DÚ 2

(1) $\int \frac{1}{x(x+2)\underbrace{\sqrt{1-x^2}}_{\sqrt{(1-x)(1+x)}}}\,dx = \int \frac{1}{x(x+2)\sqrt{(1-x)(1+x)\frac{(1-x)}{(1-x)}}}$

$$= \int \frac{1}{x(x+2)(1-x)}\sqrt{\frac{1-x}{1+x}}\,dx$$

$t=\sqrt{\frac{1-x}{1+x}}$ $\quad t^2(1+x)=1-x \quad x(t^2+1)=1-t^2 \quad x=\frac{1-t^2}{1+t^2}$

$dx = \frac{-4t}{(1+t^2)^2}\,dt \qquad x+2=\frac{1-t^2+2+2t^2}{1+t^2}=\frac{3+t^2}{1+t^2}$

$1-x=\frac{1+t^2-1+t^2}{1+t^2}=\frac{2t^2}{1+t^2}$

↳ $\int \frac{\quad}{\frac{1-t^2}{1+t^2}\cdot\frac{3+t^2}{1+t^2}\cdot\frac{2t^2}{1+t^2}}\cdot t\cdot\frac{-4t}{(1+t^2)^2}\,dt = \int\frac{-2(1+t^2)}{(1-t)(1+t)(3+t^2)}\,dt$

$$= \int \frac{\overset{A}{-1/2}}{1-t} + \frac{\overset{B}{-1/2}}{1+t} + \frac{\overset{Ct+D}{1}}{3+t^2}\,dt$$

$A(1+t)(3+t^2) + B(1-t)(3+t^2) + (Ct+D)(1-t)(1+t) = -2(1+t^2)$

$t=1 \qquad 8A=-4 \qquad A=-1/2$

$t=-1 \qquad 8B=-4 \qquad B=-1/2$

$t=0 \qquad -3/2-3/2+D=-2 \qquad D=1$

$t=2 \qquad -1/2\cdot3\cdot7 + -1/2(-1)\cdot7 + (2C+1)(-1)\cdot3 = -10$

$\qquad -3(2C+1) = -3 \qquad C=0$

$= +\frac{1}{2}\log|1-t| - \frac{1}{2}\log|1+t| + \int\frac{1}{3(1+(\frac{t}{\sqrt3})^2)}\,dt =$

$\stackrel{c}{=} \frac{1}{2}\log|1-t| - 1/2\log|1+t| + \frac{1}{3}\sqrt3\arctan\frac{t}{\sqrt3}$

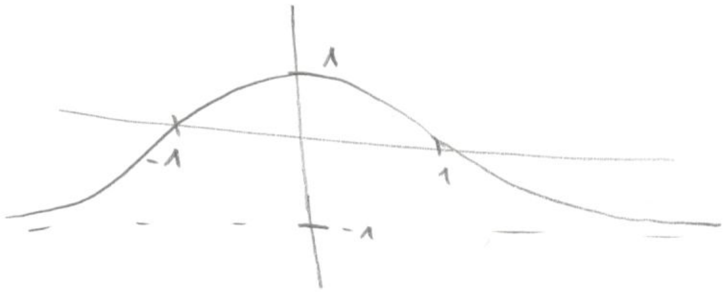
$= \frac{1}{2}\log\left|1-\sqrt{\frac{1-x}{1+x}}\right| - 1/2\log\left|1+\sqrt{\frac{1-x}{1+x}}\right| + \frac{\sqrt3}{3}\arctan\frac{1}{\sqrt3}\sqrt{\frac{1-x}{1+x}}$

$x\in(-1,0),(0,1)$

2 VoS

$$\varphi(t) = \frac{1-t^2}{1+t^2}$$

$$\varphi'(t) = \frac{-4t}{(1+t^2)^2}$$



$$f(x) = \frac{1}{x(x+2)\sqrt{1-x^2}} \qquad x \in \underset{\alpha_1, \beta_1}{(-1,0)}, \underset{\alpha_2, \beta_2}{(0,1)}$$

