

① Mín na gradn línke [realiz, tj. pro $k=1$] Gr1

Nadl $U \subset \mathbb{R}^{n-1}$ je otevřená a $h: U \rightarrow \mathbb{R}^k$ je
br. $\mathcal{C}^1$ funkce

$$S := \text{graf } h = \{(u, h(u)) \mid u \in U\}.$$

Víme, že $\varphi(u) := (u, h(u))$, $u \in U$ je mapa S a

$$D\varphi = \begin{array}{|c|} \hline I_{n-1} \\ \hline \nabla h \\ \hline \end{array} \quad \text{a} \quad D\varphi^T D\varphi = \left(\delta_{ij} + \frac{\partial h}{\partial u_i} \frac{\partial h}{\partial u_j} \right)_{i,j=1}^{n-1} \\ = I_{n-1} + \nabla h^T \nabla h$$

m $m-1$ vodek vodek

c) Sylvestrov *: $\det(D\varphi^T D\varphi) = 1 + \nabla h \nabla h^T = 1 + \|\nabla h\|^2$
 $\times$ $\in \mathbb{R}^{n-1}$

tudíž $J\varphi = \sqrt{1 + \|\nabla h\|^2}$ a

$$\int_S f(x) dS(x) = \int_U f(u, h(u)) \sqrt{1 + \|\nabla h(u)\|^2} d\lambda^{n-1}(u)$$

* Sylvestrov věta: Je-li $a, b \in \mathbb{R}^n$, potom
 sloupce

$$\det(I_n + ab^T) = 1 + a^T b$$

Důkaz: $A :=$ def.

$$\begin{pmatrix} I_n & -a \\ b^T & 1 \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ b^T & 1 \end{pmatrix} \cdot \begin{pmatrix} I_n & 0 \\ 0 & 1 + b^T a \end{pmatrix} \begin{pmatrix} I_n & -a \\ 0 & 1 \end{pmatrix}$$

Potom $\det A =$

$$\begin{array}{l} 1 \cdot (1 + b^T a) \cdot 1 = \\ 1 \cdot \det(I_n + ab^T) \cdot 1 \end{array} = \begin{pmatrix} I_n & -a \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} I_n + ab^T & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} I_n & 0 \\ b^T & 1 \end{pmatrix}$$

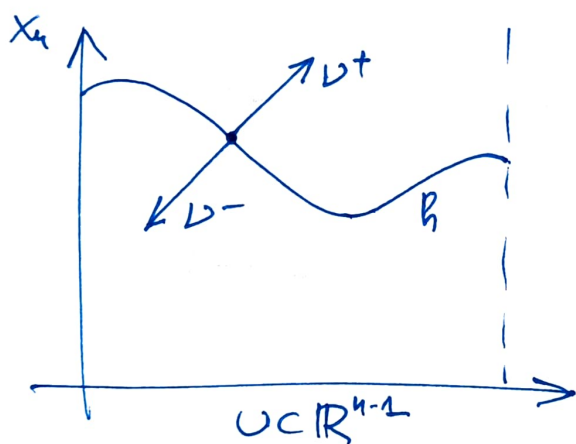
(i) Důležité: $J\varphi = \underbrace{\left\| \frac{\partial \varphi}{\partial u_1} \times \dots \times \frac{\partial \varphi}{\partial u_{n-1}} \right\|}_{\substack{\text{"1"} \\ N}} = \sqrt{1 + \|\nabla h\|^2} \quad \text{Gr 2}$
 $\nwarrow$
 Euklidovská v $\mathbb{R}^{n-1}$

• $N_h = (-1)^{n+1} |F_{n-1}|$

• U je kolmý ke vlně $\frac{\partial \varphi}{\partial u_i} = (0, \dots, 1, \dots, 0, \frac{\partial h}{\partial u_i})$, tm.

