(1) $$\lim_{x \to 0} \frac{e^x \sin(2x) - 2\log(1+\sin x) - 3x^2}{x \cos(\sin x) - \sin x}$$

$x, y \to 0$

- $\sin x = x - \frac{x^3}{6} + o(x^4)$ (stačilo do 3. řádu)

$\cos y = 1 - \frac{y^2}{2} + \frac{y^4}{24} + o(y^5)$

$$\cos(\sin x) = 1 - \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^4)\right)^2 + \frac{1}{24}\left(x - \frac{x^3}{6} + o(x^4)\right)^4 + o\left(\left(x - \frac{x^3}{6} + o(x^4)\right)^5\right)$$

$$= 1 - \frac{1}{2}\left(x^2 - \frac{2}{6}x^4 + o(x^4)\right) + \frac{x^4}{24} + o(x^5) + o(x^5)$$

$$= 1 - \frac{x^2}{2} + x^4\underbrace{\left(\frac{1}{6} + \frac{1}{24}\right)}_{\frac{5}{24}} + o(x^4)$$

$$x\cos(\sin x) - \sin x = \cancel{x} - \frac{x^3}{2} + \frac{5}{24}x^5 - \cancel{x} + \frac{x^3}{6} + o(x^4)$$

$$= -\frac{x^3}{3} + o(x^4)$$

- $\sin(2x) = 2x - \frac{8x^3}{6} + o(x^4)$

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + o(x^4)$

$$e^x \sin(2x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + o(x^4)\right)\left(2x - \frac{4}{3}x^3 + o(x^4)\right)$$

$$= 2x - \frac{4}{3}x^3 + o(x^4) + 2x^2 - \frac{4}{3}x^4 + x^3 + \frac{2x^4}{6}$$

$$= 2x + 2x^2 + x^3\underbrace{\left(1 - \frac{4}{3}\right)}_{-1/3} + x^4\underbrace{\left(-\frac{4}{3} + \frac{2}{6}\right)}_{-1} + o(x^4)$$

$\log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + o(y^4)$

$$\log(1+\sin x) = \left(x - \frac{x^3}{6} + o(x^4)\right) - \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^4)\right)^2 + \frac{1}{3}\left(x - \frac{x^3}{6} + o(x^4)\right)^3 - \frac{1}{4}\left(x - \frac{x^3}{6} + o(x^4)\right)^4$$
$$+ o\left(\left(x - \frac{x^3}{6} + o(x^4)\right)^4\right)$$

$$= x - \frac{x^3}{6} + o(x^4) - \frac{1}{2}\left(-\frac{2}{6}x^4 + x^2\right) + \frac{1}{3}x^3 - \frac{1}{4}x^4 + o(x^4)$$

$$= x - \frac{x^2}{2} + x^3\underbrace{\left(-\frac{1}{6} + \frac{1}{3}\right)}_{1/6} + x^4\underbrace{\left(\frac{1}{6} - \frac{1}{4}\right)}_{\frac{2-3}{12} = -1/12} + o(x^4)$$

$$e^x \sin(2x) - 2\log(1+\sin x) - 3x^2 =$$

$$= 2x + 2x^2 - \frac{1}{3}x^3 - x^4 - 2\left(x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12}\right) - 3x^2 + o(x^4)$$

$$= x^3\underbrace{\left(-1/3 - 1/3\right)}_{-2/3} + x^4\underbrace{\left(-1 + 1/6\right)}_{-5/6} + o(x^4)$$

$$\lim_{x\to 0} \frac{-\frac{2}{3}x^3 - \frac{5}{6}x^4 + o(x^4)}{-\frac{x^3}{3} + o(x^4)} = \lim_{x\to 0} \cancel{\frac{x^3}{x^3}} \cdot \frac{-\frac{2}{3} - \frac{5}{6}x + \frac{o(x^4)}{x^3 \cdot x}\cdot x}{-\frac{1}{3} + \frac{o(x^4)}{x^3 \cdot x}\cdot x} \overset{AL}{=} \frac{-\frac{2}{3} + 0 + 0}{-1/3 + 0}$$

$$= \underline{\underline{2}}$$

(2) $\int 2\,\sqrt[3]{x}\,\frac{\arctan \sqrt[3]{x}}{\sqrt[3]{x^2}}\,dx = 3\int \arctan \sqrt[3]{x} \cdot 2\sqrt[3]{x} \cdot \frac{1}{3\sqrt[3]{x^2}}\,dx$

$y = \sqrt[3]{x} \quad (= x^{1/3})$

$dy = \frac{1}{3}\,\frac{1}{\sqrt[3]{x^2}}\,dx$

$= 3\int \arctan y \cdot 2y\,dy$

$v \qquad u'$

$v' = \frac{1}{1+y^2} \qquad u = y^2$

Substituce

$x \in (-\infty, 0),\ (0, \infty)$
$(\alpha, \beta) \quad (\tilde{\alpha}, \tilde{\beta})$

$y \in \mathbb{R}$
(a, b)

