open. Then a one-to-one map $f : G \overset{onto}{\to} \Omega$ is

- conformal if and only if f and f^{-1} are both holomorphic;
- diffeomorphism if and only if f and f^{-1} are continuously differentiable;
- homeomorphism if and only if f and f^{-1} are both continuous.

We know conformal $\implies$ diffeomorphism $\implies$ homeomorphism.

9 Preservation of angles

Definition 9.1. For $z \in \mathbb{C} \setminus \{0\}$ put $A(z) = \frac{z}{|z|}$.

Definition 9.2. Let $G \subseteq \mathbb{C}$ be open, $f : G \to \mathbb{C}$, $z_0 \in G$ and $P(z_0) \subseteq G$ be such that $\forall z \in P(z_0) \colon f(z) \neq f(z_0)$. Then we say that f *preserves angles (and orientation) at* z_0 if

$$\forall \theta \in \mathbb{R} \colon (\text{♖}) :- \lim_{r \to 0+} e^{-i\theta} A\left(f(z_0 + re^{i\theta}) - f(z_0)\right) \in \mathbb{C} \text{ exists and is independent of } \theta.$$

Example 9.3. $f(z) = \overline{z}$ (reflection along the real axis) fails the previous condition, but preserves angles. Hence the note in the definition regarding orientation.

Example 9.4. Let f be a non-constant holomorphic function in a neighbourhood $U(z_0)$ of the point z_0. Then there is $p \in \mathbb{N}$ such that

$$f(z) = f(z_0) + a_p(z - z_0)^p + \cdots, z \in U(z_0), \text{ with } a_p = \frac{f^{(p)}(z_0)}{p!} \neq 0.$$

Then the limit from the definition can be calculated as follows:

$$(\text{♖}) = \lim_{r \to 0+} e^{-i\theta} \frac{a_p r^p e^{ip\theta} + \cdots}{|a_p r^p e^{ip\theta} + \cdots|} = \frac{a_p}{|a_p|} e^{i\theta(p-1)}, \quad \theta \in \mathbb{R}.$$

Conclusion: Such f preserves angles at z_0 iff $f'(z_0) \neq 0$; in which case the limit is equal to $f'(z_0)/|f'(z_0)|$.

Notation 9.5. Let $f : \mathbb{R}^2(\mathbb{C}) \to \mathbb{R}^2(\mathbb{C})$ have the total differential $df(z_0)$ at $z_0 \in \mathbb{R}^2 = \mathbb{C}$, i.e.

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0) - df(z_0)h}{|h|} = 0.$$

Then $df(z_0)h = \frac{\partial f}{\partial x}(z_0)h_1 + \frac{\partial f}{\partial y}(z_0)h_2$, $h = (h_1, h_2) = h_1 + ih_2 \in \mathbb{R}^2 = \mathbb{C}$. We have $h_1 = (h + \overline{h})/2, h_2 = (h - \overline{h})/2i$ and

$$df(z_0)h = \partial f(z_0)h + \overline{\partial} f(z_0)\overline{h}, \text{ where } \begin{aligned} \partial f(z_0) &= \tfrac{1}{2}\left(\tfrac{\partial f}{\partial x}(z_0) - i\tfrac{\partial f}{\partial y}(z_0)\right), \\ \overline{\partial} f(z_0) &= \tfrac{1}{2}\left(\tfrac{\partial f}{\partial x}(z_0) + i\tfrac{\partial f}{\partial y}(z_0)\right). \end{aligned}$$

The Cauchy-Riemann theorem states that $f'(z_0)$ exists iff $df(z_0)$ exists and $\overline{\partial} f(z_0) = 0$; in this case $f'(z_0) = \partial f(z_0)$.

