

The Gauss–Ostrogradsky Theorem

Flux, divergence, and proofs on elementary domains

Geometry 2 – 120-minute exercise class

Companion to *Geometry 2, Lectures 1–4*

Teaching plan (120 min). Normal vectors and boundaries (25 min); flux, divergence and theorem (20 min); concrete computations (20 min); *proof for the cube* (15 min); *proof for i -special sets* (20 min); *proof for the ball* (20 min). Material marked [\[optional / independent study\]](#) may be left for further study.

1 Bodies and outward normals

25 min

Definition 1.1 (Body with an almost smooth boundary). For $n \geq 2$, a set $M \subset \mathbb{R}^n$ is a *body with an almost smooth boundary* if it is bounded,

$$M = \overline{\text{int } M},$$

and ∂M is a generalized $(n - 1)$ -surface: a finite union of simple $(n - 1)$ -surfaces together with finitely many lower-dimensional surface pieces, subject to the separation condition from the lectures. The points in the $(n - 1)$ -dimensional components are called *regular boundary points*. We denote the surface measure on ∂M by $\lambda_{\partial M}$ and its integral by $\int_{\partial M} \cdots dS$.

We always assume $\lambda_{\partial M}(\partial M) < \infty$ when applying the divergence theorem. Lower-dimensional boundary pieces have zero $(n - 1)$ -dimensional surface measure. For $n = 1$, the analogues are compact intervals, with endpoint counting measure.

Example 1.2 (Regular and singular points). The sphere $\overline{\partial B(0, R)} = S_R^{n-1}$ is smooth. For the cube $[0, 1]^n$, the relative interiors of the $(n - 1)$ -dimensional faces are regular; edges, corners, and higher-codimension faces are singular. Their surface measure is zero.

Definition 1.3 (Outward and inward unit normals). Let $M \subset \mathbb{R}^n$ be a body with an almost smooth boundary, and let $x \in \partial M$ be a regular point. A unit vector $\nu \in (T_x \partial M)^\perp$ is called *outward* if there is $\delta > 0$ such that

$$x + t\nu \notin M \quad \text{for every } 0 < t < \delta.$$

It is called *inward* if there is $\delta > 0$ such that

$$x + t\nu \in \text{int } M \quad \text{for every } 0 < t < \delta.$$

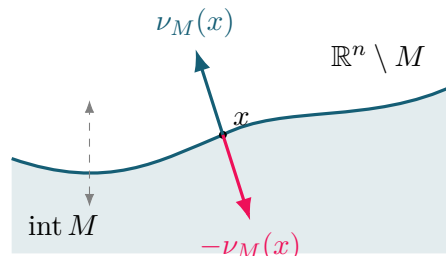


Figure 1: The normal points toward the exterior, regardless of how the boundary is parametrized.

Lemma 1.4 (Existence and uniqueness of the outward unit normal). *At every regular point x of the boundary of a body M with an almost smooth boundary, there exists exactly one outward unit normal $\nu_M(x)$ and exactly one inward unit normal. They are opposite vectors.*

Proof. [optional / independent study]

By regularity, x lies on a simple $(n-1)$ -surface component S_i of ∂M . The separation condition in the definition of a generalized surface, and the finiteness of the decomposition, yield a neighborhood V_0 of x such that $\partial M \cap V_0 = S_i \cap V_0$. By the local graph representation theorem, after permuting coordinates and shrinking the neighborhood if necessary, there are an open neighborhood V of x , an open set $U \subset \mathbb{R}^{n-1}$, and a C^1 function $h : U \rightarrow \mathbb{R}$ for which

$$\partial M \cap V = \{(u, h(u)) : u \in U\} \cap V, \quad x = (u_0, h(u_0)).$$

Choose a ball $B \subset U$ about u_0 and $\varepsilon > 0$ so small that

$$W = \{(u, s) : u \in B, |s - h(u)| < \varepsilon\} \subset V.$$

The sets

$$W^+ = \{(u, s) \in W : s > h(u)\}, \quad W^- = \{(u, s) \in W : s < h(u)\}$$

are connected and open, and neither meets ∂M . Because M is closed,

$$\mathbb{R}^n \setminus \partial M = \text{int } M \cup (\mathbb{R}^n \setminus M)$$

is a disjoint union of open sets. Thus each of W^+ and W^- lies entirely either in $\text{int } M$ or in $\mathbb{R}^n \setminus M$. They cannot both be exterior: otherwise $W \cap \text{int } M = \emptyset$, contrary to $x \in M = \overline{\text{int } M}$. Nor can they both be interior: as $W^+ \cup W^-$ is dense in W and M is closed, this would imply $W \subset M$ and hence $x \in \text{int } M$, a contradiction. Consequently, exactly one of W^+ and W^- is interior and the other is exterior.