$f = 2y \arctan y$

$\varphi(x) = \sqrt[3]{x}$

$\varphi'(x) = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{x^2}}$

$\varphi(\alpha, \beta) = (-\infty, 0) \subseteq (a, b)$

$\varphi(\tilde{\alpha}, \tilde{\beta}) = (0, \infty) \subset (a, b)$ ✓

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$= 3\left(y^2 \arctan y - \int \frac{y^2}{1+y^2}\,dy\right)$

$= 3y^2 \arctan y - 3\int \underbrace{\frac{y^2+1}{y^2+1}}_{1} - \frac{1}{1+y^2}\,dy$

F(y)

$\stackrel{c}{=} 3y^2 \arctan y - 3y + 3 \arctan y$

$= 3\sqrt[3]{x^2} \arctan \sqrt[3]{x} - 3\sqrt[3]{x} + 3 \arctan \sqrt[3]{x}$

$x \in (-\infty, 0),\ (0, \infty)$

$F(\varphi(x))$

$$(3)\quad \int^{P/Q} \frac{x^3-3x+36}{(x^2-6x+9)(x^2+2x+3)}\,dx = \int \frac{\overset{0}{A}}{x-3} + \frac{\overset{3}{B}}{(x-3)^2} + \frac{\overset{x\ +3}{Cx+D}}{x^2+2x+3}\,dx$$

st $P = 3 < 4 =$ st Q ✓

$$(x^2-6x+9) = (x-3)^2$$

$$x^3-3x+36 = A(x-3)(x^2+2x+3) + B(x^2+2x+3) + (Cx+D)(x-3)^2$$

$x=3$ $\quad 27-9+36 = B(9+6+3) \quad 54 = 18B \quad \boxed{B=3}$

$x=0$ $\quad 36 = A(-3)(3) + 3\cdot 3 + D9$

$27 = -9A + 9D$

$\boxed{3 = D - A}$

$x=1$ $\quad 1-3+36 = A(-2)6 + 3\cdot 6 + (C+D)\cdot 4$

$16 = -12A + 4C + 4D$

$\boxed{4 = -3A + C + D}$

$x=2$ $\quad 8-6+36 = -A\cdot 11 + 33 + 2C + D$

$\boxed{5 = -11A + 2C + D}$

$$\begin{array}{c} \begin{array}{ccc} A & C & D \end{array} \\ \left(\begin{array}{ccc|c} -1 & 0 & 1 & 3 \\ -3 & 1 & 1 & 4 \\ -11 & 2 & 1 & 5 \end{array}\right) \end{array} \sim \left(\begin{array}{ccc|c} -1 & 0 & 1 & 3 \\ -2 & 1 & 0 & 1 \\ -10 & 2 & 0 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 0 & 1 & 3 \\ -2 & 1 & 0 & 1 \\ -6 & 0 & 0 & 0 \end{array}\right)$$

$\boxed{A=0} \quad \boxed{C=1} \quad \boxed{D=3}$

$$\int \frac{3}{(x-3)^2}\,dx \overset{c}{=} 3\cdot \frac{-1}{x-3}$$

$$\int \frac{x+3}{x^2+2x+3}\,dx = \frac{1}{2}\int \frac{2x+2-2+6}{x^2+2x+3}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+3}\,dx + \int \frac{2}{x^2+2x+3}\,dx$$

$$\overset{c}{=} \frac{1}{2}\log(x^2+2x+3)$$

$$\int \frac{2}{x^2+2x+3}\,dx = \int \frac{2}{(x+1)^2+2}\,dx = \frac{1}{\cancel{2}}\int \frac{\cancel{2}}{\left(\frac{x+1}{\sqrt{2}}\right)^2+1}\,dx$$

$$\stackrel{c}{=} \sqrt{2}\,\arctan \frac{x+1}{\sqrt{2}}$$

Závěr

$$\int \stackrel{c}{=} \frac{-3}{x-3} + \frac{1}{2}\log\left(x^2+2x+3\right) + \sqrt{2}\,\arctan\frac{x+1}{\sqrt{2}}$$

$$x \in (-\infty, 3), (3, \infty)$$

(4) $m = \dfrac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ $E_k = mc^2 - m_0c^2$

cíl: $E_k = \frac{1}{2} m_0 v^2$ v velmi malé vůči c.

$$E_k = c^2 (m - m_0) = c^2 \left(\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} - m_0 \right) = c^2 m_0 \left(\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1 \right)$$

$$(1+y)^{-1/2} = 1 + -\frac{1}{2} y + \frac{-1/2(-1/2-1)}{2!} y^2 + o(y^2)$$

$y = -\frac{v^2}{c^2}$ $\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + o\left(\frac{v^4}{c^4}\right)$

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$$E_k = c^2 m_0 \left(\cancel{1} + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + o\left(\frac{v^4}{c^4}\right) \cancel{-1} \right)$$

pokud vezmeme jen 1. člen, dostaneme

$$E_k \approx c^2 m_0 \cdot \frac{1}{2}\frac{v^2}{c^2} + \underbrace{c^2 m_0 \cdot o\left(\frac{v^2}{c^2}\right)}_{\to 0 \text{ pro } v \to 0}$$

$$= \underline{\underline{\frac{1}{2} m_0 v^2}}$$