Example 9.6. Let f have the total differential $df(z_0)$ at $z_0 \in \mathbb{C}$ and $df(z_0) \neq 0$. We have $df(z_0)h = \alpha h + \beta \overline{h}, h \in \mathbb{C}$ with $\alpha = \partial f(z_0) \neq 0$ or $\beta = \overline{\partial} f(z_0) \neq 0$. Then for $\theta \in \mathbb{R}$ we have

$$(♖) = \lim_{r \to 0+} e^{-i\theta} \frac{df(z_0)(re^{i\theta}) + o(r)}{|df(z_0)(re^{i\theta}) + o(r)|} = e^{-i\theta} \frac{df(z_0)(e^{i\theta})}{|df(z_0)(e^{i\theta})|} = \frac{\alpha + \beta e^{2i\theta}}{|\alpha + \beta e^{2i\theta}|}$$

if $df(z_0)(e^{i\theta}) \neq 0$. There are three possible cases:

(i) $\alpha = 0, \beta \neq 0$: (♖) depends on θ,

(ii) $\alpha \neq 0, \beta = 0$: (♖) $= \frac{\alpha}{|\alpha|}$,

(iii) $\alpha \neq 0, \beta \neq 0$: (♖) depends on θ (HW).

Conclusion: Such f preserves angles at z_0 iff $f'(z_0)$ exists and $f'(z_0) \neq 0$.

By Examples 9.4 and 9.6 we get

Theorem 9.7. *Let $G, \Omega \subseteq \mathbb{C}$ be open. Then $f : G \overset{onto}{\to} \Omega$ is conformal if and only if f is a diffeomorphism of G onto Ω preserving angles at any point of G.*

Example 9.8. $f(z) = z\,|z|$ preserves angles at 0, $f'(0) = 0 = df(0)$, but f is not holomorphic at any neighbourhood of 0.

9.1 Examples of conformal mappings

Example 9.9. $f : \mathbb{S} \overset{onto}{\to} \mathbb{S}$ is conformal iff

$$f(z) = \frac{az+b}{cz+d} \text{ for some } a, b, c, d \text{ with } ad - bc \neq 0.$$

Note that $ad - bc = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$ implies f is constant or not well-defined.

⌜ $\Longleftarrow$ is obvious, $\Longrightarrow$: Since $f \in \mathcal{M}(\mathbb{S})$, $f = \frac{P}{Q}$ rational. Since f is one-to-one, f has just one simple zero point and one simple pole. ⌟

Example 9.10. By Theorem 8.5 we know that $f : \mathbb{D} \overset{onto}{\to} \mathbb{D}$ is conformal iff there are $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that $f(z) = e^{i\theta} \frac{z - \alpha}{1 - \overline{\alpha} z}, z \in \mathbb{D}$.

Example 9.11. $f : \mathbb{C} \overset{onto}{\to} \mathbb{C}$ is conformal iff there are $a, b \in \mathbb{C}, a \neq 0$ such that $f(z) = az + b, z \in \mathbb{C}$.

⌜ $\Longleftarrow$ ✓, $\Longrightarrow$: At ∞ f has an isolated singularity which is not essential. Actually, $\lim_{z \to \infty} f(z) = \infty$. Hence $f : \mathbb{S} \overset{onto}{\to} \mathbb{S}$ and $f(\infty) = \infty$. ⌟

10 Linear fractional transformations

Definition 10.1. We say that $f : \mathbb{S} \to \mathbb{S}$ is a linear fractional transformation if

$$f(z) = \frac{az+b}{cz+d} \text{ for some } a, b, c, d \text{ with } ad - bc \neq 0. \tag{LFT}$$

Proposition 10.2. *The set $\mathcal{M}_2$ of all LFT's endowed with composition forms a group, called the* Möbius[18] *group. The Möbius group is generated by transformations of the following types:*

- *translation: $z \mapsto z + b, b \in \mathbb{C}$,*
- *homotheties: $z \mapsto rz, r > 0$,*
- *rotation: $z \mapsto az, |a| = 1$,*
- *inversions: $z \mapsto \frac{1}{z}$.*

Proof. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2 \times 2}, \det A \neq 0$ put $f_A(z) = \frac{az+b}{cz-d}$. Then $f_E = Id, f_A \cdot f_{A'} = f_{AA'}, f_A^{-1} = f_{A^{-1}}$.