Put $f(u, s) = s - h(u)$, so that $f(x) = 0$ and $\nabla f(x) = (-\nabla h(u_0), 1) \neq 0$. The tangent space of the graph is the orthogonal complement of $\nabla f(x)$; hence its only two unit normal vectors are

$$\nu_+ = \frac{\nabla f(x)}{\|\nabla f(x)\|}, \quad \nu_- = -\frac{\nabla f(x)}{\|\nabla f(x)\|}.$$

By differentiability of f , as $t \downarrow 0$,

$$f(x + t\nu_{\pm}) = f(x) + t\langle \nabla f(x), \nu_{\pm} \rangle + o(t) = \pm t\|\nabla f(x)\| + o(t).$$

For all sufficiently small $t > 0$, both points $x + t\nu_{\pm}$ belong to W , with $x + t\nu_+ \in W^+$ and $x + t\nu_- \in W^-$. Since these sets are respectively the interior and exterior sides (in one order or the other), one of ν_+, ν_- satisfies the outward condition in the definition, and the other satisfies the inward condition. They are the only unit normal vectors, proving existence and uniqueness. $\square$

Remark 1.5 (How to compute a normal). For a smooth level surface $g(x) = 0$ with $\nabla g(x) \neq 0$, the two unit normals are $\pm \nabla g / \|\nabla g\|$. Choose the sign using the local interior side. In particular, for $M = \{\|x\| \leq R\}$, $\nu_M(x) = x/R$ on ∂M ; on the face $\{x_i = 1\}$ of the cube, $\nu_M = e_i$, and on $\{x_i = 0\}$, $\nu_M = -e_i$.

2 Flux, divergence and the theorem

20 min

Definition 2.1 (Flux). For a continuous vector field F on the boundary of a body as above, its *outward flux* is

$$\Phi_{\partial M}(F) = \int_{\partial M} \langle F(x), \nu_M(x) \rangle dS. \quad (2.1)$$

The scalar product $\langle F, \nu_M \rangle$ extracts the component of F perpendicular to the boundary. It is positive for outward flow, negative for inward flow, and zero for tangential flow. The flux is the *signed total* of these contributions. Values of the normal on the singular subset do not affect the integral.

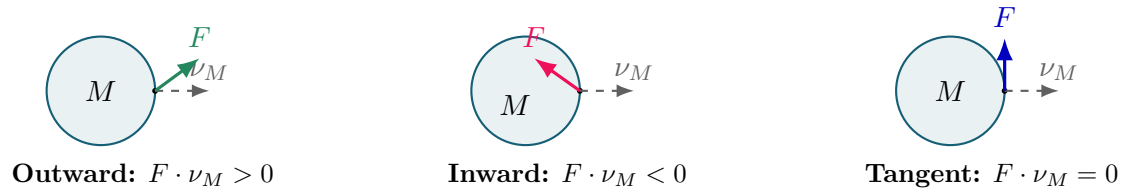


Figure 2: The sign of the local flux depends on the direction of F relative to the outward normal.

Definition 2.2 (Divergence). For $F = (F_1, \dots, F_n)$ of class C^1 on an open subset of $\mathbb{R}^n$, its *divergence* is

$$\operatorname{div} F = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}. \quad (2.2)$$

Example 2.3 (Three elementary fields on $\mathbb{R}^2$). The fields $F_+(x) = x$, $F_-(x) = -x$, and $F_0(x_1, x_2) = (-x_2, x_1)$ satisfy

$$\operatorname{div} F_+ = 2, \quad \operatorname{div} F_- = -2, \quad \operatorname{div} F_0 = 0.$$

The first points radially outwards, the second radially inwards, while the third rotates tangentially to concentric circles. This contrasts *local net outflow* with motion along streamlines.