$(a_2, b_2) = (0,1)$ $\qquad \varphi(a_2, b_2) = (0,1) = (\alpha_2, \beta_2)$

$(a_1, b_1) = (1,\infty)$ $\qquad \varphi(a_1, b_1) = (-1,0) = (\alpha_1, \beta_1)$

$\varphi' \neq 0$ na $(a_1, b_1), (a_2, b_2)$

(2) $\displaystyle\int_0^{2\pi} \frac{1+\sin x}{2+\sin x-\cos x}\,dx = \int_0^{\pi} + \int_{\pi}^{2\pi}$

spoj na ℝ

$(\alpha,\beta) = (0,\pi)$ $\quad \varphi(\alpha,\beta) = (0,\infty)$

$(\tilde\alpha,\tilde\beta) = (\pi, 2\pi)$ $\quad \varphi(\tilde\alpha,\tilde\beta) = (-\infty,0)$

$\varphi' \neq 0$ na $(\alpha,\beta), (\tilde\alpha,\tilde\beta)$

$\varphi(x) = \tan\frac{x}{2}$ $\qquad$ (nebo $\int_{-\pi}^{\pi}$)

$\varphi' = \frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}$

$$\to \int \frac{1+\frac{2t}{1+t^2}}{2+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt = \int \frac{1+t^2+2t}{2+2t^2+2t-1+t^2}\cdot\frac{2}{1+t^2}\,dt$$

$$= \int \frac{t^2+2t+1}{3t^2+2t+1}\cdot\frac{2}{1+t^2}\,dt = \int \frac{\overset{At+B}{3t+1}}{3t^2+2t+1} + \frac{\overset{Ct+D}{-t+1}}{1+t^2}\,dt$$

$$(At+B)(1+t^2) + (Ct+D)(3t^2+2t+1) = 2(t+1)^2$$

$$At^3 + At + Bt^2 + B + 3Ct^3 + 2Ct^2 + Ct + 3Dt^2 + 2Dt + D = 2t^2 + 4t + 2$$

$A + 3C = 0$

$B + 2C + 3D = 2$

$A + C + 2D = 4$

$B + D = 2$

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 \\ 1 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 2 & -2 & -4 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right)$$

$D = 1 \qquad C = -1$

$B = 1 \qquad A = 3$

$$\int \frac{3t+1}{3t^2+2t+1} = \frac{1}{2}\int \frac{6t+2}{3t^2+2t+1}\, dt \overset{c}{=} \frac{1}{2} \log |3t^2+2t+1|$$

" $y = 3t^2+2t+1$ "

$dy = 6t+2$

$$\int \frac{-t+1}{1+t^2}\, dt = -\frac{1}{2}\int \frac{2t}{1+t^2} dt + \int \frac{1}{1+t^2}\, dt \overset{c}{=} -\tfrac{1}{2}\log|1+t^2| + \arctan t$$

$$\int_0^{\pi} \dots dx \to \int_0^{\infty} \dots dt = \left[\tfrac{1}{2}\log|3t^2+2t+1| - \tfrac{1}{2}\log|1+t^2| + \arctan t\right]_0^{\infty}$$

$$= \left[\frac{1}{2}\log\left|\frac{3t^2+2t+1}{1+t^2}\right| + \arctan t\right]_0^{\infty} = \frac{1}{2}\log 3 + \frac{\pi}{2} - 0$$

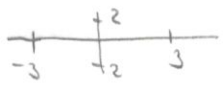
$$\int_{\pi}^{2\pi} \dots dx \to \int_{-\infty}^{0} \dots dt = \left[\frac{1}{2}\log \frac{3t^2+2t+1}{1+t^2} + \arctan\right]_{-\infty}^{0} = 0 - \left(\frac{1}{2}\log 3 - \pi/2\right)$$

Závěr: $\int_0^{2\pi} = \underline{\underline{\pi}}$

(3) S pláště, elipsa $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$

$$f(x) = \sqrt{4\left(1 - \frac{x^2}{9}\right)} = \sqrt{4 - \frac{4}{9}x^2} = 2\sqrt{1-\left(\frac{x}{3}\right)^2}$$

$$f' = \frac{1}{2}\,\frac{1}{\sqrt{4-\frac{4}{9}x^2}} \cdot -\frac{8}{9}x = -\frac{2}{9}\,\frac{x}{\sqrt{1-\left(\frac{x}{3}\right)^2}}$$



$$S = 2\pi \int_{-3}^{3} \sqrt{4 - \frac{4}{9}x^2}\ \sqrt{1 + \left(\frac{-4/9\, x}{\sqrt{4 - 4/9\, x^2}}\right)^2}\ dx$$