If $c = 0$, the second claim is obvious. If $c \neq 0$, then

$$f(z) = \frac{az+b}{cz-d} = \frac{\frac{a}{c}cz + \frac{a}{c}d - \frac{a}{c}d + b}{cz+d} = \frac{a}{c} + \frac{\lambda}{cz+d}, \text{ with } \lambda = \frac{bc - ad}{c}.$$

□

[18]August Ferdinand Möbius (17 November 1790, Schulpforta, Electorate of Saxony – 26 September 1868, Leipzig, German Confederation)

Definition 10.3. A *generalized circle* in $\mathbb{S} = \mathbb{C} \cup \{\infty\}$ is either a circle in $\mathbb{C}$ or a straight line (including the endpoint ∞). Denote by $\mathcal{F}$ the family of all generalized circles in $\mathbb{S}$.

Proposition 10.4. *Every LFT preserves $\mathcal{F}$. Via stereographic projection, $\mathcal{F}$ corresponds to the family of all circles on the unit sphere S^2 in $\mathbb{R}^3$.*

Proof. We will show that every generalized circle is given by an equation of the form $\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0$ for some $\alpha, \gamma \in \mathbb{R}, \beta \in \mathbb{C}$. First, consider a circle given by $|z - z_0| = r$ for some $z_0 \in \mathbb{C}, r \in [0, \infty)$. Then

$$\begin{aligned} |z - z_0| &= r \\ (z - z_0)\overline{(z - z_0)} &= r^2 \\ z\overline{z} - z_0\overline{z} - z\overline{z_0} + z_0\overline{z_0} - r^2 &= 0. \end{aligned}$$

Thus we can take $\alpha = 1, \beta = -\overline{z_0}, \gamma = |z_0|^2 - r^2$. Second, consider a line given by $2a\Re z + 2b\Im z + c = 0$ for some $a, b, c \in \mathbb{R}$. Then adding and subtracting $ai\Im z$ and $bi\Re z$ gives

$$\begin{aligned} a\Re z + b\Im z + ia\Im z - ib\Re z + a\Re z + b\Im z - ia\Im z + ib\Re z + c &= 0 \\ (a - ib)(\Re z + i\Im z) + (a + ib)(\Re z - i\Im z) + c &= 0 \\ (a - ib)z + \overline{(a - ib)}\overline{z} + c &= 0. \end{aligned}$$

Thus we can take $\alpha = 0, \beta = a - ib, \gamma = c$.

Replacing z with $\frac{1}{z}$ yields an equation of the same type. □

Proposition 10.5. *Let (z_1, z_2, z_3) and (w_1, w_2, w_3) be two triples of different points of $\mathbb{S}$. Then there is a unique LFT f with $f(z_j) = w_j, 1 \leq j \leq 3$.*

Proof. If $(z_1, z_2, z_3) \in \mathbb{C}^3$, then $\varphi_{z_1,z_2,z_3}(z) = \frac{z-z_1}{z-z_3}\frac{z_2-z_3}{z_2-z_1}$ maps (z_1, z_2, z_3) to $(0, 1, \infty)$. Otherwise without loss of generality $z_3 = \infty$ and we can define $\varphi_{z_1,z_2,z_3}(z) = \frac{z-z_1}{z_2-z_1}$. Then $f :- \varphi^{-1}_{w_1,w_2,w_3} \circ \varphi_{z_1,z_2,z_3}$ is the desired LFT. □

Conclusion: Every generalized circle can be mapped onto every generalized circle by an LFT. In particular, every circle can be mapped onto every straight line. Also, every open disc can be mapped onto any open half-space.

Example 10.6. Let $H = \{z \in \mathbb{C} \colon \Im z > 0\}$. $f(z) = \frac{z-i}{z+i}$ is a conformal mapping of the upper half-plane onto the unit disc.

Example 10.7. Let $\Omega = \mathbb{D} \setminus [0, 1]$. By Riemann's theorem, there is a conformal map $h : \Omega \overset{onto}{\to} \mathbb{D}$. Find such an h.

Set, for example, $h_1 : \mathbb{D} \setminus [0, 1] \to \{\Re z < -1/2\}, h_2(z) = \frac{1}{z-1}$, $h_2(z) = -(z + \frac{1}{2})$, $h_3(z) = z^2$, $h_4(z) = \frac{1}{z}$, $h_5(z) = \sqrt{z - 4}$, $h_6(z) = \frac{z-1}{z+1}$ and finally put $h = h_6 \circ h_5 \circ h_4 \circ h_3 \circ h_2 \circ h_1$.