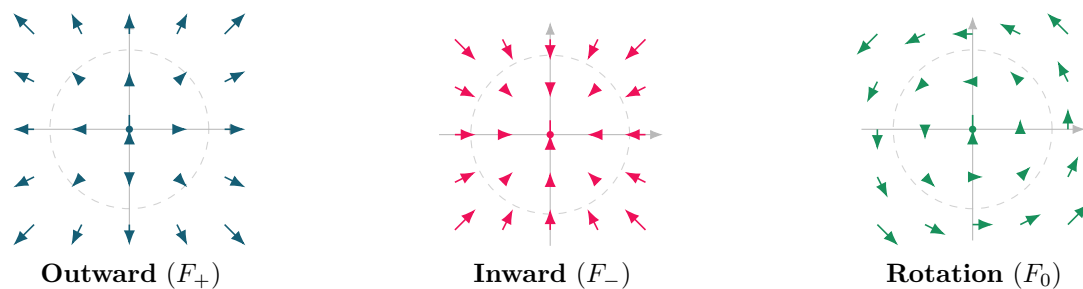


Figure 3: Vector fields with positive, negative, and zero divergence. The arrows have a common scale; the dashed circle highlights their radial or tangential directions.

Remark 2.4 (Local interpretation of divergence). In words, $\operatorname{div} F(x)$ measures the *net outward flux per unit volume* near x . Positive divergence means local net outflow (a source), negative divergence means local net inflow (a sink), and zero divergence means no net local outflow—not necessarily that the vector field vanishes.

[optional / independent study]

More precisely, for a C^1 field near x , applying the divergence theorem to small balls $\overline{B(x, r)}$ gives

$$\operatorname{div} F(x) = \lim_{r \downarrow 0} \frac{1}{\lambda^n(B(x, r))} \int_{\partial B(x, r)} \langle F, \nu \rangle dS.$$

Indeed, the ratio equals the average of $\operatorname{div} F$ over the ball, and continuity makes this average converge to $\operatorname{div} F(x)$.

Theorem 2.5 (Gauss–Ostrogradsky; coordinate form). *Let $M \subset \mathbb{R}^n$ be a body with an almost smooth boundary and $\lambda_{\partial M}(\partial M) < \infty$. If f is of class C^1 on a neighborhood of M , then, for $i = 1, \dots, n$,*

$$\boxed{\int_M \frac{\partial f}{\partial x_i} d\lambda^n = \int_{\partial M} f \nu_M^i dS.} \quad (2.3)$$

Here ν_M^i is the i th coordinate of the outward unit normal, defined on the regular part of ∂M .

Corollary 2.6 (Divergence theorem). *For any C^1 vector field $F = (F_1, \dots, F_n)$ on a neighborhood of M ,*

$$\boxed{\int_M \operatorname{div} F d\lambda^n = \int_{\partial M} \langle F, \nu_M \rangle dS.} \quad (2.4)$$

Proof. Apply (2.3) to $f = F_i$ for each i and sum over $i = 1, \dots, n$. □

Key point. Formula (2.4) equates the total outward flux through the boundary with the integral of the local divergence inside the body. It is a multidimensional analogue of $\int_a^b f' = f(b) - f(a)$.

3 Concrete calculations

20 min

In each example, compute the volume integral and the boundary flux *independently*, then compare the results.

Example 3.1 (Cube: direct verification). Let $M = [0, 1]^3$ and $F(x, y, z) = (x^2, y^2, z^2)$. Since $\operatorname{div} F = 2x + 2y + 2z$,

$$\int_M \operatorname{div} F dx dy dz = 1 + 1 + 1 = 3.$$

Boundary flux. On the three faces $x = 1$, $y = 1$, and $z = 1$, the respective fluxes are 1, 1, and 1; on the opposite faces they vanish. Since edges have zero surface measure, $\Phi_{\partial M}(F) = 3$.