$N := (-1)^{n+1} \left(-\frac{\partial h}{\partial u_1} \mid \dots \mid -\frac{\partial h}{\partial u_{n-1}} \mid 1 \right)$

(iii) Dále $\nu^\pm(x) := \pm \frac{(-1)^{n+1} N(\varphi^{-1}(x))}{\|N(\varphi^{-1}(x))\|}$, $x \in S$ je spojitě pole jednotkových normálových vektorů, které "směřuje" do nadgrubu / podgrubu h ! (Proč?)
 [ZADÍČEK]



Tedy $\nu^\pm(x) = \pm \frac{1}{\sqrt{1 + \|\nabla h(u)\|^2}} \cdot (-\nabla h(u), 1)$,
 $x = (u, h(u)), u \in U$

(Pr.) $S_+^{n-1} := \{x \in \mathbb{R}^n \mid \|x\|=1, x_n > 0\}$ horní polokoule

• $S_+^{n-1} = \text{graf } h$, kde $h(u) := \sqrt{1 - u_1^2 - \dots - u_{n-1}^2}$, $u \in B_{n-1}$ a

$B_{n-1} := \{u \in \mathbb{R}^{n-1} \mid \|u\| < 1\}$ je jednotk. koule v $\mathbb{R}^{n-1}$

• $\nabla h(u) = -\frac{u^\top}{\sqrt{1 - \|u\|^2}}$, $J\varphi(u) = \frac{1}{\sqrt{1 - \|u\|^2}}$ a pro $S = S_+^{n-1}$ je

$\lambda_S(S) = \int_{B_{n-1}} \frac{du}{\sqrt{1 - \|u\|^2}}$ [$n=2$: π ; $n=3$: 2π , polární souřadn.]

Točny prostor

Popišto točny prostor $T_x S$ k ploše S v bode $x \in S$, pokud S je zadáno (lokalně)

① mapou, tm. $\varphi: U \subset \mathbb{R}^k \rightarrow S$, $x = \varphi(u)$:

Potom $T_x S = D\varphi(u)(\mathbb{R}^k) = \text{Lo} \left\{ \frac{\partial \varphi}{\partial u_1}(u), \dots, \frac{\partial \varphi}{\partial u_k}(u) \right\}$.

② jako graf funkce, tm. $S = \text{graf}(f)$,

$f: U \subset \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$ tm. $\mathcal{C}^1$: Potom je

$\varphi(u) := (u, f(u))$, $u \in U$ mapa S a

$$D\varphi(u) = \begin{array}{|c|} \hline \begin{array}{c} k \\ \hline I_k \\ \hline \end{array} \\ \begin{array}{c} \hline Df(u) \\ \hline n-k \end{array} \\ \hline k \end{array} \quad \begin{array}{l} \text{jednotkový} \\ \text{matri} \end{array}$$

③ implicitně, tm. $S = \{x \in \mathbb{R}^n \mid F(x) = 0\}$,

kde $F: G \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$ ^{*}: Potom platí

$$T_x S^\perp = \text{Lo} \left\{ \underbrace{\nabla F_1(x), \dots, \nabla F_{n-k}(x)}_{\text{lin. nezávislé}} \right\}, \quad x \in S.$$

radky v $DF(x)$

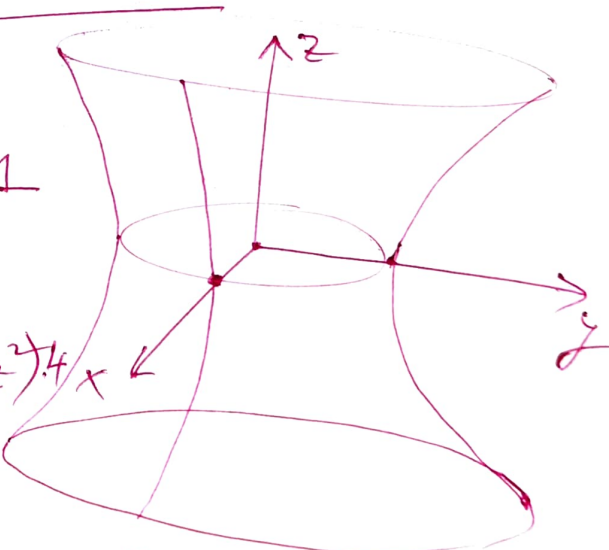
^{*} je aspoň $\mathcal{C}^1$ a $\text{rank}(DF) = n-k$ na $S \neq \emptyset$.