11 Harmonic functions

We study $f : \mathbb{C} \to \mathbb{C}$. Since $\mathbb{C} \simeq \mathbb{R}^2$ we have $z = x + iy$, $x = \Re z, y = \Im z$ and $f = u + iv$ with $u = \Re f, v = \Im f$.

Observation: If $G \subseteq \mathbb{C}$ is a domain, $f, g \in \mathcal{H}(G)$ and $\Re f = \Re g$ on G, then there is $c \in \mathbb{R}$ such that $\Im f = \Im g + c$ on G.

⌜ It follows from the Cauchy-Riemann conditions: $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. ⌟

Question 11.1. What are the real parts of holomorphic functions?

Recall Notation 9.5.

Lemma 11.2. *If $f \in C^2(G)$, $G \subseteq \mathbb{C}$ open, then $\partial\overline{\partial}f = \overline{\partial}\partial f = \frac{1}{4}\Delta f$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.*

Proof.

$$\partial\overline{\partial}f = \frac{1}{4}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f = \frac{1}{4}\Delta f.$$

□

Definition 11.3. If $G \subseteq \mathbb{C}$ is open, we say that $u \in C^2(G)$ is *harmonic* if $\Delta u = 0$ on G.

Example 11.4. If $f \in \mathcal{H}(G)$, then $\Re f$ and $\Im f$ are harmonic on G.
⌜
$$0 = \partial\overline{\partial} f = \frac{1}{4}\Delta f = \frac{1}{4}\left(\Delta(\Re f) + i\Delta(\Im f)\right) \implies \Delta(\Re f) = 0 = \Delta(\Im f).$$
⌟

Example 11.5. If $G \subseteq \mathbb{C}$ is a simply connected domain, $f \in \mathcal{H}(G)$ and $f \neq 0$ on G, then $\log|f| = \Re F$ for some $F \in \mathcal{H}(G)$. In particular $\log|f|$ is harmonic by the previous example.
⌜ We know there is a holomorphic branch F of $\log f$, i.e. F is holomorphic and $f = e^F$. But $|f| = e^{\Re F}$. ⌟

Corollary 11.6. *If $G \subseteq \mathbb{C}$ is open, $f \in \mathcal{H}(G), f \neq 0$, then $\log|f|$ is harmonic.*

Proof. By Example 11.5 $\log|f|$ is harmonic on any open ball U in G. Since "being harmonic" is a local property, this proves the statement. □

Example 11.7. $f(z) = \log|z|, z \in \mathbb{C}\setminus\{0\}$ is harmonic on $\mathbb{C}\setminus\{0\}$, but f is not the real part of any holomorphic function on $\mathbb{C}\setminus\{0\}$.
⌜ Assume $F \in \mathcal{H}(\mathbb{C}\setminus\{0\})$ and $\Re F = f$. On $\mathbb{C}\setminus(-\infty, 0]$, $\Re F = \Re(\log)$ implies $\Im F = \Im(\log) + c$ for some $c \in \mathbb{R}$. So we have $F = \log + ic$ on $\mathbb{C}\setminus(-\infty, 0]$, but unlike F, $\log + ic$ is not continuous at, say, $z = -1$. ⌟

Theorem 11.8. *If $G \subseteq \mathbb{C}$ is a simply connected domain and $u : G \to \mathbb{R}$ is harmonic, then there is $f \in \mathcal{H}(G)$ such that $\Re f = u$.*

Remark 11.9. (i) Every harmonic function is locally (but not necessarily globally) the real part of some holomorphic function.

(ii) If $f \in \mathcal{H}(G)$, then $f' = \partial f = \partial(f + \overline{f})$, because $\partial\overline{f} = \overline{(\overline{\partial} f)} = 0$. Hence $f' = \partial(f + \overline{f}) = 2\partial(\Re f)$.