Example 3.2 (Unit ball: a nonconstant normal). Let $M = \overline{B(0, 1)} \subset \mathbb{R}^3$ and $F(x, y, z) = (0, 0, z^3)$. We have $\operatorname{div} F = 3z^2$. **Volume integral:** slicing the ball by horizontal planes gives

$$\int_M \operatorname{div} F d\lambda^3 = \int_{-1}^1 3z^2 \pi(1 - z^2) dz = \frac{4\pi}{5}.$$

Boundary flux: on S^2 , $\nu_M = (x, y, z)$, so $\langle F, \nu_M \rangle = z^4$. Parametrize the sphere away from the poles by $(\theta, z) \mapsto (\sqrt{1 - z^2} \cos \theta, \sqrt{1 - z^2} \sin \theta, z)$, for which $dS = d\theta dz$. Hence

$$\int_{S^2} z^4 dS = \int_0^{2\pi} \int_{-1}^1 z^4 dz d\theta = \frac{4\pi}{5}.$$

Example 3.3 (Tangential field on a disk). For $D = \overline{B(0, 1)} \subset \mathbb{R}^2$, let $F(x, y) = (-y, x)$. Since $\operatorname{div} F = 0$ and $\nu_D(x, y) = (x, y)$ on ∂D , one has $F \cdot \nu_D = -xy + xy = 0$ pointwise. Both sides of (2.4) vanish.

Proposition 3.4 (Volume and flux). *For every body covered by Theorem 2.5,*

$$\lambda^n(M) = \frac{1}{n} \int_{\partial M} \langle x, \nu_M(x) \rangle dS. \quad (3.1)$$

Proof. Take $F(x) = x/n$, so $\operatorname{div} F = 1$. Then (2.4) gives (3.1). □

4 A complete proof for the unit cube

15 min

The main idea is to integrate one coordinate derivative at a time. Each integration produces the difference of the values on two opposite faces.

Theorem 4.1 (Divergence theorem for the cube). *Let $Q = [0, 1]^n$, $n \geq 1$, and let $F: U \rightarrow \mathbb{R}^n$ be C^1 on an open neighborhood U of Q . Then*

$$\int_Q \operatorname{div} F \, d\lambda^n = \int_{\partial Q} \langle F, \nu_Q \rangle \, dS.$$

Proof. For $n = 1$ this is the fundamental theorem of calculus, with outward normals -1 and 1 at the endpoints. Let $n \geq 2$. Fix $i \in \{1, \dots, n\}$. Put

$$H_i^- := \{x \in Q : x_i = 0\}, \quad H_i^+ := \{x \in Q : x_i = 1\}.$$

On the relative interior of $H_i^\pm$ the normal is $\pm e_i$; on every other open face its i th component vanishes. Intersections of faces have $(n - 1)$ -dimensional surface measure zero. Writing $\hat{x}$ for the remaining $n - 1$ coordinates and applying Fubini's theorem and the one-dimensional fundamental theorem of calculus,

$$\begin{aligned} \int_Q \frac{\partial F_i}{\partial x_i}(x) \, d\lambda^n &= \int_{[0,1]^{n-1}} \left(\int_0^1 \frac{\partial F_i}{\partial x_i}(\hat{x}, t) \, dt \right) \, d\hat{x} \\ &= \int_{[0,1]^{n-1}} (F_i(\hat{x}, 1) - F_i(\hat{x}, 0)) \, d\hat{x} \\ &= \int_{H_i^+} F_i \, dS - \int_{H_i^-} F_i \, dS = \int_{\partial Q} F_i \nu_Q^i \, dS. \end{aligned} \quad (4.1)$$

The formula holds for every i . Summing (4.1) from $i = 1$ to n proves the assertion. $\square$

Key point. The proof does not invoke the general divergence theorem. For each coordinate, the flux difference across two opposite faces *is exactly* the fundamental theorem of calculus.

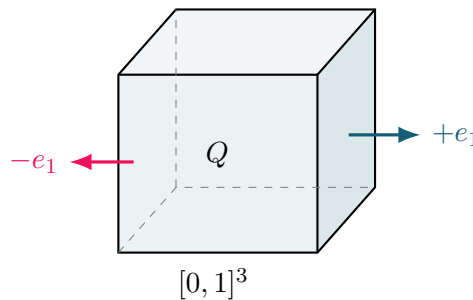


Figure 4: Opposite faces carry opposite unit normals; face intersections have zero surface measure.

5 The coordinate formula for special sets

20 min

For the cube, every line parallel to a coordinate axis meets the interior in an interval (or not at all). The same proof works when the interval endpoints vary with the other coordinates.