$4(1 - \frac{1}{9}x^2)$

$$= 2\pi \int_{-3}^{3} \sqrt{4-\tfrac{4}{9}x^2} \cdot \frac{\sqrt{4-\tfrac{4}{9}x^2+\left(\tfrac{4}{9}x\right)^2}}{\sqrt{4-4/9x^2}} =$$

$$\frac{16}{81}-\frac{4}{9}$$

$$= 2\pi \int_{-3}^{3} \sqrt{4-\frac{20}{81}x^2}\, dx = 2\pi \int_{-3}^{3} 2\sqrt{1-\frac{5}{81}x^2}$$

$$\sqrt{\frac{5}{81}}x = \frac{\sqrt5}{9}x = \sin t$$

$$= 4\pi \int_{-\arcsin\frac{\sqrt5}{3}}^{\arcsin\frac{\sqrt5}{3}} \sqrt{1-\sin^2 t}\,\frac{9}{\sqrt5}\cos t\, dt =$$

$$x = \frac{9}{\sqrt5}\sin t$$

$$dx = \frac{9}{\sqrt5}\cos t\, dt$$

$$= \frac{36\pi}{\sqrt5}\int_{-\arcsin\frac{\sqrt5}{3}}^{\arcsin\frac{\sqrt5}{3}} \cos^2 t\, dt = \frac{36\pi}{\sqrt5}\int_{-\arcsin\frac{\sqrt5}{3}}^{\arcsin\frac{\sqrt5}{3}} \frac{1}{2}(1+\cos(2t))\, dt$$

$$= \frac{18\pi}{\sqrt5}\Big[t + \underbrace{\tfrac{1}{2}\sin(2t)}_{\sin t \cos t}\Big]_{-\arcsin\frac{\sqrt5}{3}}^{\arcsin\frac{\sqrt5}{3}} = \frac{18\pi}{\sqrt5}\left(2\arcsin\frac{\sqrt5}{3} + \sin\left(2\arcsin\frac{\sqrt5}{3}\right)\right)$$

$$= 2\,\frac{18\pi}{\sqrt5}\left(\arcsin\frac{\sqrt5}{3} + \sin\left(\arcsin\frac{\sqrt5}{3}\right)\underbrace{\cos\left(\arcsin\frac{\sqrt5}{3}\right)}_{\cos t = \sqrt{1-\sin^2 t}}\right)$$

$$\sqrt{1-\frac{5}{9}} = \frac{2}{3}$$

$$= \frac{36\pi}{\sqrt5}\left(\arcsin\frac{\sqrt5}{3} + \frac{\sqrt5}{3}\cdot\frac{2}{3}\right)$$

(4) $f(x) = \sqrt{x(D-x)}$ $x \in [0, D]$ | palivo 100π km

$$f' = \frac{1}{2} \frac{1}{\sqrt{x(D-x)}} (D-2x)$$

$$l = \int_0^D \sqrt{1 + \frac{(D-2x)^2}{4(x(D-x))}}\, dx =$$

$$= \int_0^D \frac{\sqrt{4Dx - 4x^2 + D^2 + 4x^2 - 4Dx}}{2\sqrt{x(D-x)}} = \frac{D}{2} \int_0^D \frac{1}{\sqrt{x(D-x)}}$$

$x = D \sin^2 t$

$dx = D\, 2 \sin t \cos t$

$x(D-x) = D \sin^2 t (D - D \sin^2 t) = D^2 \sin^2 \cos^2 t$

$$= \frac{D}{2} \int_0^{\pi/2} \frac{1}{D \sin t \cos t} D\, 2 \sin t \cos t\, dt = \int_0^{\pi/2} D = \frac{D\pi}{2}$$

$100\pi = \frac{D\pi}{2} \rightarrow \underline{\underline{D = 200 \text{ km}}}$