Γ skutočné, nechť $v \in T_x S$. Potom ex.
 existuje $c: I \rightarrow S$ a $t_0 \in I$ tak, že
 $c(t_0) = x$ a $c'(t_0) = v$. Potom $F(c(t)) = 0$,
 $t \in I$ a $(F(c(t)))'|_{t=t_0} = DF(x)v = 0$, neboli
 $v \in \mathbb{R}^n$ je kotný me v. $\nabla F_i(x)$. $\square$
 radky $DF(x)$

Pr. Napište rovnice tečnej roviny
 jednodílného hyperboloidu $S \subset \mathbb{R}^3$ / zadaného
 implicitne $x^2 + y^2 - z^2 = 1$
 v bode (i) $(1, 0, 0)$, (ii) $(1, 1, 1)$, (iii) $A = (a, b, c)$.

$$x^2 + y^2 = 1 + z^2$$

1. $F(x, y, z) := x^2 + y^2 - z^2 - 1$
 $\nabla F = 2(x, y, -z) \neq 0$ na S
 $\|\nabla F\|^2 = 4(x^2 + y^2 + z^2) = (1 + 2z^2) \cdot 4$



$\neq 0$ me S

Potom $(T_A S)^\perp = \text{Lo}\{(a, b, -c)\}$ a tečnej rovne
 v $A \in S$ je $a(x-a) + b(y-b) - c(z-c) = 0$.

Pro $A = (1, 0, 0)$: $x = 1$

$A = (1, 1, 1)$: $(x-1) + (y-1) - (z-1) = 0$
 $x + y - z - 1 = 0$

Defn: $N(X) := \nabla F(X) / \|\nabla F(X)\| =$ TP3

$\frac{1}{\sqrt{1+2z^2}}(x|y| - z)$ je jedinečný normálový vektor
 $\forall X \in S, X = (x|y|z)$.

(2) Pro $z > 0$ je $z = f(x|y) := \sqrt{x^2 + y^2 - 1}$,
 $(x|y) \in U := \mathbb{R}^2 \setminus \overline{B(0,1)}$.

Pro $\varphi(x|y) = (x|y|f(x|y))$ je

$\frac{\partial \varphi}{\partial x} = \left(1 \mid 0 \mid \frac{x}{\sqrt{x^2 + y^2 - 1}}\right) = \left(1 \mid 0 \mid \frac{x}{z}\right)$ tace

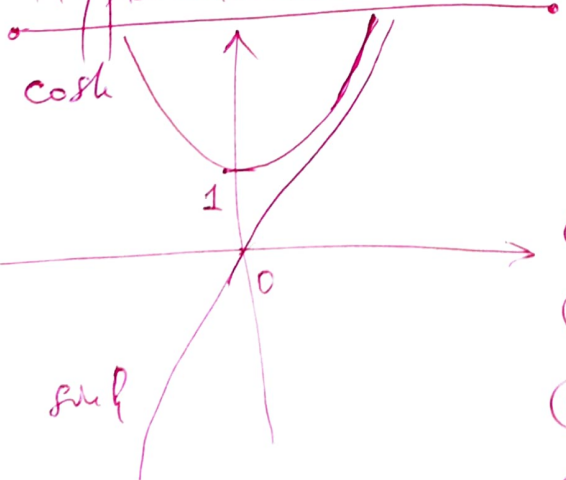
$\frac{\partial \varphi}{\partial y} = \left(0 \mid 1 \mid \frac{y}{\sqrt{x^2 + y^2 - 1}}\right) = \left(0 \mid 1 \mid \frac{y}{z}\right)$ vlastný