Definition 5.1 (i -special body). Fix $1 \leq i \leq n$. A body M satisfying the hypotheses of Theorem 2.5 is called i -special if there are a bounded open set $U \subset \mathbb{R}^{n-1}$ and functions $h^\pm \in C^1(U)$, $h^- < h^+$ on U , such that

$$\operatorname{int} M = \{x \in \mathbb{R}^n : \hat{x} \in U, \quad h^-(\hat{x}) < x_i < h^+(\hat{x})\}. \quad (5.1)$$

Here $\hat{x}$ denotes the $n - 1$ coordinates other than x_i .

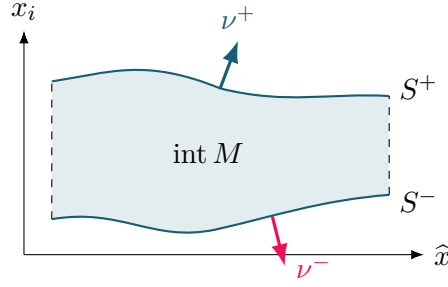


Figure 5: An i -special body: upper and lower graphs, with a possible lateral boundary above ∂U .

Theorem 5.2 (Coordinate formula for i -special bodies). *Let M be i -special and let f be C^1 on a neighborhood of M . Then*

$$\int_M \frac{\partial f}{\partial x_i} d\lambda^n = \int_{\partial M} f \nu_M^i dS. \quad (5.2)$$

Proof. Renumber coordinates so that $i = n$ and write $x = (u, t)$, $u \in \mathbb{R}^{n-1}$. Since ∂M has zero n -dimensional Lebesgue measure, the volume integral equals the integral over $\text{int } M$. Fubini and the fundamental theorem of calculus give

$$\int_M \frac{\partial f}{\partial x_n} d\lambda^n = \int_U (f(u, h^+(u)) - f(u, h^-(u))) du. \quad (5.3)$$

Let $S^\pm = \{(u, h^\pm(u)) : u \in U\}$. For the graph parametrizations $\varphi^\pm(u) = (u, h^\pm(u))$, the surface Jacobians are

$$J_{n-1}\varphi^\pm(u) = \sqrt{1 + \|\nabla h^\pm(u)\|^2},$$

and the outward unit normals, respectively, are

$$\nu_M(u, h^+(u)) = \frac{(-\nabla h^+(u), 1)}{\sqrt{1 + \|\nabla h^+(u)\|^2}}, \quad \nu_M(u, h^-(u)) = \frac{(\nabla h^-(u), -1)}{\sqrt{1 + \|\nabla h^-(u)\|^2}}.$$

Consequently, by the surface-area formula,

$$\int_{S^+} f \nu_M^n dS = \int_U f(u, h^+(u)) du, \quad \int_{S^-} f \nu_M^n dS = - \int_U f(u, h^-(u)) du. \quad (5.4)$$

These equalities remain valid even if the derivatives of $h^\pm$ become unbounded near ∂U , because the graph parametrizations are C^1 on U , and f and the normal components are bounded on the boundary, which has finite surface measure.

We now account for the remaining *lateral boundary*

$$L := \partial M \setminus (S^+ \cup S^-).$$

Every $x = (u, t) \in L$ has $u \in \partial U$: indeed, if $u \in U$, the vertical section of $\text{int } M$ is precisely $(h^-(u), h^+(u))$, so the only boundary points above u belong to S^- or S^+ ; if $u \notin \bar{U}$, there are no points of M above u because $M = \overline{\text{int } M}$.

At every regular point $x \in L$, the outward unit normal satisfies

$$\nu_M^n(x) = 0.$$

The geometric justification is deferred to Remark 5.3. Since the singular points carry zero surface measure, it follows that

$$\int_L f \nu_M^n dS = 0.$$

Combining this identity with (5.3) and (5.4) establishes (5.2). $\square$

Remark 5.3 (Normal component along the lateral boundary). [\[optional / independent study\]](#)

Here is a proof of the fact used above: $\nu_M^n(x) = 0$ at each regular point $x = (u, t)$ of L . Since $u \in \partial U$, neither $x + se_n$ nor $x - se_n$ can lie in $\text{int } M$, for any $s \in \mathbb{R}$, by (5.1). Suppose that $\nu_M^n(x) \neq 0$. At the regular point x , choose a C^1 local defining function g for ∂M with $g(x) = 0$ and $\nabla g(x) \neq 0$. Its gradient is parallel to $\nu_M(x)$, so $\partial_n g(x) \neq 0$. For $s \downarrow 0$,

$$g(x \pm se_n) = \pm s \partial_n g(x) + o(s).$$

Thus the two points lie on opposite sides of the boundary for sufficiently small $s > 0$. By the local separation property in Lemma 1.4, exactly one side is contained in $\text{int } M$. Hence one of $x + se_n$ and $x - se_n$ belongs to $\text{int } M$, a contradiction.