Proof. We have $\partial u \in \mathcal{H}(G)$, because $\overline{\partial}(\partial u) = \frac{1}{4}\Delta u = 0$. Then there is $f_0 \in \mathcal{H}(G)$ such that $f_0' = 2\partial u$. Using the second remark we get $2\partial(\Re f_0) = f_0' = 2\partial u$, rearranging the terms gives $\partial(\Re f_0 - u) = 0$, but $\Re f_0 - u$ is real and so $\frac{\partial}{\partial x}(\Re f_0 - u) = 0$ and $\frac{\partial}{\partial y}(\Re f_0 - u) = 0$ on the domain G. Hence $u = \Re f_0 + c$ for some $c \in \mathbb{R}$. Put $f = f_0 + c$. □

Corollary 11.10. *Let $G \subseteq \mathbb{C}$ be open and $u : G \to \mathbb{R}$ be harmonic. Then $u \in C^\infty(G)$ and u satisfies the* mean value property*:*
$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,\mathrm{dt} = u(z_0) \tag{MV}$$
whenever $\overline{U(z_0, r)} \subseteq G$.

Proof. Let $\overline{U(z_0, r)} \subseteq G$. Take $R \in (r, \infty)$ such that $U(z_0, R) \subseteq G$. Then $u = \Re F$ for some $F \in \mathcal{H}(U(z_0, R))$. So $u \in C^\infty(U(z_0, R))$ and by the Cauchy integral formula,
$$F(z_0) = \frac{1}{2\pi i}\int_\varphi \frac{F(z)}{z - z_0}\,\mathrm{dz} = \frac{1}{2\pi i}\int_0^{2\pi} F(z_0 + re^{it})\frac{ire^{it}}{re^{it}}\,\mathrm{dt} = \frac{1}{2\pi}\int_0^{2\pi} F(z_0 + re^{it})\,\mathrm{dt},$$
where $\varphi(t) = z_0 + re^{it}, t \in [0, 2\pi]$, which implies (MV). □

Theorem 11.11 (The maximum principle)**.** *Let $G \subseteq \mathbb{C}$ be a domain and $u : G \to \mathbb{R}$ be a continuous function satisfying the mean value property. If u is not constant, then u does not attain an extremum in G.*

Proof. Let us assume that $z_0 \in G$ and, say, $u(z_0) \geq u$ on G. Put $M = \{z \in G : u(z) = u(z_0)\}$. Obviously, $\emptyset \neq M$ is closed in G. If we show that M is open, then $M = G$.

Let $z_1 \in M$ and $U(z_1, r) \subseteq G$. We show that $U(z_1, r) \subseteq M$. Assume there is $z_2 \in U(z_1, r) \setminus M$. By (MV), $u(z_0) = u(z_1) = \frac{1}{2\pi}\int_0^{2\pi} u(z_1 + \rho e^{it})\,\mathrm{dt} \leq u(z_0)$, where $\rho = |z_1 - z_2|$. If we show that the last inequality is strict, we will arrive to a contradiction. It is in fact the case, because $u \leq u(z_0)$ on G and $u < u(z_0)$ in a neighbourhood of z_2. □

Corollary 11.12. *Let $G \subseteq \mathbb{C}$ be bounded and open, $u \in C(\overline{G})$ and u be harmonic on G. Then u attains its extrema on ∂G, i.e. $\min_{\partial G} \leq u \leq \max_{\partial G} u$ on G. The assumption of harmonicity can be replaced with the (MV) property.*

Proof. Let $z_0 \in G$ and $u(z_0) = \max_{\overline{G}} u$. Then u is constant on the component G_0 of G containing z_0. So u attains the maximum on the boundary. □

11.1 The Poisson Integral

Let $u : \overline{\mathbb{D}} \to \mathbb{R}$ be harmonic, i.e. u is harmonic on $U(0,r)$ fo some $r \in (1,\infty)$. Then there is $f \in \mathcal{H}(\overline{\mathbb{D}})$ such that $u = \Re f$ on $\overline{\mathbb{D}}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n, |z| \le 1$. For $|z| \le 1, z = re^{i\theta}$, we have

$$u(z) = \Re f(z) = \Re a_0 + \sum_{n=1}^{\infty} \frac{1}{2}\left(a_n r^n e^{in\theta} + \overline{a_n} r^n e^{-in\theta}\right).$$