Potom $T_A S = \text{Lo} \{ (e \mid 0 \mid a), (0 \mid e \mid b) \}$,

$T_A S^\perp = \text{Lo} \{ (a \mid b \mid -c) \}$, protože

$(e \mid 0 \mid a) \times (0 \mid e \mid b) = (-ae \mid -be \mid e^2)$, c.d.o.

Hyperbolické funkce:



$e^u = \cosh u + \sinh u$, kde

$\cosh u := \frac{e^u + e^{-u}}{2}$

$\sinh u := \frac{e^u - e^{-u}}{2}, u \in \mathbb{R}$

(i) $\cosh' = \sinh, \sinh' = \cosh$

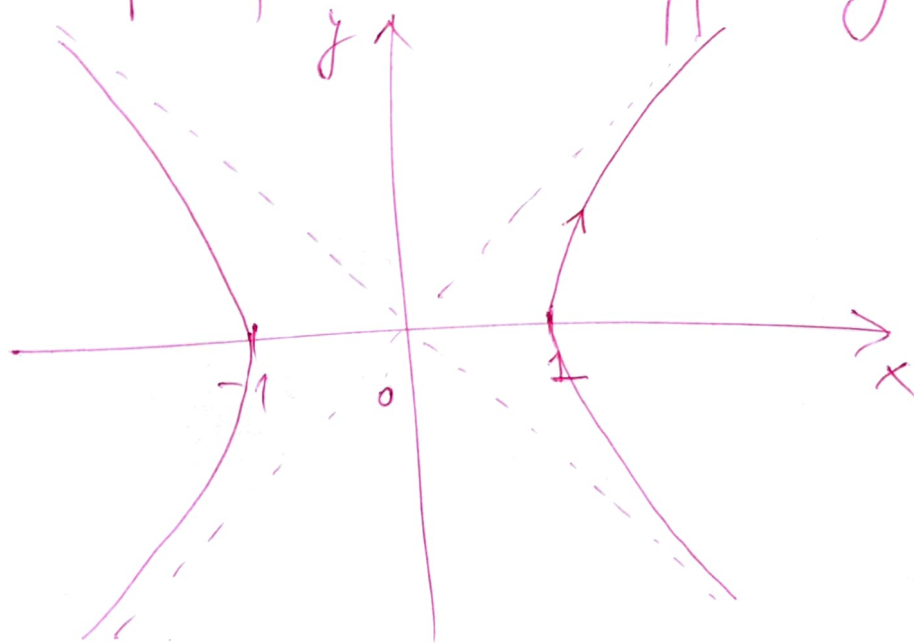
(ii) $\cosh^2 + \sinh^2 = 1$

(iii) $\cosh + \sinh = \cosh(2u)$

atd.