Corollary 5.4 (Special in every coordinate). *If M is i -special for every $i = 1, \dots, n$, then the full divergence theorem holds on M .*

Proof. Apply Theorem 5.2 to $f = F_i$ for each coordinate and sum. □

6 A complete proof for the unit ball

20 min

Here we repeat the coordinate-by-coordinate argument directly on a curved boundary; we do not invoke the theorem for special sets.

Theorem 6.1 (Divergence theorem for the unit ball). *Let $n \geq 1$ and $M = \overline{B(0, 1)} \subset \mathbb{R}^n$. For every C^1 vector field F on an open neighborhood of M ,*

$$\boxed{\int_M \text{div } F(x) \, d\lambda^n = \int_{S^{n-1}} \langle F(x), x \rangle \, dS.} \quad (6.1)$$

Here the outward unit normal to S^{n-1} is $\nu_M(x) = x$.

Proof. For $n = 1$ this is the fundamental theorem of calculus on $[-1, 1]$: $\int_{-1}^1 F'(t) \, dt = F(1) - F(-1)$, with outward normals 1 and -1 at the endpoints.

Assume $n \geq 2$. Fix $i \in \{1, \dots, n\}$. Write $x = (u, t)$, where $t = x_i$ and $u = \hat{x} \in \mathbb{R}^{n-1}$. Define

$$U = B^{n-1}(0, 1), \quad h^\pm(u) = \pm \sqrt{1 - \|u\|^2}, \quad u \in U.$$

The interior of M lies between the graphs of h^- and h^+ . By Fubini's theorem and the fundamental theorem of calculus,

$$\int_M \frac{\partial F_i}{\partial x_i} \, d\lambda^n = \int_U (F_i(u, h^+(u)) - F_i(u, h^-(u))) \, du. \quad (6.2)$$

For $S^\pm = \{(u, h^\pm(u)) : u \in U\}$, the graphs have derivatives and $(n-1)$ -dimensional surface Jacobians

$$\nabla h^\pm(u) = \mp \frac{u}{\sqrt{1 - \|u\|^2}}, \quad J_{n-1} \varphi^\pm(u) = \sqrt{1 + \|\nabla h^\pm(u)\|^2} = \frac{1}{\sqrt{1 - \|u\|^2}}.$$

On the unit sphere the normal is $\nu_M(x) = x$. Therefore $\nu_M^i(u, h^\pm(u)) = h^\pm(u)$, and consequently

$$\nu_M^i(u, h^\pm(u)) J_{n-1} \varphi^\pm(u) = \pm 1. \quad (6.3)$$

The surface-area formula now gives

$$\int_{S^\pm} F_i \nu_M^i \, dS = \pm \int_U F_i(u, h^\pm(u)) \, du.$$

The two graphs cover S^{n-1} except for the equator

$$E_i = \{x \in S^{n-1} : x_i = 0\}.$$

This equator has zero $(n-1)$ -dimensional surface measure: for $n=2$ it consists of two points, and for $n \geq 3$ it is an $(n-2)$ -sphere. Thus, on comparing the graph integrals with (6.2),

$$\int_M \frac{\partial F_i}{\partial x_i} d\lambda^n = \int_{S^{n-1}} F_i \nu_M^i dS.$$

The derivatives $\nabla h^\pm$ are unbounded near ∂U , but the graphs are C^1 on U . The integrals are finite because $F_i \nu_M^i$ is bounded and the unit sphere has finite area. The cancellation in (6.3) removes the apparent singularity.