Hence

$$u(z) = \sum_{n=-\infty}^{\infty} b_n r^{|n|} e^{in\theta}, \text{ where } b_n = \begin{cases} \Re a_0, & n = 0 \\ \frac{1}{2} a_n, & n > 0 \\ \frac{1}{2}\overline{a_n}, & n < 0 \end{cases}. \tag{1}$$

In addition, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{it}) e^{-imt}\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty} b_n r^{|n|} e^{int}\right) e^{-imt}\,\mathrm{d}t = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} b_n \int_{-\pi}^{\pi} e^{i(n-m)t}\,\mathrm{d}t = b_m. \tag{2}$$

Putting (2) into (1) we get for $z = re^{i\theta}$ with $r \in [0,1)$ that

$$u(z) = \sum_{n=-\infty}^{\infty} r^{|n|} \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{it}) e^{in(\theta-t)}\,\mathrm{d}t \overset{r \le 1}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)}\right) u(e^{it})\,\mathrm{d}t.$$

Definition 11.13. We define

(i) *the Poisson*[19] *kernel*: For $0 \le r < 1, \theta \in \mathbb{R}$ put $P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$;

(ii) *the Poisson integral*:

$$[Pu](re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) u(e^{it})\,\mathrm{d}t \tag{PI}$$

for $0 \le r < 1, \theta \in \mathbb{R}$.

Fact 11.14.

$$P_r(\theta) = \Re\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right) = \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}.$$

Proof.

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \implies 2\Re\left(\frac{1}{1-z}\right) = 2 + \sum_{n=1}^{\infty}\left(r^n e^{in\theta} + r^n e^{-in\theta}\right) = 1 + P_r(\theta)$$

and so

$$P_r(\theta) = \Re\left(\frac{2}{1-z}\right) - 1 = \Re\left(\frac{1+z}{1-z}\right) = \Re\left(\frac{(1+z)(1-\overline{z})}{(1-z)(1-\overline{z})}\right) = \Re\left(\frac{1 - \overline{z} + z - z\overline{z}}{1 - z - \overline{z} + z\overline{z}}\right) = \Re\left(\frac{1 - |z|^2 + 2i\Im z}{1 - 2\Re z + |z|^2}\right).$$

∎

Theorem 11.15 (Poisson formula)**.** *If u is harmonic on $\overline{\mathbb{D}}$, then $u = Pu$ on $\mathbb{D}$.*

Remark 11.16. (i) The Poisson formula is an analogue of the Cauchy integral formula.

(ii) $P1 = 1$.

Theorem 11.17 (Properties of Poisson integral)**.** *Let $g \in L^1(\mathbb{T})$.*

(i) Then Pg is harmonic on $\mathbb{D}$.

(ii) If g is continuous at $z_0 \in \mathbb{T}$, then $\lim_{z \to z_0, z \in \mathbb{D}} Pg(z) = g(z_0)$. In particular, if $g \in C(\mathbb{T})$ and we define a function u as Pg in $\mathbb{D}$ and as g in $\mathbb{T}$, then u is continuous on $\overline{\mathbb{D}}$ and harmonic in $\mathbb{D}$.

(iii) For a.e. $\theta \in \mathbb{R}$ holds $\lim_{r \to 1-} Pg(re^{i\theta}) = g(e^{i\theta})$ (Fatou).

[19] Siméon Denis Poisson (21 June 1781, Pithiviers, Kingdom of France – 25 April 1840, Sceaux, Hauts-de-Seine, Kingdom of France)

Proof. Without loss of generality assume g is real-valued.

(i): For $|z| < 1, z = re^{i\theta}$ we have

$$Pg(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Re \left(\frac{1 + re^{i(\theta - t)}}{1 - re^{i(\theta - t)}} \right) g(e^{it}) \, \mathrm{dt} = \Re \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + re^{i(\theta - t)}}{1 - re^{i(\theta - t)}} g(e^{it}) \, \mathrm{dt} \right) =: \Re f.$$

We have $f \in \mathcal{H}(\mathbb{D})$ and $Pg = \Re f$, so Pg is harmonic.