Pozn: (i) $x = \cosh v$, $y = \sinh v$, $v \in \mathbb{R}$ je TP4

1-mepe pravo kotro hyperboly $x^2 - y^2 = 1$ v $\mathbb{R}^2$

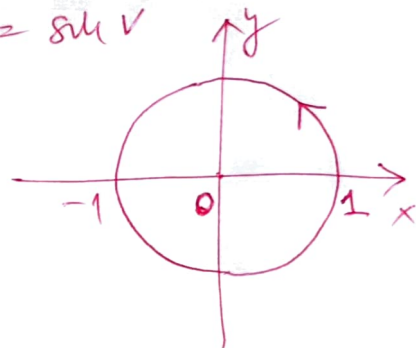


$$y = \pm \sqrt{x^2 - 1}$$

(ii) $x^2 + y^2 = 1$

$$x = \cos v$$

$$y = \sin v$$



3. $x^2 + y^2 = 1 + z^2 = r^2$, kjer $r = r(z) := \sqrt{1 + z^2} \geq 1$

$$x = r \cdot \cos v$$

$$y = r \cdot \sin v$$

$$r^2 - z^2 = 1$$

$$r = \cosh u$$

$$z = \sinh u$$

$$x = \cosh(u) \cdot \cos(v)$$

$$\varphi: y = \cosh(u) \cdot \sin(v) \quad u, v \in \mathbb{R}$$

$$z = \sinh(u)$$

(i) φ je C^∞ , $\varphi: \mathbb{R}^{2 \times 1} \rightarrow S \subset \mathbb{R}^3$

$$\frac{\partial \varphi}{\partial u} = (\sinh u \cos v, \sinh u \sin v, \cosh u)$$

$$\frac{\partial \varphi}{\partial v} = (-\cosh u \sin v, \cosh u \cos v, 0)$$

$$\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} = (-\cosh^2 u \cos v, -\cosh^2 u \sin v, \cosh u \sinh u)$$

$$\left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\| = \cosh u \cdot (\cosh(2u))^{1/2} > 0. \quad \boxed{TPJ}$$

Pozor: $N(X) = - \left(\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right) / \left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\| \cdot X$
 je-li $N(X)$ jako v (1).

(ii) $\psi := \varphi|_U$ je meřa, je-li $U := \mathbb{R} \times (-\pi + v_0, v_0 + \pi)$ pro libovolné $v_0 \in \mathbb{R}$.

Např. pro $v_0 = 0$: $u = \sinh^{-1} z$

$v = \arg(x, y)$, kde

$\arg: \mathbb{R}^2 \xrightarrow{u, v} (-\pi, \pi]$ je tr. klesající hodnota argumentu $\neq 0$! Víme, že $\arg$ je spojitá na $\mathbb{R}^2 \setminus (-\infty, 0] \times \{0\}$.

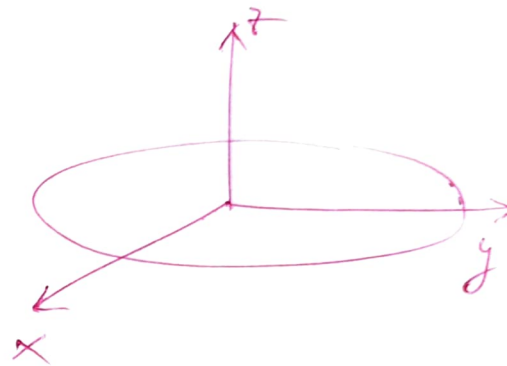
Kvadry v $\mathbb{R}^3$ (viz G1)

KV1

I. Neodpuzneme:

- Elipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Pozn: sféra pro $a=b=c$.