Summing the coordinate identities for $i=1, \dots, n$ proves (6.1). $\square$

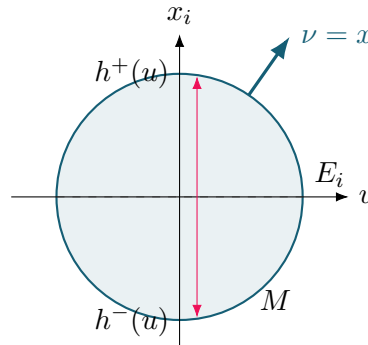


Figure 6: A two-dimensional cross-section of the unit ball: the graphs meet along an equator of lower surface dimension.

Remark 6.2 (Arbitrary balls). The theorem for any closed ball $\overline{B(a, R)}$ follows by the affine change of variables $x = a + Ry$ from the unit-ball case.

Corollary 6.3 (Surface area and volume of the unit ball). *[optional / independent study]* For $n \geq 2$,

$$\lambda_{S^{n-1}}(S^{n-1}) = n \lambda^n(B(0, 1)).$$

Proof. Take $F(x) = x$. Its divergence is n , while $\langle F, \nu_M \rangle = 1$ on S^{n-1} ; apply Theorem 6.1. $\square$

7 Further exercises and a general proof strategy *[optional / independent study]*

Exercise 7.1 (Constant fields and balance of normals). Show that for every constant vector $v \in \mathbb{R}^n$,

$$\int_{\partial M} \langle v, \nu_M \rangle dS = 0.$$

Deduce that $\int_{\partial M} \nu_M dS = 0$ as a vector identity.

Exercise 7.2 (Flux through the unit sphere). Let $M = \overline{B(0, 1)} \subset \mathbb{R}^n$, $n \geq 2$, and $F(x) = x$. Compute the flux first using $\nu = x$ and then by the divergence theorem. Explain why this is consistent with the volume formula (3.1).

Exercise 7.3 (A simple nonzero solenoidal field). For $F(x, y, z) = (-y, x, 0)$ and $M = [-1, 1]^3$, show directly that the total outward flux is zero by pairing opposite faces. Compare with $\operatorname{div} F$.

Remark 7.4 (Cutting a domain into special pieces). For a body that is not i -special, try to subdivide it into finitely many i -special bodies $M_1, \dots, M_k$ with pairwise disjoint interiors. Assuming the common interfaces are regular apart from sets of zero surface measure, the flux integrals over every internal interface cancel because the unit normals of the adjoining pieces are opposite. In a planar annulus, a suitable cut into upper, lower, left, and right regions illustrates this strategy. A full proof for arbitrary bodies with almost smooth boundary requires an appropriate decomposition theorem, which is *not* established here.

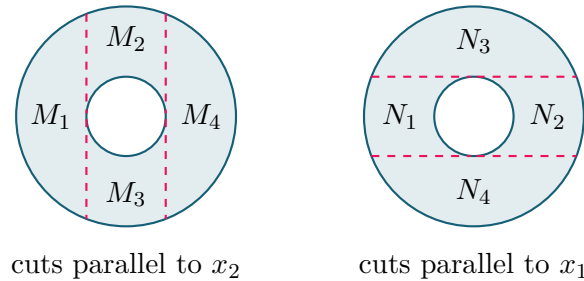


Figure 7: Schematic cutting of an annulus in two coordinate directions. Each interior interface appears twice with opposite normals.

Solutions to the further exercises

Exercise 7.1. The constant field $F(x) = v$ has $\operatorname{div} F = 0$, so its flux is zero by (2.4). Take $v = e_i$ for $i = 1, \dots, n$ to obtain the vector identity componentwise.

Exercise 7.2. On S^{n-1} , $\langle x, \nu \rangle = 1$, hence the flux equals $\lambda_{S^{n-1}}(S^{n-1})$. As $\operatorname{div} x = n$, the divergence theorem gives the same flux as $n\lambda^n(B(0, 1))$. Division by n recovers (3.1).

Exercise 7.3. On $z = \pm 1$, the normal is $\pm e_3$ and $F \cdot \nu = 0$. On $x = \pm 1$, the normal flux is $\mp y$, whose integral over the square in (y, z) vanishes. On $y = \pm 1$, the normal flux is $\pm x$, whose integral over the square in (x, z) also vanishes. Thus the total flux is zero, agreeing with $\operatorname{div} F = \partial_x(-y) + \partial_y x = 0$.