(ii): Without loss of generality assume that $g(z_0) = 0$ (otherwise pass to $g - g(z_0)$ and use that $P1 = 1$). Let $\varepsilon > 0$ be given and $z_0 = e^{i\theta_0}$. Take $\delta_0 \in (0, \pi)$ such that $\forall t \in (\theta_0 - \delta_0, \theta_0 + \delta_0) : \left| g(e^{it}) \right| < \varepsilon$. Let $z \in \mathbb{D}, z = re^{it}$ and $|\theta - \theta_0| < \delta/2$. Then

$$Pg(z) = \frac{1}{2\pi} \int_{\theta_0 - \delta_0}^{\theta_0 + \delta_0} P_r(\theta - t) g(e^{it}) \, \mathrm{dt} + \frac{1}{2\pi} \int_A P_r(\theta - t) g(e^{it}) \, \mathrm{dt} =: I_1 + I_2,$$

where $A = (-\pi, \pi) \setminus (\theta_0 - \delta_0, \theta_0 + \delta_0)$. Of course, $P_r > 0$ and $P1 = 1$, hence $|I_1| < \varepsilon$.

Next we have

$$|I_2| \leq \frac{1}{2\pi} \int_A P_r(\theta - t) \left| g(e^{it}) \right| \mathrm{dt} \leq P_r \left(\frac{\delta}{2} \right) \frac{1}{2\pi} \left\| g \right\|_1 \stackrel{r \to 1-}{\to} 0.$$

since for $t \in A$ holds $\frac{\delta}{2} \leq |\theta - t| \leq \frac{3\pi}{2}$ $(-\frac{\delta}{2} < \theta_0 - \theta < \frac{\delta}{2}$ and $\delta < t - \theta_0 < \pi)$ and thus

$$0 < P_r(\theta - t) = \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)} \leq P_r \left(\frac{\delta}{2} \right).$$

Take $r_0 \in (0, 1)$ such that $|I_2| < \varepsilon$ for $r \in (r_0, 1)$. Then for all $z = re^{i\theta}$ with $|\theta - \theta_0| < \delta/2$ and $r \in (r_0, 1)$ we have $|Pg| \leq |I_1| + |I_2| < 2\varepsilon$.

(iii): Left without proof. □

11.2 The Dirichlet problem

Let $G \subseteq \mathbb{C}$ be open and bounded. Let $g \in C(\partial G)$. The Dirichlet problem (DP) is to find $u \in C(\overline{G})$ such that u is harmonic on G and $u = g$ on ∂G.

Remark 11.18.

(i) The DP corresponds to many problems in physics, e.g. a charge distribution g on ∂G gives an electric potential u on G.

(ii) By the maximum principle (DP) has at most one solution. (Given two solutions u_1, u_2, consider $v = u_1 - u_2$, which is harmonic in G and zero on the boundary).

(iii) The Dirichlet problem does not always have a (classical) solution (e.g. for $G = \mathbb{D} \setminus \{0\}$, $u = 0$ on $\mathbb{T}$ and $u(0) = 1$ - Zaremba's example).

(iv) (DP) has a unique solution on "nice" domains, e.g. Lipschitz domains.

(v) In particular, (DP) has a unique classical solution on any open disc in $\mathbb{C}$. For $\mathbb{D}$ this follows directly from the previous theorem. For a general disc use a transformation of the form $z \mapsto rz + b$ for $r > 0, b \in \mathbb{C}$.

Theorem 11.19. *Let $G \subseteq \mathbb{C}$ be open and $u : G \to \mathbb{R}$. Then u is harmonic on G if and only if u is continuous on G and satisfies (MV) from Corollary 11.10.*

Proof. $\implies$: We know this.

$\impliedby$: Let $u \in C(G)$ satisfy (MV) on G. Let $U :- U(z_0, r)$ with $\overline{U} \subseteq G$. Let h be the unique solution of (DP) on U with the boundary data given by $u|_{\partial U}$. Then $v :- u - h$ is continuous on $\overline{U}$, satisfies (MV) on U and $v = 0$ on ∂U. By the maximal principle, we get that $v \equiv 0$ on U, so $u = h$ on U, so u is harmonic on U. □