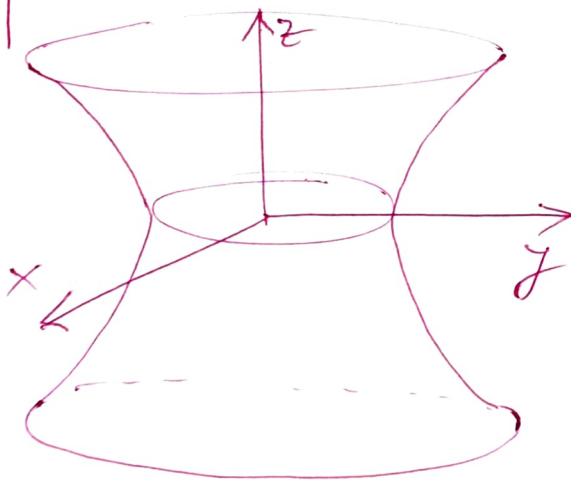


- hyperboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \pm 1$

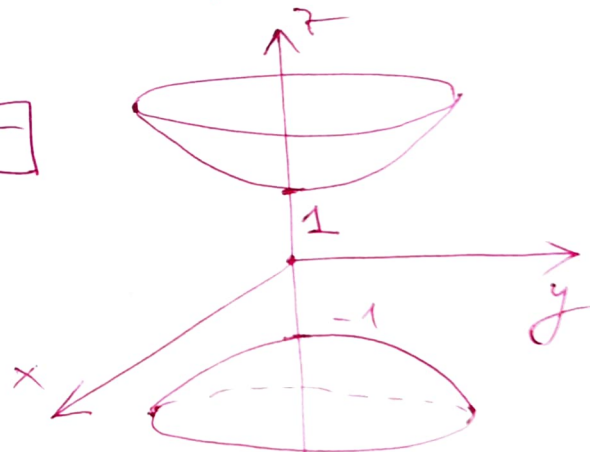
jednodílný / dvoudílný

$$x^2 + y^2 = z^2 \pm 1$$

[+]



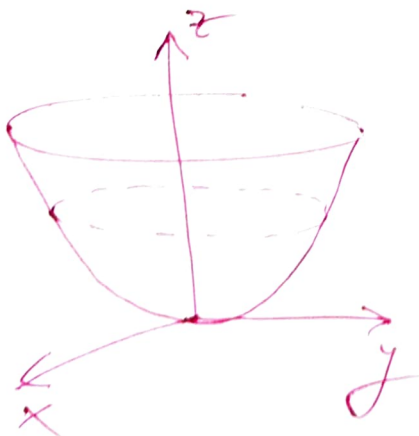
[-]



- Paraboloid: $z = \frac{x^2}{a^2} \pm \frac{y^2}{b^2}$

eliptický /
hyperbolický

[+]



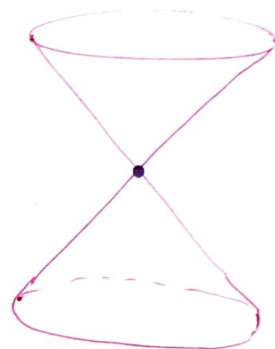
[-]



II. Degenerované :

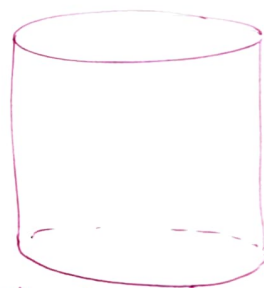
KV2

- Kružok : $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ (*)
eliptický, $(z \neq 0)$



- Válčové plochy : eliptické / hyperbolické / parabolické

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1, \quad x^2 + 2ay = 0$$

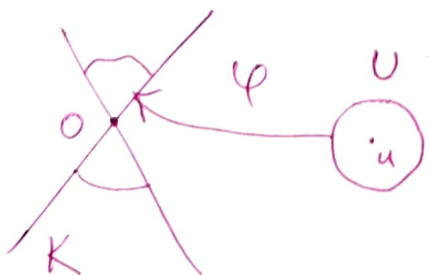


Ukazuje, že jde o 2-plochy v $\mathbb{R}^3$. Najdeťo
akékoľvek plochy.

[WIKIPEDIA, Quadrics]

*) Pozn: Kružok $K := \{x^2 + y^2 = z^2\} \subset \mathbb{R}^3$ nie je 2-plocha,
ale $K_0 := K \setminus \{(0,0,0)\}$ áno.

Sporné. Nechť $\varphi: U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ je mapa K a
 $0 \in \langle \varphi \rangle$, tm. $0 = \varphi(u)$, $u \in U$. Buď: Prádp., že U
je rovnice. Tíež vzniká ot. medzi kruhom $B(u, \delta)$.
 $\subset U$.



Potom $\langle \varphi \rangle \setminus \{0\} = \varphi(U \setminus \{u\})$
nie je sour.

coť je spor. $\llcorner$